Analysis of linear operators

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Abstract. This is an introduction to the theory of linear operators. We first discuss the basic examples of Hilbert spaces, for the most appearing in relation with questions from physics, and the relevant linear operators on them. Then we get into a systematic study of such operators, notably featuring the spectral theorem in infinite dimensions, in its various formulations, and its applications. We then discuss in detail, with some further theory and examples, the case of the compact operators, which are quite close to the usual, finite matrices. Finally, we provide an introduction to the theory of operator algebras, with the basics here explained, by using the spectral theorem.

Preface

Operator theory is a wide business, and this for good reason, the point being that such operators are in fact infinite matrices, and so all the linear algebra that you know, or perhaps that you don't know yet, is naturally part of the theory. To be more precise, linear algebra is the "trivial" part of operator theory, corresponding to the case where that infinite matrix, that we are trying hard to understand, happens to be finite.

Besides linear algebra, operator theory knows as well how to swallow probability, and this because one of the most beautiful and advanced objects of modern probability, which are the random matrices, can be thought of as being linear operators too.

Finally, differential geometry, analysis on manifolds, Fourier transform theory, ordinary and partial differential equations, and so on, are all related to operator theory too, the general principle being that, no matter what geometry or analysis problem you are looking at, there is always a key operator there, waiting to be diagonalized.

So, this is the general principle of operator theory, by definition some sort of linear algebra in infinite dimensions, and with this point of view providing the main tools for the study of operators, in analogy what we know from linear algebra, and in practice, with all this being related to mathematics and physics in a large sense, for all tastes.

The present book is a standard introduction to operator theory, with the aim being, as usual for operator theory books, double. On one hand we would like to present some tools for the study of linear operators, as much and as sharp as we can, inspired by usual linear algebra. And on the other hand we would like to keep en eye on the main examples of operators, coming from geometry, analysis, probability and physics.

More in detail now, the book is organized in 4 parts, as follows:

Part I discusses the basic examples of Hilbert spaces, for the most appearing in relation with questions from physics, and the relevant linear operators on them.

Part II gets into a systematic study of such operators, notably featuring the spectral theorem in infinite dimensions, in its various formulations, and its applications.

Part III discusses in detail, with some further theory and examples, the case of the compact operators, which are quite close to the usual, finite matrices.

Part IV is an introduction to the theory of the algebras that the linear operators can form, with the basics here explained, by using the spectral theorem.

In the hope that you will find this book useful. As briefly explained above, operator theory is a wide discipline, and there are many ways of getting introduced to it, depending on the main examples of operators that you have in mind. Here we will be rather guided by old quantum mechanics, as developed by Heisenberg, Schrödinger and others, and it is my hope is that you will like this approach, which is something quite natural.

Many thanks to my linear algebra undergraduate students, and to my functional analysis graduate students too, operator theory and related physics aspects are fun and interesting, and are a good topic for side discussions, and even for some systematic teaching, when things in linear algebra or in functional analysis get a bit boring.

Thanks as well to my cats, operator theory is an art, involving never-ending learning, and nothing more inspiring, than watching a cat quickly diagonalizing a mouse.

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Teo Banica

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Part I

Linear operators

We're leaving together
But still, it's farewell
And maybe we'll come back
To Earth, who can tell

CHAPTER 1

Linear algebra

1a. Linear maps

According to various findings in physics, starting with those of Heisenberg from the early 1920s, basic quantum mechanics involves linear operators $T : H \to H$ from a complex Hilbert space H to itself. The space H is typically infinite dimensional, a basic example being the Schrödinger space $H = L^2(\mathbb{R}^3)$ of the wave functions $\psi : \mathbb{R}^3 \to \mathbb{C}$ of the electron. In fact, in what regards the electron, this space $H = L^2(\mathbb{R}^3)$ is basically the correct one, with the only adjustment needed, due to Pauli and others, being that of tensoring with a copy of $K = \mathbb{C}^2$, in order to account for the electron spin.

But more on this later. Let us start this book more modestly, as follows:

Fact 1.1. *We are interested in quantum mechanics, taking place in infinite dimensions, but as a main source of inspiration we will have $H = \mathbb{C}^N$, with scalar product*

$$< x, y >= \sum_i x_i \bar{y}_i$$

with the linearity at left being the standard mathematical convention. More specifically, we will be interested in the mathematics of the linear operators $T : H \to H$.

The point now, that you surely know about, is that the above operators $T : H \to H$ correspond to the square matrices $A \in M_N(\mathbb{C})$. Thus, as a preliminary to what we want to do in this book, we need a good knowledge of linear algebra over $\mathbb{C}$.

You probably know well linear algebra, but always good to recall this, and this will be the purpose of the present chapter. Let us start with the very basics:

Theorem 1.2. *The linear maps $T : \mathbb{C}^N \to \mathbb{C}^N$ are in correspondence with the square matrices $A \in M_N(\mathbb{C})$, with the linear map associated to such a matrix being*

$$Tx = Ax$$

and with the matrix associated to a linear map being $A_{ij} =< Te_j, e_i >$.

Proof. The first assertion is clear, because a linear map $T : \mathbb{C}^N \to \mathbb{C}^N$ must send a vector $x \in \mathbb{C}^N$ to a certain vector $Tx \in \mathbb{C}^N$, all whose components are linear combinations of the components of x. Thus, we can write, for certain complex numbers $A_{ij} \in \mathbb{C}$:

$$T\begin{pmatrix}x_1\\ \vdots\\ \vdots\\ x_N\end{pmatrix} = \begin{pmatrix}A_{11}x_1 + \ldots + A_{1N}x_N\\ \vdots\\ \vdots\\ A_{N1}x_1 + \ldots + A_{NN}x_N\end{pmatrix}$$

Now the parameters $A_{ij} \in \mathbb{C}$ can be regarded as being the entries of a square matrix $A \in M_N(\mathbb{C})$, and with the usual convention for matrix multiplication, we have:

$$Tx = Ax$$

Regarding the second assertion, with $Tx = Ax$ as above, if we denote by $e_1, \ldots, e_N$ the standard basis of $\mathbb{C}^N$, then we have the following formula:

$$Te_j = \begin{pmatrix}A_{1j}\\ \vdots\\ \vdots\\ A_{Nj}\end{pmatrix}$$

But this gives the second formula, $< Te_j, e_i >= A_{ij}$, as desired. □

Our claim now is that, no matter what we want to do with T or A, of advanced type, we will run at some point into their adjoints T^* and A^*, constructed as follows:

THEOREM 1.3. *The adjoint operator $T^* : \mathbb{C}^N \to \mathbb{C}^N$, which is given by*

$$< Tx, y >=< x, T^*y >$$

corresponds to the adjoint matrix $A^* \in M_N(\mathbb{C})$, given by

$$(A^*)_{ij} = \bar{A}_{ji}$$

via the correspondence between linear maps and matrices constructed above.

PROOF. Given a linear map $T : \mathbb{C}^N \to \mathbb{C}^N$, fix $y \in \mathbb{C}^N$, and consider the linear form $\varphi(x) =< Tx, y >$. This form must be as follows, for a certain vector $T^*y \in \mathbb{C}^N$:

$$\varphi(x) =< x, T^*y >$$

Thus, we have constructed a map $y \to T^*y$ as in the statement, which is obviously linear, and that we can call T^*. Now by taking the vectors $x, y \in \mathbb{C}^N$ to be elements of the standard basis of $\mathbb{C}^N$, our defining formula for T^* reads:

$$< Te_i, e_j >=< e_i, T^*e_j >$$

By reversing the scalar product on the right, this formula can be written as:

$$< T^*e_j, e_i >= \overline{< Te_i, e_j >}$$

But this means that the matrix of T^* is given by $(A^*)_{ij} = \bar{A}_{ji}$, as desired. □

Getting back to our claim, the adjoints $*$ are indeed ubiquitous, as shown by:

THEOREM 1.4. *The following happen:*

(1) $T(x) = Ux$ *with* $U \in M_N(\mathbb{C})$ *is an isometry precisely when* $U^* = U^{-1}$.
(2) $T(x) = Px$ *with* $P \in M_N(\mathbb{C})$ *is a projection precisely when* $P^2 = P^* = P$.

