Spectral measures and beyond

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"Introduction to operator algebras", 6/6

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Quantum algebra 1

There are basically 4 types of quantum algebra beasts:

1. Quantum groups. Basic examples $G \subset U_N^+$.
2. Tensor categories. $C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$.
3. Planar algebras. Simplest $P_k = End(u^{\otimes k})$.
4. Subfactors. Simplest $A^G \subset (M_N(\mathbb{C}) \otimes A)^G$.

$\Longrightarrow$ For full fun, classify the finite index subfactors of R!

Quantum algebra 2

Further quantum beasts, more advanced/specialized:

1. Quantum permutation groups: $G \subset S_N^+$.

2. Tensor categories: $C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$.

3. Spin planar algebras: $P_k = Fix(u^{\otimes k})$.

4. Fixed point subfactors: $A^G \subset (\mathbb{C}^N \otimes A)^G$.

$\implies$ Also with $\{1, \ldots, N\}$ replaced by finite NC spaces.

Spectral measures

<u>Definition</u>. The spectral measure of $G \subset U_N^+$ is the law:

$$\mu = law(\chi) \quad , \quad \chi = Tr(u)$$

For a planar algebra P, this is the measure having as moments:

$$M_k = \dim(P_k)$$

Similar definitions for tensor categories and subfactors.

<u>Comment</u>. A popular invariant in "algebra" is the Poincaré series:

$$f(z) = \sum_{k=0}^{\infty} \dim(P_k) z^k$$

This is the Stieltjes transform of the spectral measure μ.

Free probability 1/4

Definition. Two subalgebras $B, C \subset A$ are called:

(1) Independent, if $tr(b) = tr(c) = 0$ implies $tr(bc) = 0$.

(2) Free, if $tr(b_i) = tr(c_i) = 0$ implies $tr(b_1 c_1 b_2 c_2 \ldots) = 0$.

Theorem. We have the following results:

(1) $C^*(\Gamma), C^*(\Lambda)$ are independent inside $C^*(\Gamma \times \Lambda)$.

(2) $C^*(\Gamma), C^*(\Lambda)$ are free inside $C^*(\Gamma * \Lambda)$.

$\implies$ Linearization of the usual/free convolution: $\log F/R$.

Free probability 2/4

<u>Theorem</u>. Assuming that $x_1, x_2, x_3, \ldots \in A$ are i.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim \mathcal{N}(0, t)$$

where $\mathcal{N}(0, t)$ is the normal law, having density $\frac{1}{\sqrt{2\pi t}} e^{-y^2/2t} dy$.

<u>Theorem</u>. Assuming that $x_1, x_2, x_3, \ldots \in A$ are f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim \gamma_t$$

where γ_t is the Wigner law, having density $\frac{1}{2\pi t} \sqrt{4t^2 - x^2} dx$.

Free probability 3/4

Theorem. We have the following convergence,

$$\left(\left(1-\frac{1}{n}\right)\delta_0+\frac{1}{n}\delta_t\right)^{*n}\to p_t$$

with p_t being the Poisson law of parameter $t>0$.

Theorem. We have the following convergence,

$$\left(\left(1-\frac{1}{n}\right)\delta_0+\frac{1}{n}\delta_t\right)^{\boxplus n}\to \pi_t$$

with π_t being the Marchenko-Pastur law of parameter $t>0$,

$$\pi_t=\max(1-t,0)\delta_0+\frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx$$

also called free Poisson law of parameter $t>0$.

Free probability 4/4

Definition. Associated to any compactly supported positive measure ρ on $\mathbb{R}$, with mass $c = mass(\rho)$, are the probability measures

$$p_\rho = \lim_{n\to\infty}\left(\left(1-\frac{c}{n}\right)\delta_0 + \frac{1}{n}\rho\right)^{*n}$$

$$\pi_\rho = \lim_{n\to\infty}\left(\left(1-\frac{c}{n}\right)\delta_0 + \frac{1}{n}\rho\right)^{\boxplus n}$$

called compound Poisson and compound free Poisson laws.

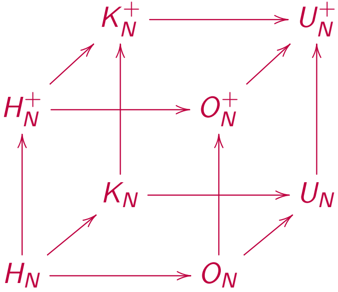
Definition. Let ε_s be the uniform mesure on the s-th roots of unity.

(1) With $\rho = t\varepsilon_s$, we get the classical and free Bessel laws.

(2) At $s = 2, \infty$, we call these laws "real" and "complex".

Core objects

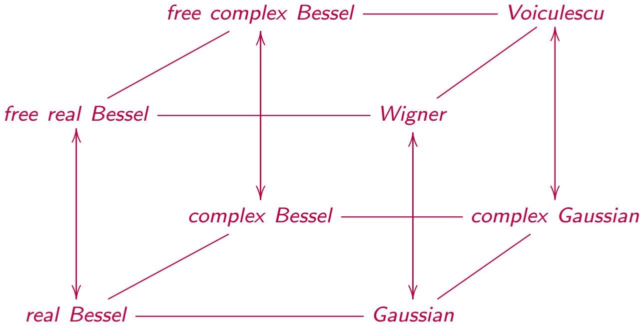
The main unitary and reflection groups, where $H_N = \mathbb{Z}_2 \wr S_N$, $K_N = \mathbb{T} \wr S_N$, and H_N^+, K_N^+ are their free analogues:



In planar algebra terms, the free objects correspond to TL and FC, depending on the correspondence which is chosen.

Core measures

The asymptotic laws of truncated characters $\chi_t = \sum_{i=1}^{[tN]} u_{ii}$ are



with the vertical arrows standing for the Bercovici-Pata bijection.

Beyond

These were very basic results, concerning the most basic quantum groups, and the most basic planar algebras. Problems:

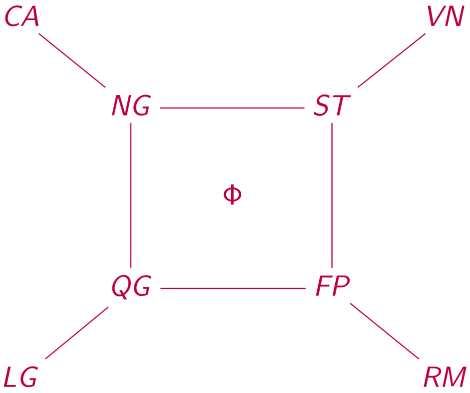
(1) Quantum groups. Easy quantum groups, and beyond. Meander determinants. Advanced probability, a la Diaconis.

(2) Subfactors. Poincaré series obstructions. Spectral measure blow-up. Big index subfactors. Also, what is $t > 0$.

(3) Free probability, NCG. The tori and other manifolds are subject to Meixner/free Meixner. Unification with Bercovici-Pata.

Conclusion

The picture of modern operator algebras is as follows,



with the hot stuff and physics being in the middle.