Quantum algebra explained

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Landscape

There are basically 4 types of quantum algebra beasts:

1. Quantum groups
2. Tensor categories
3. Planar algebras
4. Subfactors

Quantum groups 1

Fact. Consider a compact group G, with structural maps:

$$m : G \times G \to G \quad , \quad u : \{.\} \to G \quad , \quad i : G \to G$$

By transposing m, u, i, we obtain certain morphisms Δ, ε, S.

Definition. A Hopf C^*-algebra is a C^*-algebra A, with morphisms

$$\Delta : A \to A \otimes A \quad , \quad \varepsilon : A \to \mathbb{C} \quad , \quad S : A \to A^{opp}$$

satisfying suitable axioms, as in the group case. We write

$$A = C(G) = C^*(\Gamma)$$

and call G compact quantum group, and Γ discrete quantum group.

Quantum groups 2

Examples. We have compact quantum groups defined via

$$C(O_N^+) = C^*\left((u_{ij})_{i,j=1\ldots N}\Big| u=\bar{u}, u^t=u^{-1}\right)$$

$$C(U_N^+) = C^*\left((u_{ij})_{i,j=1\ldots N}\Big| u^*=u^{-1}, u^t=\bar{u}^{-1}\right)$$

called free orthogonal, and free unitary quantum groups.

Remark. These quantum groups act on the free spheres

$$O_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+} \quad , \quad U_N^+ \curvearrowright S^{N-1}_{\mathbb{C},+}$$

so we have uniform integration, and von Neumann algebras.

Tensor categories 1

Definition. A corepresentation of A is a matrix $v \in M_n(A)$,

$$\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj} \quad , \quad \varepsilon(v_{ij}) = \delta_{ij} \quad , \quad S(v_{ij}) = v_{ji}^*$$

As basic examples, we have $u = (u_{ij})$, and its tensor powers $u^{\otimes k}$.

Theorem. The corepresentations are subject to a Peter-Weyl type theory. In particular, the irreducible ones appear by decomposing the tensor powers $u^{\otimes k}$ of the fundamental corepresentation u.

Tensor categories 2

Definition. The Tannakian category of a Woronowicz algebra (A, u) is the following collection $C = (C(k, l))$ of vector spaces:

$$C(k, l) = Hom(u^{\otimes k}, u^{\otimes l})$$

Definition. The Woronowicz algebra associated to a Tannakian category $C = (C(k, l))$ is constructed as follows:

$$A = C^* \left((u_{ij})_{i,j=1 \ldots N} \middle| T \in Hom(u^{\otimes k}, u^{\otimes l}), \forall T \in C(k, l) \right)$$

Theorem. These operations produce a bijection $A \leftrightarrow C$, between Woronowicz algebras, and Tannakian categories.

Planar algebras 1

<u>Definition</u>. A planar algebra is a collection of FD $*$-algebras

$$P = (P_k)_{k \in \mathbb{N}}$$

such that whenever we have a diagram consisting of

– a big circle, with k points on it

– containing small circles, with k_i points on them

– and with strings connecting the $k + \sum_i k_i$ points

we can put "input" elements of P_{k_i} on the small circles, and we obtain an "output" element of P_k on the big circle.

<u>Examples</u>. The Temperley-Lieb algebra, $P_k = TL_k$. The tensor planar algebra, $P_k = \mathcal{L}(H^{\otimes k})$, with $H = \mathbb{C}^N$.

Planar algebras 2

Theorem. Given a compact quantum group $G \subset U_N^+$, the sequence of finite dimensional algebras

$$P_k = End(u^{\otimes k})$$

form a subalgebra of the tensor planar algebra. Any subalgebra of the tensor planar algebra appears in this way.

Proof. By Tannakian duality we have a correspondence:

$$G \subset U_N^+ \quad \longleftrightarrow \quad C \subset \mathcal{L}(H^{\otimes k}, H^{\otimes l})_{kl}$$

By restricing to the diagonal (via Frobenius) we get the result.

Subfactors 1

Theorem. Let $A_0 \subset A_1$ be a subfactor of finite index, and consider its Jones tower, obtained by basic construction:

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \ldots$$

The Jones projections $e_1, e_2, e_3, \ldots$ generate then a copy of the Temperley-Lieb algebra TL, inside the ambient algebra $B(H)$.

Theorem. The sequence of higher relative commutants

$$P_k = A_0' \cap A_k$$

form a planar algebra, extending the algebra $< e_i >= TL$.

Subfactors 2

Theorem. Given a compact quantum group $G \subset U_N^+$, consider its adjoint action on the matrix algebra $M_N(\mathbb{C})$:

$$G \curvearrowright M_N(\mathbb{C})$$

Assume that G acts on a II_1 factor A, minimally, $(A^G)' \cap A = \mathbb{C}$. We have then an inclusion of II_1 factors, of index N^2,

$$A^G \subset (M_N(\mathbb{C}) \otimes A)^G$$

whose planar algebra is the previously constructed one, namely:

$$P_k = End(u^{\otimes k})$$

As examples, O_N^+, U_N^+ produce Temperley-Lieb subfactors.

Summary

Very basic examples of the 4 quantum algebra beasts:

1. Quantum groups: $G \subset U_N^+$.
2. Tensor categories: $C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$.
3. Planar algebras: $P_k = End(u^{\otimes k})$.
4. Subfactors: $A^G \subset (M_N(\mathbb{C}) \otimes A)^G$.

An interesting variation, to be discussed next time:

1. Quantum groups: $G \subset S_N^+$.
2. Tensor categories: $C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$.
3. Planar algebras: $P_k = Fix(u^{\otimes k})$.
4. Subfactors: $A^G \subset (\mathbb{C}^N \otimes A)^G$.

Extensions

There are countless extensions and variations of all this:

(1) Quantum groups. Those studied here, $G \subset U_N^+$, are technically "compact quantum Lie groups of Kac type". Many other.

(2) Tensor categories. Those studied here are over $k = \mathbb{C}$, have an involution $*$, and importantly, are "semisimple". Many other.

(3) Planar algebras. Similar comments. In addition, the world of planar algebras, even "unmodified", is substantially bigger.

(4) Subfactors. Same situation as for planar algebras. Very good question here: what are the finite index subfactors of R?