PROOF. Let us first recall that the lengths, or norms, of the vectors $x \in \mathbb{C}^N$ can be recovered from the knowledge of the scalar products, as follows:

$$||x|| = \sqrt{< x, x >}$$

Conversely, we can recover the scalar products out of norms, by using the following difficult to remember formula, called complex polarization identity:

$$4 < x, y >= ||x+y||^2 - ||x-y||^2 + i||x+iy||^2 - i||x-iy||^2$$

The proof of this latter formula is indeed elementary, as follows:

$$\begin{aligned} & ||x+y||^2 - ||x-y||^2 + i||x+iy||^2 - i||x-iy||^2 \\ = \ & ||x||^2 + ||y||^2 - ||x||^2 - ||y||^2 + i||x||^2 + i||y||^2 - i||x||^2 - i||y||^2 \\ & +2Re(< x, y >) + 2Re(< x, y >) + 2iIm(< x, y >) + 2iIm(< x, y >) \\ = \ & 4 < x, y > \end{aligned}$$

Finally, we will use Theorem 1.3, and more specifically the following formula coming from there, valid for any matrix $A \in M_N(\mathbb{C})$ and any two vectors $x, y \in \mathbb{C}^N$:

$$< Ax, y >=< x, A^*y >$$

(1) Given a matrix $U \in M_N(\mathbb{C})$, we have indeed the following equivalences, with the first one coming from the polarization identity, and the other ones being clear:

$$\begin{aligned} ||Ux|| = ||x|| & \iff < Ux, Uy >=< x, y > \\ & \iff < x, U^*Uy >=< x, y > \\ & \iff U^*Uy = y \\ & \iff U^*U = 1 \\ & \iff U^* = U^{-1} \end{aligned}$$

(2) Given a matrix $P \in M_N(\mathbb{C})$, in order for $x \to Px$ to be an oblique projection, we must have $P^2 = P$. Now observe that this projection is orthogonal when:

$$\begin{aligned} < Px - x, Py >= 0 & \iff < P^*Px - P^*x, y >= 0 \\ & \iff P^*Px - P^*x = 0 \\ & \iff P^*P - P^* = 0 \\ & \iff P^*P = P^* \end{aligned}$$

The point now is that by conjugating the last formula, we obtain $P^*P = P$. Thus we must have $P = P^*$, and this gives the result. □

Summarizing, the linear operators come in pairs T, T^*, and the associated matrices come as well in pairs A, A^*. This is something quite interesting, philosophically speaking, and will keep this in mind, and come back to it later, on numerous occasions.

1b. Diagonalization

Let us discuss now the diagonalization question for the linear maps and matrices. Again, we will be quite brief here, and for more, we refer to any standard linear algebra book. By the way, there will be some complex analysis involved too, and here we refer to Rudin [**81**]. Which book of Rudin will be in fact the one and only true prerequisite for reading the present book, but more on references and reading later.

The basic diagonalization theory, formulated in terms of matrices, is as follows:

PROPOSITION 1.5. *A vector $v \in \mathbb{C}^N$ is called eigenvector of $A \in M_N(\mathbb{C})$, with corresponding eigenvalue λ, when A multiplies by λ in the direction of v:*

$$Av = \lambda v$$

In the case where $\mathbb{C}^N$ has a basis $v_1, \ldots, v_N$ formed by eigenvectors of A, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_N$, in this new basis A becomes diagonal, as follows:

$$A \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

Equivalently, if we denote by $D = diag(\lambda_1, \ldots, \lambda_N)$ the above diagonal matrix, and by $P = [v_1 \ldots v_N]$ the square matrix formed by the eigenvectors of A, we have:

$$A = PDP^{-1}$$

In this case we say that the matrix A is diagonalizable.

PROOF. This is something which is clear, the idea being as follows:

(1) The first assertion is clear, because the matrix which multiplies each basis element v_i by a number λ_i is precisely the diagonal matrix $D = diag(\lambda_1, \ldots, \lambda_N)$.

(2) The second assertion follows from the first one, by changing the basis. We can prove this by a direct computation as well, because we have $Pe_i = v_i$, and so:

$$\begin{aligned} PDP^{-1}v_i &= PDe_i \\ &= P\lambda_i e_i \\ &= \lambda_i Pe_i \\ &= \lambda_i v_i \end{aligned}$$

Thus, the matrices A and PDP^{-1} coincide, as stated. □

Let us recall as well that the basic example of a non diagonalizable matrix, over the complex numbers as above, is the following matrix:

$$J = \begin{pmatrix}0&1\\0&0\end{pmatrix}$$

Indeed, we have $J\binom{x}{y} = \binom{y}{0}$, so the eigenvectors are the vectors of type $\binom{x}{0}$, all with eigenvalue 0. Thus, we have not enough eigenvectors for constructing a basis of $\mathbb{C}^2$.

In general, in order to study the diagonalization problem, the idea is that the eigenvectors can be grouped into linear spaces, called eigenspaces, as follows:

THEOREM 1.6. *Let $A \in M_N(\mathbb{C})$, and for any eigenvalue $\lambda \in \mathbb{C}$ define the corresponding eigenspace as being the vector space formed by the corresponding eigenvectors:*

$$E_\lambda = \left\{ v \in \mathbb{C}^N \Big| Av = \lambda v \right\}$$

These eigenspaces E_λ are then in a direct sum position, in the sense that given vectors $v_1 \in E_{\lambda_1}, \ldots, v_k \in E_{\lambda_k}$ corresponding to different eigenvalues $\lambda_1, \ldots, \lambda_k$, we have:

$$\sum_i c_i v_i = 0 \implies c_i = 0$$

In particular we have the following estimate, with sum over all the eigenvalues,

$$\sum_\lambda \dim(E_\lambda) \leq N$$

and our matrix is diagonalizable precisely when we have equality.

PROOF. We prove the first assertion by recurrence on $k \in \mathbb{N}$. Assume by contradiction that we have a formula as follows, with the scalars $c_1, \ldots, c_k$ being not all zero:

$$c_1 v_1 + \ldots + c_k v_k = 0$$

By dividing by one of these scalars, we can assume that our formula is:

$$v_k = c_1 v_1 + \ldots + c_{k-1} v_{k-1}$$

Now let us apply A to this vector. On the left we obtain:

$$Av_k = \lambda_k v_k = \lambda_k c_1 v_1 + \ldots + \lambda_k c_{k-1} v_{k-1}$$

On the right we obtain something different, as follows:

$$\begin{aligned} A(c_1 v_1 + \ldots + c_{k-1} v_{k-1}) &= c_1 A v_1 + \ldots + c_{k-1} A v_{k-1} \\ &= c_1 \lambda_1 v_1 + \ldots + c_{k-1} \lambda_{k-1} v_{k-1} \end{aligned}$$

We conclude from this that the following equality must hold:

$$\lambda_k c_1 v_1 + \ldots + \lambda_k c_{k-1} v_{k-1} = c_1 \lambda_1 v_1 + \ldots + c_{k-1} \lambda_{k-1} v_{k-1}$$

On the other hand, we know by recurrence that the vectors $v_1,\ldots,v_{k-1}$ must be linearly independent. Thus, the coefficients must be equal, at right and at left:

$$\lambda_k c_1 = c_1\lambda_1$$
$$\vdots$$
$$\lambda_k c_{k-1} = c_{k-1}\lambda_{k-1}$$

Now since at least one of the numbers c_i must be nonzero, from $\lambda_k c_i = c_i\lambda_i$ we obtain $\lambda_k = \lambda_i$, which is a contradiction. Thus our proof by recurrence of the first assertion is complete. As for the second assertion, this follows from the first one. $\square$

In order to reach now to more advanced results, we can use the characteristic polynomial, which appears via the following fundamental result:

THEOREM 1.7. *Given a matrix $A\in M_N(\mathbb{C})$, consider its characteristic polynomial:*

$$P(x)=\det(A-x1_N)$$

The eigenvalues of A are then the roots of P. Also, we have the inequality

$$\dim(E_\lambda)\leq m_\lambda$$

where m_λ is the multiplicity of λ, as root of P.

PROOF. The first assertion follows from the following computation, using the fact that a linear map is bijective when the determinant of the associated matrix is nonzero:

$$\begin{aligned}\exists v, Av=\lambda v &\iff \exists v,(A-\lambda 1_N)v=0\\ &\iff \det(A-\lambda 1_N)=0\end{aligned}$$

Regarding now the second assertion, given an eigenvalue λ of our matrix A, consider the dimension $d_\lambda=\dim(E_\lambda)$ of the corresponding eigenspace. By changing the basis of $\mathbb{C}^N$, as for the eigenspace E_λ to be spanned by the first d_λ basis elements, our matrix becomes as follows, with B being a certain smaller matrix:

$$A\sim\begin{pmatrix}\lambda 1_{d_\lambda}&0\\0&B\end{pmatrix}$$

We conclude that the characteristic polynomial of A is of the following form:

$$P_A=P_{\lambda 1_{d_\lambda}}P_B=(\lambda-x)^{d_\lambda}P_B$$

Thus the multiplicity m_λ of our eigenvalue λ, as a root of P, satisfies $m_\lambda\geq d_\lambda$, and this leads to the conclusion in the statement. $\square$

Now recall that we are over $\mathbb{C}$, which is something that we have not used yet, in our last two statements. And the point here is that we have the following key result:

THEOREM 1.8. *Any polynomial $P \in \mathbb{C}[X]$ decomposes as*

$$P = c(X - a_1) \ldots (X - a_N)$$

with $c \in \mathbb{C}$ and with $a_1, \ldots, a_N \in \mathbb{C}$.

PROOF. It is enough to prove that P has one root, and we do this by contradiction. Assume that P has no roots, and pick a number $z \in \mathbb{C}$ where $|P|$ attains its minimum:

$$|P(z)| = \min_{x \in \mathbb{C}} |P(x)| > 0$$

Since $Q(t) = P(z+t) - P(z)$ is a polynomial which vanishes at $t = 0$, this polynomial must be of the form ct^k + higher terms, with $c \neq 0$, and with $k \geq 1$ being an integer. We obtain from this that, with $t \in \mathbb{C}$ small, we have the following estimate:

$$P(z+t) \simeq P(z) + ct^k$$

Now let us write $t = rw$, with $r > 0$ small, and with $|w| = 1$. Our estimate becomes:

$$P(z + rw) \simeq P(z) + cr^k w^k$$

Now recall that we have assumed $P(z) \neq 0$. We can therefore choose $w \in \mathbb{T}$ such that cw^k points in the opposite direction to that of $P(z)$, and we obtain in this way:

$$|P(z + rw)| \simeq |P(z) + cr^k w^k| = |P(z)|(1 - |c|r^k)$$

Now by choosing $r > 0$ small enough, as for the error in the first estimate to be small, and overcame by the negative quantity $-|c|r^k$, we obtain from this:

$$|P(z + rw)| < |P(z)|$$

But this contradicts our definition of $z \in \mathbb{C}$, as a point where $|P|$ attains its minimum. Thus P has a root, and by recurrence it has N roots, as stated. □

Now by putting everything together, we obtain the following result:

THEOREM 1.9. *Given a matrix $A \in M_N(\mathbb{C})$, consider its characteristic polynomial*

$$P(X) = \det(A - X1_N)$$

then factorize this polynomial, by computing the complex roots, with multiplicities,

$$P(X) = (-1)^N (X - \lambda_1)^{n_1} \ldots (X - \lambda_k)^{n_k}$$

and finally compute the corresponding eigenspaces, for each eigenvalue found:

$$E_i = \left\{ v \in \mathbb{C}^N \Big| Av = \lambda_i v \right\}$$

The dimensions of these eigenspaces satisfy then the following inequalities,

$$\dim(E_i) \leq n_i$$

and A is diagonalizable precisely when we have equality for any i.

PROOF. This follows by combining Theorem 1.6, Theorem 1.7 and Theorem 1.8. Indeed, the statement is well formulated, thanks to Theorem 1.8. By summing the inequalities $\dim(E_\lambda)\leq m_\lambda$ from Theorem 1.7, we obtain an inequality as follows:

$$\sum_\lambda\dim(E_\lambda)\leq\sum_\lambda m_\lambda\leq N$$

On the other hand, we know from Theorem 1.6 that our matrix is diagonalizable when we have global equality. Thus, we are led to the conclusion in the statement. □

This was for the main result of linear algebra. There are countless applications of this, and generally speaking, advanced linear algebra consists in building on Theorem 1.9.

In practice, diagonalizing a matrix remains something quite complicated. Let us record a useful algorithmic version of the above result, as follows:

THEOREM 1.10. *The square matrices $A\in M_N(\mathbb C)$ can be diagonalized as follows:*

(1) *Compute the characteristic polynomial.*
(2) *Factorize the characteristic polynomial.*
(3) *Compute the eigenvectors, for each eigenvalue found.*
(4) *If there are no N eigenvectors, A is not diagonalizable.*
(5) *Otherwise, A is diagonalizable, $A=PDP^{-1}$.*

PROOF. This is an informal reformulation of Theorem 1.9, with (4) referring to the total number of linearly independent eigenvectors found in (3), and with $A=PDP^{-1}$ in (5) being the usual diagonalization formula, with P,D being as before. □

As an illustration for all this, which is a must-know computation, we have:

THEOREM 1.11. *The rotation of angle $t\in\mathbb R$ in the plane diagonalizes as:*

$$\begin{pmatrix}\cos t&-\sin t\\ \sin t&\cos t\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&1\\ i&-i\end{pmatrix}\begin{pmatrix}e^{-it}&0\\ 0&e^{it}\end{pmatrix}\begin{pmatrix}1&-i\\ 1&i\end{pmatrix}$$

Over the reals this is impossible, unless $t=0,\pi$, where the rotation is diagonal.

PROOF. Observe first that, as indicated, unlike we are in the case $t=0,\pi$, where our rotation is $\pm 1_2$, our rotation is a "true" rotation, having no eigenvectors in the plane. Fortunately the complex numbers come to the rescue, via the following computation:

$$\begin{pmatrix}\cos t&-\sin t\\ \sin t&\cos t\end{pmatrix}\begin{pmatrix}1\\ i\end{pmatrix}=\begin{pmatrix}\cos t-i\sin t\\ i\cos t+\sin t\end{pmatrix}=e^{-it}\begin{pmatrix}1\\ i\end{pmatrix}$$

We have as well a second complex eigenvector, coming from:

$$\begin{pmatrix}\cos t&-\sin t\\ \sin t&\cos t\end{pmatrix}\begin{pmatrix}1\\ -i\end{pmatrix}=\begin{pmatrix}\cos t+i\sin t\\ -i\cos t+\sin t\end{pmatrix}=e^{it}\begin{pmatrix}1\\ -i\end{pmatrix}$$

Thus, we are led to the conclusion in the statement. □

As another basic illustration, we have the following result:

THEOREM 1.12. *The all-one, or flat matrix, namely*

$$\mathbb{I}_N = \begin{pmatrix} 1 & \ldots & 1 \\ \vdots & & \vdots \\ 1 & \ldots & 1 \end{pmatrix}$$

diagonalizes as follows, over the complex numbers,

$$\mathbb{I}_N = \frac{1}{N} F_N Q F_N^*$$

with $F_N = (w^{ij})_{ij}$ *with* $w = e^{2\pi i/N}$ *being the Fourier matrix, and* $Q = diag(N, 0, \ldots, 0)$*.*

PROOF. It is clear that we have $\mathbb{I}_N = NP_N$, with P_N being the projection on the all-1 vector $\xi = (1)_i \in \mathbb{R}^N$. Thus, $\mathbb{I}_N$ diagonalizes over $\mathbb{R}$, as follows:

$$\mathbb{I}_N \sim \begin{pmatrix} N & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$$

The problem, however, is that when looking for 0-eigenvectors, in order to have an explicit diagonalization formula, we must solve the following equation:

$$x_1 + \ldots + x_N = 0$$

And this is not an easy task, if we want a nice basis for the space of solutions. Fortunately, complex numbers come to the rescue, and we are led to the conclusion in the statement. We will leave the verifications here as an instructive exercise. □

1c. Matrix tricks

At the level of basic examples of diagonalizable matrices, we first have the following result, which provides us with the "generic" examples:

THEOREM 1.13. *For a matrix* $A \in M_N(\mathbb{C})$ *the following conditions are equivalent,*

1. *The eigenvalues are different,* $\lambda_i \neq \lambda_j$,
2. *The characteristic polynomial* P *has simple roots,*
3. *The characteristic polynomial satisfies* $(P, P') = 1$,
4. *The resultant of* P, P' *is nonzero,* $R(P, P') \neq 0$,
5. *The discriminant of* P *is nonzero,* $\Delta(P) \neq 0$,

and in this case, the matrix is diagonalizable.

PROOF. The last assertion holds indeed, due to Theorem 1.9. As for the equivalences in the statement, these are all standard, the idea for their proofs, along with some more theory, needed for using in practice the present result, being as follows:

(1) $\iff$ (2) This follows from Theorem 1.9.

(2) $\iff$ (3) This is standard, the double roots of P being roots of P'.

(3) $\iff$ (4) The idea here is that associated to any two polynomials P,Q is their resultant $R(P,Q)$, which checks whether P,Q have a common root. Let us write:

$$P = c(X-a_1)\ldots(X-a_k)$$

$$Q = d(X-b_1)\ldots(X-b_l)$$

We can define then the resultant as being the following quantity:

$$R(P,Q) = c^l d^k \prod_{ij}(a_i - b_j)$$

The point now, that we will explain as well, is that this is a polynomial in the coefficients of P,Q, with integer coefficients. Indeed, this can be checked as follows:

– We can expand the formula of $R(P,Q)$, and in what regards $a_1,\ldots,a_k$, which are the roots of P, we obtain in this way certain symmetric functions in these variables, which will be therefore polynomials in the coefficients of P, with integer coefficients.

– We can then look what happens with respect to the remaining variables $b_1,\ldots,b_l$, which are the roots of Q. Once again what we have here are certain symmetric functions, and so polynomials in the coefficients of Q, with integer coefficients.

– Thus, we are led to the above conclusion, that $R(P,Q)$ is a polynomial in the coefficients of P,Q, with integer coefficients, and with the remark that the $c^l d^k$ factor is there for these latter coefficients to be indeed integers, instead of rationals.

Alternatively, let us write our two polynomials in usual form, as follows:

$$P = p_k X^k + \ldots + p_1 X + p_0$$

$$Q = q_l X^l + \ldots + q_1 X + q_0$$

The corresponding resultant appears then as the determinant of an associated matrix, having size $k+l$, and having 0 coefficients at the blank spaces, as follows:

$$R(P,Q) = \begin{vmatrix} p_k & & & q_l & & \\ \vdots & \ddots & & \vdots & \ddots & \\ p_0 & & p_k & q_0 & & q_l \\ & \ddots & \vdots & & \ddots & \vdots \\ & & p_0 & & & q_0 \end{vmatrix}$$

(4) $\iff$ (5) Once again this is something standard, the idea here being that the discriminant $\Delta(P)$ of a polynomial $P \in \mathbb{C}[X]$ is, modulo scalars, the resultant $R(P, P')$. To be more precise, let us write our polynomial as follows:

$$P(X) = cX^N + dX^{N-1} + \ldots$$

Its discriminant is then defined as being the following quantity:

$$\Delta(P) = \frac{(-1)^{\binom{N}{2}}}{c} R(P, P')$$

This is a polynomial in the coefficients of P, with integer coefficients, with the division by c being indeed possible, under $\mathbb{Z}$, and with the sign being there for various reasons, including the compatibility with some well-known formulae, at small values of N. $\square$

All the above might seem a bit complicated, so as an illustration, let us work out an example. Consider the case of a polynomial of degree 2, and a polynomial of degree 1:

$$P = ax^2 + bx + c \quad , \quad Q = dx + e$$

In order to compute the resultant, let us factorize our polynomials:

$$P = a(x-p)(x-q) \quad , \quad Q = d(x-r)$$

The resultant can be then computed as follows, by using the two-step method:

$$\begin{aligned} R(P, Q) &= ad^2(p-r)(q-r) \\ &= ad^2(pq - (p+q)r + r^2) \\ &= cd^2 + bd^2r + ad^2r^2 \\ &= cd^2 - bde + ae^2 \end{aligned}$$

Observe that $R(P, Q) = 0$ corresponds indeed to the fact that P, Q have a common root. Indeed, the root of Q is $r = -e/d$, and we have:

$$P(r) = \frac{ae^2}{d^2} - \frac{be}{d} + c = \frac{R(P, Q)}{d^2}$$

We can recover as well the resultant as a determinant, as follows:

$$R(P, Q) = \begin{vmatrix} a & d & 0 \\ b & e & d \\ c & 0 & e \end{vmatrix} = ae^2 - bde + cd^2$$

Finally, in what regards the discriminant, let us see what happens in degree 2. Here we must compute the resultant of the following two polynomials:

$$P = aX^2 + bX + c \quad , \quad P' = 2aX + b$$

The resultant is then given by the following formula:

$$\begin{aligned}R(P,P') &= ab^2-b(2a)b+c(2a)^2\\ &= 4a^2c-ab^2\\ &= -a(b^2-4ac)\end{aligned}$$

Now by doing the discriminant normalizations, we obtain, as we should:

$$\Delta(P)=b^2-4ac$$

As already mentioned, one can prove that the matrices having distinct eigenvalues are "generic", and so the above result basically captures the whole situation. We have in fact the following collection of density results, which are quite advanced:

THEOREM 1.14. *The following happen, inside $M_N(\mathbb{C})$:*

1. *The invertible matrices are dense.*
2. *The matrices having distinct eigenvalues are dense.*
3. *The diagonalizable matrices are dense.*

PROOF. These are quite advanced results, which can be proved as follows:

(1) This is clear, intuitively speaking, because the invertible matrices are given by the condition $\det A\neq 0$. Thus, the set formed by these matrices appears as the complement of the hypersurface $\det A=0$, and so must be dense inside $M_N(\mathbb{C})$, as claimed.

(2) Here we can use a similar argument, this time by saying that the set formed by the matrices having distinct eigenvalues appears as the complement of the hypersurface given by $\Delta(P_A)=0$, and so must be dense inside $M_N(\mathbb{C})$, as claimed.

(3) This follows from (2), via the fact that the matrices having distinct eigenvalues are diagonalizable, that we know from Theorem 1.13. There are of course some other proofs as well, for instance by putting the matrix in Jordan form. □

As an application of the above results, and of our methods in general, we have:

THEOREM 1.15. *The following happen:*

1. *We have $P_{AB}=P_{BA}$, for any two matrices $A,B\in M_N(\mathbb{C})$.*
2. *AB,BA have the same eigenvalues, with the same multiplicities.*
3. *If A has eigenvalues $\lambda_1,\ldots,\lambda_N$, then $f(A)$ has eigenvalues $f(\lambda_1),\ldots,f(\lambda_N)$.*

PROOF. These results can be deduced by using Theorem 1.14, as follows:

(1) It follows from definitions that the characteristic polynomial of a matrix is invariant under conjugation, in the sense that we have the following formula:

$$P_C=P_{ACA^{-1}}$$

Now observe that, when assuming that A is invertible, we have:

$$AB=A(BA)A^{-1}$$

Thus, we have the result when A is invertible. By using now Theorem 1.14 (1), we conclude that this formula holds for any matrix A, by continuity.

(2) This is a reformulation of (1), via the fact that P encodes the eigenvalues, with multiplicities, which is hard to prove with bare hands.

(3) This is something quite informal, clear for the diagonal matrices D, then for the diagonalizable matrices PDP^{-1}, and finally for all matrices, by using Theorem 1.14 (3), provided that f has suitable regularity properties. We will be back to this. □

Let us go back to the main problem raised by the diagonalization procedure, namely the computation of the roots of characteristic polynomials. We have here:

THEOREM 1.16. *The complex eigenvalues of a matrix $A \in M_N(\mathbb{C})$, counted with multiplicities, have the following properties:*

1. *Their sum is the trace.*
2. *Their product is the determinant.*

PROOF. Consider indeed the characteristic polynomial P of the matrix:

$$\begin{aligned} P(X) &= \det(A - X1_N) \\ &= (-1)^N X^N + (-1)^{N-1} Tr(A) X^{N-1} + \ldots + \det(A) \end{aligned}$$

We can factorize this polynomial, by using its N complex roots, and we obtain:

$$\begin{aligned} P(X) &= (-1)^N (X - \lambda_1) \ldots (X - \lambda_N) \\ &= (-1)^N X^N + (-1)^{N-1} \left(\sum_i \lambda_i \right) X^{N-1} + \ldots + \prod_i \lambda_i \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Regarding now the intermediate terms, we have here:

THEOREM 1.17. *Assume that $A \in M_N(\mathbb{C})$ has eigenvalues $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$, counted with multiplicities. The basic symmetric functions of these eigenvalues, namely*

$$c_k = \sum_{i_1 < \ldots < i_k} \lambda_{i_1} \ldots \lambda_{i_k}$$

are then given by the fact that the characteristic polynomial of the matrix is:

$$P(X) = (-1)^N \sum_{k=0}^{N} (-1)^k c_k X^k$$

Moreover, all symmetric functions of the eigenvalues, such as the sums of powers

$$d_s = \lambda_1^s + \ldots + \lambda_N^s$$

appear as polynomials in these characteristic polynomial coefficients c_k.

PROOF. These results can be proved by doing some algebra, as follows:

(1) Consider indeed the characteristic polynomial P of the matrix, factorized by using its N complex roots, taken with multiplicities. By expanding, we obtain:

$$\begin{aligned}P(X) &= (-1)^N(X-\lambda_1)\ldots(X-\lambda_N)\\ &= (-1)^N X^N + (-1)^{N-1}\left(\sum_i \lambda_i\right)X^{N-1}+\ldots+\prod_i \lambda_i\\ &= (-1)^N X^N + (-1)^{N-1}c_1X^{N-1}+\ldots+(-1)^0 c_N\\ &= (-1)^N\left(X^N - c_1X^{N-1}+\ldots+(-1)^N c_N\right)\end{aligned}$$

With the convention $c_0 = 1$, we are led to the conclusion in the statement.

(2) This is something standard, coming by doing some abstract algebra. Working out the formulae for the sums of powers $d_s = \sum_i \lambda_i^s$, at small values of the exponent $s \in \mathbb{N}$, is an excellent exercise, which shows how to proceed in general, by recurrence. □

1d. Spectral theorems

Let us go back now to the diagonalization question. Here is a key result:

THEOREM 1.18. *Any matrix $A \in M_N(\mathbb{C})$ which is self-adjoint, $A = A^*$, is diagonalizable, with the diagonalization being of the following type,*

$$A = UDU^*$$

with $U \in U_N$, and with $D \in M_N(\mathbb{R})$ diagonal. The converse holds too.

PROOF. As a first remark, the converse trivially holds, because if we take a matrix of the form $A = UDU^*$, with U unitary and D diagonal and real, then we have:

$$\begin{aligned}A^* &= (UDU^*)^*\\ &= UD^*U^*\\ &= UDU^*\\ &= A\end{aligned}$$

In the other sense now, assume that A is self-adjoint, $A = A^*$. Our first claim is that the eigenvalues are real. Indeed, assuming $Av = \lambda v$, we have:

$$\begin{aligned}\lambda < v,v > &= < \lambda v, v >\\ &= < Av, v >\\ &= < v, Av >\\ &= < v, \lambda v >\\ &= \bar{\lambda} < v, v >\end{aligned}$$

Thus we obtain $\lambda\in\mathbb R$, as claimed. Our next claim now is that the eigenspaces corresponding to different eigenvalues are pairwise orthogonal. Assume indeed that:

$$Av=\lambda v\quad,\quad Aw=\mu w$$

We have then the following computation, using $\lambda,\mu\in\mathbb R$:

$$\begin{aligned}\lambda<v,w> &= <\lambda v,w>\\ &= <Av,w>\\ &= <v,Aw>\\ &= <v,\mu w>\\ &= \mu<v,w>\end{aligned}$$

Thus $\lambda\neq\mu$ implies $v\perp w$, as claimed. In order now to finish the proof, it remains to prove that the eigenspaces of A span the whole space $\mathbb C^N$. For this purpose, we will use a recurrence method. Let us pick an eigenvector of our matrix:

$$Av=\lambda v$$

Assuming now that we have a vector w orthogonal to it, $v\perp w$, we have:

$$\begin{aligned}<Aw,v> &= <w,Av>\\ &= <w,\lambda v>\\ &= \lambda<w,v>\\ &= 0\end{aligned}$$

Thus, if v is an eigenvector, then the vector space $v^\perp$ is invariant under A. Moreover, since a matrix A is self-adjoint precisely when $<Av,v>\in\mathbb R$ for any vector $v\in\mathbb C^N$, as one can see by expanding the scalar product, the restriction of A to the subspace $v^\perp$ is self-adjoint. Thus, we can proceed by recurrence, and we obtain the result. □

As basic examples of self-adjoint matrices, we have the orthogonal projections. The diagonalization result regarding them is as follows:

Proposition 1.19. *The matrices $P\in M_N(\mathbb C)$ which are projections,*

$$P^2=P^*=P$$

are precisely those which diagonalize as follows,

$$P=UDU^*$$

with $U\in U_N$, and with $D\in M_N(0,1)$ being diagonal.

Proof. The equation for the projections being $P^2 = P^* = P$, the eigenvalues λ are real, and we have as well the following condition, coming from $P^2 = P$:

$$\begin{aligned}\lambda < v, v > &= < \lambda v, v > \\ &= < Pv, v > \\ &= < P^2 v, v > \\ &= < Pv, Pv > \\ &= < \lambda v, \lambda v > \\ &= \lambda^2 < v, v >\end{aligned}$$

Thus we obtain $\lambda \in \{0, 1\}$, as claimed, and as a final conclusion here, the diagonalization of the self-adjoint matrices is as follows, with $e_i \in \{0, 1\}$:

$$P \sim \begin{pmatrix} e_1 & & \\ & \ddots & \\ & & e_N \end{pmatrix}$$

To be more precise, the number of 1 values is the dimension of the image of P, and the number of 0 values is the dimension of space of vectors sent to 0 by P. □

An important class of self-adjoint matrices, which includes for instance all the projections, are the positive matrices. The theory here is as follows:

Theorem 1.20. *For a matrix $A \in M_N(\mathbb{C})$ the following conditions are equivalent, and if they are satisfied, we say that A is positive:*

1. *$A = B^2$, with $B = B^*$.*
2. *$A = CC^*$, for some $C \in M_N(\mathbb{C})$.*
3. *$< Ax, x > \geq 0$, for any vector $x \in \mathbb{C}^N$.*
4. *$A = A^*$, and the eigenvalues are positive, $\lambda_i \geq 0$.*
5. *$A = UDU^*$, with $U \in U_N$ and with $D \in M_N(\mathbb{R}_+)$ diagonal.*

Proof. The idea is that the equivalences in the statement basically follow from some elementary computations, with only Theorem 1.18 needed, at some point:

(1) $\implies$ (2) This is clear, because we can take $C = B$.

(2) $\implies$ (3) This follows from the following computation:

$$\begin{aligned}< Ax, x > &= < CC^* x, x > \\ &= < C^* x, C^* x > \\ &\geq 0\end{aligned}$$

(3) $\implies$ (4) By using the fact that $< Ax, x >$ is real, we have:

$$\begin{aligned}< Ax, x > &= < x, A^* x > \\ &= < A^* x, x >\end{aligned}$$

Thus we have $A=A^*$, and the remaining assertion, regarding the eigenvalues, follows from the following computation, assuming $Ax=\lambda x$:

$$\begin{aligned} <Ax,x> &= <\lambda x,x> \\ &= \lambda <x,x> \\ &\geq 0 \end{aligned}$$

(4) $\implies$ (5) This follows indeed by using Theorem 1.18.

(5) $\implies$ (1) Assuming $A=UDU^*$, with $U\in U_N$, and with $D\in M_N(\mathbb{R}_+)$ being diagonal, we can set $B=U\sqrt{D}U^*$. Then B is self-adjoint, and its square is given by:

$$\begin{aligned} B^2 &= U\sqrt{D}U^*\cdot U\sqrt{D}U^* \\ &= UDU^* \\ &= A \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Let us record as well the following technical version of the above result:

THEOREM 1.21. *For a matrix $A\in M_N(\mathbb{C})$ the following conditions are equivalent, and if they are satisfied, we say that A is strictly positive:*

1. *$A=B^2$, with $B=B^*$, invertible.*
2. *$A=CC^*$, for some $C\in M_N(\mathbb{C})$ invertible.*
3. *$<Ax,x>>0$, for any nonzero vector $x\in\mathbb{C}^N$.*
4. *$A=A^*$, and the eigenvalues are strictly positive, $\lambda_i>0$.*
5. *$A=UDU^*$, with $U\in U_N$ and with $D\in M_N(\mathbb{R}_+^*)$ diagonal.*

PROOF. This follows either from Theorem 1.20, by adding the various extra assumptions in the statement, or from the proof of Theorem 1.20, by modifying where needed. □

Let us discuss now the case of the unitary matrices. We have here:

THEOREM 1.22. *Any matrix $U\in M_N(\mathbb{C})$ which is unitary, $U^*=U^{-1}$, is diagonalizable, with the eigenvalues on $\mathbb{T}$. More precisely we have*

$$U=VDV^*$$

with $V\in U_N$, and with $D\in M_N(\mathbb{T})$ diagonal. The converse holds too.

PROOF. As a first remark, the converse trivially holds, because given a matrix of type $U=VDV^*$, with $V\in U_N$, and with $D\in M_N(\mathbb{T})$ being diagonal, we have:

$$\begin{aligned}U^* &= (VDV^*)^*\\ &= VD^*V^*\\ &= VD^{-1}V^{-1}\\ &= (V^*)^{-1}D^{-1}V^{-1}\\ &= (VDV^*)^{-1}\\ &= U^{-1}\end{aligned}$$

Let us prove now the first assertion, stating that the eigenvalues of a unitary matrix $U\in U_N$ belong to $\mathbb{T}$. Indeed, assuming $Uv=\lambda v$, we have:

$$\begin{aligned}<v,v> &= <U^*Uv,v>\\ &= <Uv,Uv>\\ &= <\lambda v,\lambda v>\\ &= |\lambda|^2<v,v>\end{aligned}$$

Thus we obtain $\lambda\in\mathbb{T}$, as claimed. Our next claim now is that the eigenspaces corresponding to different eigenvalues are pairwise orthogonal. Assume indeed that:

$$Uv=\lambda v\quad,\quad Uw=\mu w$$

We have then the following computation, using $U^*=U^{-1}$ and $\lambda,\mu\in\mathbb{T}$:

$$\begin{aligned}\lambda<v,w> &= <\lambda v,w>\\ &= <Uv,w>\\ &= <v,U^*w>\\ &= <v,U^{-1}w>\\ &= <v,\mu^{-1}w>\\ &= \mu<v,w>\end{aligned}$$

Thus $\lambda\neq\mu$ implies $v\perp w$, as claimed. In order now to finish the proof, it remains to prove that the eigenspaces of U span the whole space $\mathbb{C}^N$. For this purpose, we will use a recurrence method. Let us pick an eigenvector of our matrix:

$$Uv=\lambda v$$

Assuming that we have a vector w orthogonal to it, $v \perp w$, we have:

$$\begin{aligned} < Uw, v > &= < w, U^* v > \\ &= < w, U^{-1} v > \\ &= < w, \lambda^{-1} v > \\ &= \lambda < w, v > \\ &= 0 \end{aligned}$$

Thus, if v is an eigenvector, then the vector space $v^\perp$ is invariant under U. Now since U is an isometry, so is its restriction to this space $v^\perp$. Thus this restriction is a unitary, and so we can proceed by recurrence, and we obtain the result. □

The self-adjoint matrices and the unitary matrices are particular cases of the general notion of a "normal matrix", and we have here:

THEOREM 1.23. *Any matrix $A \in M_N(\mathbb{C})$ which is normal, $AA^* = A^*A$, is diagonalizable, with the diagonalization being of the following type,*

$$A = UDU^*$$

with $U \in U_N$, and with $D \in M_N(\mathbb{C})$ diagonal. The converse holds too.

PROOF. As a first remark, the converse trivially holds, because if we take a matrix of the form $A = UDU^*$, with U unitary and D diagonal, then we have:

$$\begin{aligned} AA^* &= UDU^* \cdot UD^*U^* \\ &= UDD^*U^* \\ &= UD^*DU^* \\ &= UD^*U^* \cdot UDU^* \\ &= A^*A \end{aligned}$$

In the other sense now, this is something more technical. Our first claim is that a matrix A is normal precisely when the following happens, for any vector v:

$$||Av|| = ||A^*v||$$

Indeed, the above equality can be written as follows:

$$< AA^*v, v > = < A^*Av, v >$$

But this is equivalent to $AA^* = A^*A$, by expanding the scalar products. Our next claim is that A, A^* have the same eigenvectors, with conjugate eigenvalues:

$$Av = \lambda v \implies A^*v = \bar{\lambda} v$$

Indeed, this follows from the following computation, and from the trivial fact that if A is normal, then so is any matrix of type $A-\lambda 1_N$:

$$\begin{aligned}||(A^*-\bar{\lambda}1_N)v|| &= ||(A-\lambda 1_N)^*v||\\ &= ||(A-\lambda 1_N)v||\\ &= 0\end{aligned}$$

Let us prove now, by using this, that the eigenspaces of A are pairwise orthogonal. Assume that we have two eigenvectors, corresponding to different eigenvalues, $\lambda\neq\mu$:

$$Av=\lambda v\quad,\quad Aw=\mu w$$

We have the following computation, which shows that $\lambda\neq\mu$ implies $v\perp w$:

$$\begin{aligned}\lambda<v,w> &= <\lambda v,w>\\ &= <Av,w>\\ &= <v,A^*w>\\ &= <v,\bar{\mu}w>\\ &= \mu<v,w>\end{aligned}$$

In order to finish, it remains to prove that the eigenspaces of A span the whole $\mathbb{C}^N$. This is something that we have already seen for the self-adjoint matrices, and for unitaries, and we will use here these results, in order to deal with the general normal case. As a first observation, given an arbitrary matrix A, the matrix AA^* is self-adjoint:

$$(AA^*)^*=AA^*$$

Thus, we can diagonalize this matrix AA^*, as follows, with the passage matrix being a unitary, $V\in U_N$, and with the diagonal form being real, $E\in M_N(\mathbb{R})$:

$$AA^*=VEV^*$$

Now observe that, for matrices of type $A=UDU^*$, which are those that we supposed to deal with, we have the following formulae:

$$V=U\quad,\quad E=D\bar{D}$$

In particular, the matrices A and AA^* have the same eigenspaces. So, this will be our idea, proving that the eigenspaces of AA^* are eigenspaces of A. In order to do so, let us pick two eigenvectors v,w of the matrix AA^*, corresponding to different eigenvalues, $\lambda\neq\mu$. The eigenvalue equations are then as follows:

$$AA^*v=\lambda v\quad,\quad AA^*w=\mu w$$

We have the following computation, using the normality condition $AA^* = A^*A$, and the fact that the eigenvalues of AA^*, and in particular μ, are real:

$$\begin{aligned}\lambda < Av, w > &= < \lambda Av, w >\\ &= < A\lambda v, w >\\ &= < AAA^*v, w >\\ &= < AA^*Av, w >\\ &= < Av, AA^*w >\\ &= < Av, \mu w >\\ &= \mu < Av, w >\end{aligned}$$

We conclude that we have $< Av, w >= 0$. But this reformulates as follows:

$$\lambda \neq \mu \implies A(E_\lambda) \perp E_\mu$$

Now since the eigenspaces of AA^* are pairwise orthogonal, and span the whole $\mathbb{C}^N$, we deduce from this that these eigenspaces are invariant under A:

$$A(E_\lambda) \subset E_\lambda$$

But with this result in hand, we can finish. Indeed, we can decompose the problem, and the matrix A itself, following these eigenspaces of AA^*, which in practice amounts in saying that we can assume that we only have 1 eigenspace. Now by rescaling, this is the same as assuming that we have $AA^* = 1$. But with this, we are now into the unitary case, that we know how to solve, as explained in Theorem 1.22, and so done. □

As a first application, we have the following result:

THEOREM 1.24. *Given a matrix $A \in M_N(\mathbb{C})$, we can construct a matrix $|A|$ as follows, by using the fact that A^*A is diagonalizable, with positive eigenvalues:*

$$|A| = \sqrt{A^*A}$$

This matrix $|A|$ is then positive, and its square is $|A|^2 = A^*A$. In the case $N = 1$, we obtain in this way the usual absolute value of the complex numbers.

PROOF. Consider indeed the matrix A^*A, which is normal. According to Theorem 1.23, we can diagonalize this matrix as follows, with $U \in U_N$, and with D diagonal:

$$A = UDU^*$$

From $A^*A \geq 0$ we obtain $D \geq 0$. But this means that the entries of D are real, and positive. Thus we can extract the square root $\sqrt{D}$, and then set:

$$\sqrt{A^*A} = U\sqrt{D}U^*$$

Thus, we are basically done. Indeed, if we call this latter matrix $|A|$, then we are led to the conclusions in the statement. Finally, the last assertion is clear from definitions. □

We can now formulate a first polar decomposition result, as follows:

THEOREM 1.25. *Any invertible matrix $A\in M_N(\mathbb C)$ decomposes as*

$$A=U|A|$$

with $U\in U_N$, and with $|A|=\sqrt{A^*A}$ as above.

PROOF. This is routine, and follows by comparing the actions of $A,|A|$ on the vectors $v\in\mathbb C^N$, and deducing from this the existence of a unitary $U\in U_N$ as above. We will be back to this, later on, directly in the case of the linear operators on Hilbert spaces. □

Observe that at $N=1$ we obtain in this way the usual polar decomposition of the nonzero complex numbers. More generally now, we have the following result:

THEOREM 1.26. *Any square matrix $A\in M_N(\mathbb C)$ decomposes as*

$$A=U|A|$$

with U being a partial isometry, and with $|A|=\sqrt{A^*A}$ as above.

PROOF. Again, this follows by comparing the actions of $A,|A|$ on the vectors $v\in\mathbb C^N$, and deducing from this the existence of a partial isometry U as above. Alternatively, we can get this from Theorem 1.25, applied on the complement of the 0-eigenvectors. □

This was for our basic presentation of linear algebra. There are of course many other things that can be said, but we will come back to some of them in what follows, directly in the case of the linear operators on the arbitrary Hilbert spaces.

1e. Exercises

Exercises:

EXERCISE 1.27.

EXERCISE 1.28.

EXERCISE 1.29.

EXERCISE 1.30.

EXERCISE 1.31.

EXERCISE 1.32.

EXERCISE 1.33.

EXERCISE 1.34.

Bonus exercise.

CHAPTER 2

Linear operators

2a. Hilbert spaces

We discuss in what follows an extension of the linear algebra results from the previous chapter, obtained by looking at the linear operators $T : H \to H$, with the space H being no longer assumed to be finite dimensional. Our motivations come from quantum mechanics, and in order to get motivated, here is some suggested reading:

(1) Generally speaking, physics is best learned from Feynman [**39**]. If you already know some, and want to learn quantum mechanics, go with Griffiths [**45**]. And if you're already a bit familiar with quantum mechanics, a good book is Weinberg [**96**].

(2) A look at classics like Dirac [**32**], von Neumann [**93**] or Weyl [**97**] can be instructive too. On the opposite, you have as well modern, fancy books on quantum information, such as Bengtsson-Życzkowski [**14**], Nielsen-Chuang [**76**] or Watrous [**94**].

(3) In short, many ways of getting familiar with this big mess which is quantum mechanics, and as long as you stay away from books advertised as "rigorous", "axiomatic", "mathematical", things fine. By the way, you can try as well my book [**12**].

Getting to work now, physics tells us to look at infinite dimensional complex spaces, such as the space of wave functions $\psi : \mathbb{R}^3 \to \mathbb{C}$ of the electron. In order to do some mathematics on these spaces, we will need scalar products. So, let us start with:

Definition 2.1. *A scalar product on a complex vector space H is a binary operation $H \times H \to \mathbb{C}$, denoted $(x, y) \to < x, y >$, satisfying the following conditions:*

(1) *$< x, y >$ is linear in x, and antilinear in y.*
(2) *$\overline{< x, y >} = < y, x >$, for any x, y.*
(3) *$< x, x > > 0$, for any $x \neq 0$.*

As before in chapter 1, we use here mathematicians' convention for scalar products, that is, $< , >$ linear at left, as opposed to physicists' convention, $< , >$ linear at right. The reasons for this are quite subtle, coming from the fact that, while basic quantum mechanics looks better with $< , >$ linear at right, advanced quantum mechanics looks better with $< , >$ linear at left. Or at least that's what my cats say.

As a basic example for Definition 2.1, we have the finite dimensional vector space $H=\mathbb{C}^N$, with its usual scalar product, namely:

$$<x,y>=\sum_i x_i\bar{y}_i$$

There are many other examples, and notably various spaces of L^2 functions, which naturally appear in problems coming from physics. We will discuss them later on. In order to study now the scalar products, let us formulate the following definition:

Definition 2.2. *The norm of a vector $x\in H$ is the following quantity:*

$$||x||=\sqrt{<x,x>}$$

We also call this number length of x, or distance from x to the origin.

The terminology comes from what happens in $\mathbb{C}^N$, where the length of the vector, as defined above, coincides with the usual length, given by:

$$||x||=\sqrt{\sum_i |x_i|^2}$$

In analogy with what happens in finite dimensions, we have two important results regarding the norms. First we have the Cauchy-Schwarz inequality, as follows:

Theorem 2.3. *We have the Cauchy-Schwarz inequality*

$$|<x,y>|\leq ||x||\cdot||y||$$

and the equality case holds precisely when x,y are proportional.

Proof. This is something very standard. Consider indeed the following quantity, depending on a real variable $t\in\mathbb{R}$, and on a variable on the unit circle, $w\in\mathbb{T}$:

$$f(t)=||twx+y||^2$$

By developing f, we see that this is a degree 2 polynomial in t:

$$\begin{aligned}f(t)&=&<twx+y,twx+y>\\&=&t^2<x,x>+tw<x,y>+t\bar{w}<y,x>+<y,y>\\&=&t^2||x||^2+2tRe(w<x,y>)+||y||^2\end{aligned}$$

Since f is obviously positive, its discriminant must be negative:

$$4Re(w<x,y>)^2-4||x||^2\cdot||y||^2\leq 0$$

But this is equivalent to the following condition:

$$|Re(w<x,y>)|\leq ||x||\cdot||y||$$

Now the point is that we can arrange for the number $w \in \mathbb{T}$ to be such that the quantity $w < x, y >$ is real. Thus, we obtain the following inequality:

$$| < x, y > | \leq ||x|| \cdot ||y||$$

Finally, the study of the equality case is straightforward, by using the fact that the discriminant of f vanishes precisely when we have a root. But this leads to the conclusion in the statement, namely that the vectors x, y must be proportional. □

As a second main result now, we have the Minkowski inequality:

THEOREM 2.4. *We have the Minkowski inequality*

$$||x + y|| \leq ||x|| + ||y||$$

and the equality case holds precisely when x, y are proportional.

PROOF. This follows indeed from the Cauchy-Schwarz inequality, as follows:

$$\begin{aligned} & ||x + y|| \leq ||x|| + ||y|| \\ \iff & ||x + y||^2 \leq (||x|| + ||y||)^2 \\ \iff & ||x||^2 + ||y||^2 + 2Re < x, y > \leq ||x||^2 + ||y||^2 + 2||x|| \cdot ||y|| \\ \iff & Re < x, y > \leq ||x|| \cdot ||y|| \end{aligned}$$

As for the equality case, this is clear from Cauchy-Schwarz as well. □

As a consequence of this, we have the following result:

THEOREM 2.5. *The following function is a distance on H,*

$$d(x, y) = ||x - y||$$

in the usual sense, that of the abstract metric spaces.

PROOF. This follows indeed from the Minkowski inequality, which corresponds to the triangle inequality, the other two axioms for a distance being trivially satisfied. □

The above result is quite important, because it shows that we can do geometry and analysis in our present setting, with distances and angles, a bit as in the finite dimensional case. In order to do such abstract geometry, we will often need the following key result, which shows that everything can be recovered in terms of distances:

PROPOSITION 2.6. *The scalar products can be recovered from distances, via the formula*

$$4 < x, y > = ||x + y||^2 - ||x - y||^2 + i||x + iy||^2 - i||x - iy||^2$$

called complex polarization identity.

PROOF. This is something that we have already met in finite dimensions. In arbitrary dimensions the proof is similar, as follows:

$$\begin{aligned} & ||x+y||^2 - ||x-y||^2 + i||x+iy||^2 - i||x-iy||^2 \\ = \; & ||x||^2 + ||y||^2 - ||x||^2 - ||y||^2 + i||x||^2 + i||y||^2 - i||x||^2 - i||y||^2 \\ & +2Re(< x,y >) + 2Re(< x,y >) + 2iIm(< x,y >) + 2iIm(< x,y >) \\ = \; & 4 < x,y > \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

In order to do analysis on our spaces, we need the Cauchy sequences that we construct to converge. This is something which is automatic in finite dimensions, but in arbitrary dimensions, this can fail. It is convenient here to formulate a detailed new definition, as follows, which will be the starting point for our various considerations to follow:

DEFINITION 2.7. *A Hilbert space is a complex vector space H given with a scalar product $< x,y >$, satisfying the following conditions:*

(1) *$< x,y >$ is linear in x, and antilinear in y.*
(2) *$\overline{< x,y >} =< y,x >$, for any x,y.*
(3) *$< x,x >> 0$, for any $x \neq 0$.*
(4) *H is complete with respect to the norm $||x|| = \sqrt{< x,x >}$.*

In other words, what we did here is to take Definition 2.1, and add the condition that H must be complete with respect to the norm $||x|| = \sqrt{< x,x >}$, that we know indeed to be a norm, according to the Minkowski inequality proved above. As a basic example, as before, we have the space $H = \mathbb{C}^N$, with its usual scalar product, namely:

$$< x,y >= \sum_i x_i \bar{y}_i$$

More generally now, we have the following construction of Hilbert spaces:

PROPOSITION 2.8. *The sequences of complex numbers (x_i) which are square-summable,*

$$\sum_i |x_i|^2 < \infty$$

form a Hilbert space $l^2(\mathbb{N})$, with the following scalar product:

$$< x,y >= \sum_i x_i \bar{y}_i$$

In fact, given any index set I, we can construct a Hilbert space $l^2(I)$, in this way.

PROOF. There are several things to be proved, as follows:

(1) Our first claim is that $l^2(\mathbb{N})$ is a vector space. For this purpose, we must prove that $x,y \in l^2(\mathbb{N})$ implies $x+y \in l^2(\mathbb{N})$. But this leads us into proving $||x+y|| \leq ||x|| + ||y||$,

where $||x|| = \sqrt{< x, x >}$. Now since we know this inequality to hold on each subspace $\mathbb{C}^N \subset l^2(\mathbb{N})$ obtained by truncating, this inequality holds everywhere, as desired.

(2) Our second claim is that $<,>$ is well-defined on $l^2(\mathbb{N})$. But this follows from the Cauchy-Schwarz inequality, $| < x, y > | \leq ||x|| \cdot ||y||$, which can be established by truncating, a bit like we established the Minkowski inequality in (1) above.

(3) It is also clear that $<,>$ is a scalar product on $l^2(\mathbb{N})$, so it remains to prove that $l^2(\mathbb{N})$ is complete with respect to $||x|| = \sqrt{< x, x >}$. But this is clear, because if we pick a Cauchy sequence $\{x^n\}_{n\in\mathbb{N}} \subset l^2(\mathbb{N})$, then each numeric sequence $\{x_i^n\}_{i\in\mathbb{N}} \subset \mathbb{C}$ is Cauchy, and by setting $x_i = \lim_{n\to\infty} x_i^n$, we have $x^n \to x$ inside $l^2(\mathbb{N})$, as desired.

(4) Finally, the same arguments extend to the case of an arbitrary index set I, leading to a Hilbert space $l^2(I)$, and with the remark here that there is absolutely no problem of taking about quantities of type $||x||^2 = \sum_{i\in I} |x_i|^2 \in [0, \infty]$, even if the index set I is uncountable, because we are summing positive numbers. □

Even more generally, we have the following construction of Hilbert spaces:

Theorem 2.9. *Given a measured space X, the functions $f : X \to \mathbb{C}$, taken up to equality almost everywhere, which are square-summable,*

$$\int_X |f(x)|^2 dx < \infty$$

form a Hilbert space $L^2(X)$, with the following scalar product:

$$< f, g >= \int_X f(x)\overline{g(x)}dx$$

In the case $X = I$, with the counting measure, we obtain in this way the space $l^2(I)$.

Proof. This is a straightforward generalization of Proposition 2.8, with the arguments from the proof of Proposition 2.8 carrying over in our case, as follows:

(1) The first part, regarding Cauchy-Schwarz and Minkowski, extends without problems, by using this time approximation by step functions.

(2) Regarding the fact that $<,>$ is indeed a scalar product on $L^2(X)$, there is a subtlety here, because if we want $< f, f >> 0$ for $f \neq 0$, we must declare that $f = 0$ when $f = 0$ almost everywhere, and so that $f = g$ when $f = g$ almost everywhere.

(3) Regarding the fact that $L^2(X)$ is complete with respect to $||f|| = \sqrt{< f, f >}$, this is again basic measure theory, by picking a Cauchy sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^2(X)$, and then constructing a pointwise, and hence L^2 limit, $f_n \to f$, almost everywhere.

(4) Finally, the last assertion is clear, because the integration with respect to the counting measure is by definition a sum, and so $L^2(I) = l^2(I)$ in this case. □

Quite remarkably, any Hilbert space must be of the form $L^2(X)$, and even of the particular form $l^2(I)$. This follows indeed from the following key result:

THEOREM 2.10. *Let H be a Hilbert space.*

(1) *Any algebraic basis of this space $\{f_i\}_{i\in I}$ can be turned into an orthonormal basis $\{e_i\}_{i\in I}$, by using the Gram-Schmidt procedure.*
(2) *Thus, H has an orthonormal basis, and so we have $H \simeq l^2(I)$, with I being the indexing set for this orthonormal basis.*

PROOF. All this is standard by Gram-Schmidt, the idea being as follows:

(1) First of all, in finite dimensions an orthonormal basis $\{e_i\}_{i\in I}$ is by definition a usual algebraic basis, satisfying $< e_i, e_j >= \delta_{ij}$. But the existence of such a basis follows by applying the Gram-Schmidt procedure to any algebraic basis $\{f_i\}_{i\in I}$, as claimed.

(2) In infinite dimensions, a first issue comes from the fact that the standard basis $\{\delta_i\}_{i\in\mathbb{N}}$ of the space $l^2(\mathbb{N})$ is not an algebraic basis in the usual sense, with the finite linear combinations of the functions δ_i producing only a dense subspace of $l^2(\mathbb{N})$, that of the functions having finite support. Thus, we must fine-tune our definition of "basis".

(3) But this can be done in two ways, by saying that $\{f_i\}_{i\in I}$ is a basis of H when the functions f_i are linearly independent, and when either the finite linear combinations of these functions f_i form a dense subspace of H, or the linear combinations with $l^2(I)$ coefficients of these functions f_i form the whole H. For orthogonal bases $\{e_i\}_{i\in I}$ these definitions are equivalent, and in any case, our statement makes now sense.

(4) Regarding now the proof, in infinite dimensions, this follows again from Gram-Schmidt, exactly as in the finite dimensional case, but by using this time a tool from logic, called Zorn lemma, in order to correctly do the recurrence. □

The above result, and its relation with Theorem 2.9, is something quite subtle, so let us further get into this. First, we have the following definition, based on the above:

DEFINITION 2.11. *A Hilbert space H is called separable when the following equivalent conditions are satisfied:*

(1) *H has a countable algebraic basis $\{f_i\}_{i\in\mathbb{N}}$.*
(2) *H has a countable orthonormal basis $\{e_i\}_{i\in\mathbb{N}}$.*
(3) *We have $H \simeq l^2(\mathbb{N})$, isomorphism of Hilbert spaces.*

In what follows we will be mainly interested in the separable Hilbert spaces, where most of the questions coming from quantum physics take place. In view of the above, the following philosophical question appears: why not simply talking about $l^2(\mathbb{N})$?

In answer to this, we cannot really do so, because many of the separable spaces that we are interested in appear as spaces of functions, and such spaces do not necessarily have a very simple or explicit orthonormal basis, as shown by the following result:

PROPOSITION 2.12. *The Hilbert space $H = L^2[0,1]$ is separable, having as orthonormal basis the orthonormalized version of the algebraic basis $f_n = x^n$ with $n \in \mathbb{N}$.*

PROOF. This follows from the Weierstrass theorem, which provides us with the basis $f_n = x^n$, which can be orthogonalized by using the Gram-Schmidt procedure, as explained in Theorem 2.10. Working out the details here is actually an excellent exercise. □

As a conclusion to all this, we are interested in 1 space, namely the unique separable Hilbert space H, but due to various technical reasons, it is often better to forget that we have $H = l^2(\mathbb{N})$, and say instead that we have $H = L^2(X)$, with X being a separable measured space, or simply say that H is an abstract separable Hilbert space.

2b. Linear operators

Let us get now into the study of linear operators $T : H \to H$. Before anything, we should mention that things are quite tricky with respect to quantum mechanics, and physics in general. Indeed, if there is a central operator in physics, this is the Laplace operator on the smooth functions $f : \mathbb{R}^N \to \mathbb{C}$, given by:

$$\Delta f(x) = \sum_i \frac{d^2 f}{dx_i^2}$$

And the problem is that what we have here is an operator $\Delta : C^\infty(\mathbb{R}^N) \to C^\infty(\mathbb{R}^N)$, which does not extend into an operator $\Delta : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$. Thus, we should perhaps look at operators $T : H \to H$ which are densely defined, instead of looking at operators $T : H \to H$ which are everywhere defined. We will not do so, for two reasons:

(1) Tactical retreat. When physics looks too complicated, as it is the case now, you can always declare that mathematics comes first. So, let us be pure mathematicians, simply looking in generalizing linear algebra to infinite dimensions. And from this viewpoint, it is a no-brainer to look at everywhere defined operators $T : H \to H$.

(2) Modern physics. We will see later, towards the end of the present book, when talking about various mathematical physics findings of Connes, Jones, Voiculescu and others, that a lot of interesting mathematics, which is definitely related to modern physics, can be developed by using the everywhere defined operators $T : H \to H$.

In short, you'll have to trust me here. And hang on, we are not done yet, because with this choice made, there is one more problem, mathematical this time. The problem comes from the fact that in infinite dimensions the everywhere defined operators $T : H \to H$ can be bounded or not, and for reasons which are mathematically intuitive and obvious, and physically acceptable too, we want to deal with the bounded case only.

Long story short, let us avoid too much thinking, and start in a simple way, with:

PROPOSITION 2.13. *For a linear operator $T : H \to H$, the following are equivalent:*

(1) *T is continuous.*
(2) *T is continuous at 0.*
(3) *$T(B) \subset cB$ for some $c < \infty$, where $B \subset H$ is the unit ball.*
(4) *T is bounded, in the sense that $||T|| = \sup_{||x|| \leq 1} ||Tx||$ satisfies $||T|| < \infty$.*

PROOF. This is elementary, with (1) $\iff$ (2) coming from the linearity of T, then (2) $\iff$ (3) coming from definitions, and finally (3) $\iff$ (4) coming from the fact that the number $||T||$ from (4) is the infimum of the numbers c making (3) work. $\square$

Regarding such operators, we have the following result:

THEOREM 2.14. *The linear operators $T : H \to H$ which are bounded,*

$$||T|| = \sup_{||x|| \leq 1} ||Tx|| < \infty$$

form a complex algebra with unit $B(H)$, having the property

$$||ST|| \leq ||S|| \cdot ||T||$$

and which is complete with respect to the norm.

PROOF. The fact that we have indeed an algebra, satisfying the product condition in the statement, follows from the following estimates, which are all elementary:

$$||S + T|| \leq ||S|| + ||T||$$
$$||\lambda T|| = |\lambda| \cdot ||T||$$
$$||ST|| \leq ||S|| \cdot ||T||$$

Regarding now the last assertion, if $\{T_n\} \subset B(H)$ is Cauchy then $\{T_n x\}$ is Cauchy for any $x \in H$, so we can define the limit $T = \lim_{n\to\infty} T_n$ by setting:

$$Tx = \lim_{n\to\infty} T_n x$$

Let us first check that the application $x \to Tx$ is linear. We have:

$$\begin{aligned} T(x+y) &= \lim_{n\to\infty} T_n(x+y) \\ &= \lim_{n\to\infty} T_n(x) + T_n(y) \\ &= \lim_{n\to\infty} T_n(x) + \lim_{n\to\infty} T_n(y) \\ &= T(x) + T(y) \end{aligned}$$

Similarly, we have as well the following computation:

$$\begin{aligned} T(\lambda x) &= \lim_{n\to\infty} T_n(\lambda x) \\ &= \lambda \lim_{n\to\infty} T_n(x) \\ &= \lambda T(x) \end{aligned}$$

Thus we have a linear map $T : A \to A$. It remains to prove that we have $T \in B(H)$, and that we have $T_n \to T$ in norm. For this purpose, observe that we have:

$$\begin{aligned}
& ||T_n - T_m|| \leq \varepsilon \ , \ \forall n, m \geq N \\
\implies & ||T_n x - T_m x|| \leq \varepsilon \ , \ \forall ||x|| = 1 \ , \ \forall n, m \geq N \\
\implies & ||T_n x - Tx|| \leq \varepsilon \ , \ \forall ||x|| = 1 \ , \ \forall n \geq N \\
\implies & ||T_N x - Tx|| \leq \varepsilon \ , \ \forall ||x|| = 1 \\
\implies & ||T_N - T|| \leq \varepsilon
\end{aligned}$$

As a first consequence, we obtain $T \in B(H)$, because we have:

$$\begin{aligned}
||T|| &= ||T_N + (T - T_N)|| \\
&\leq ||T_N|| + ||T - T_N|| \\
&\leq ||T_N|| + \varepsilon \\
&< \infty
\end{aligned}$$

As a second consequence, we obtain $T_N \to T$ in norm, and we are done. $\square$

In the case where H comes with a basis $\{e_i\}_{i \in I}$, we can talk about the infinite matrices $M \in M_I(\mathbb{C})$, with the remark that the multiplication of such matrices is not always defined, in the case $|I| = \infty$. In this context, we have the following result:

Theorem 2.15. *Let H be a Hilbert space, with orthonormal basis $\{e_i\}_{i \in I}$. The bounded operators $T \in B(H)$ can be then identified with matrices $M \in M_I(\mathbb{C})$ via*

$$Tx = Mx \quad , \quad M_{ij} = < Te_j, e_i >$$

and we obtain in this way an embedding as follows, which is multiplicative:

$$B(H) \subset M_I(\mathbb{C})$$

In the case $H = \mathbb{C}^N$ we obtain in this way the usual isomorphism $B(H) \simeq M_N(\mathbb{C})$. In the separable case we obtain in this way a proper embedding $B(H) \subset M_\infty(\mathbb{C})$.

Proof. We have several assertions to be proved, the idea being as follows:

(1) Regarding the first assertion, given a bounded operator $T : H \to H$, let us associate to it a matrix $M \in M_I(\mathbb{C})$ as in the statement, by the following formula:

$$M_{ij} = < Te_j, e_i >$$

It is clear that this correspondence $T \to M$ is linear, and also that its kernel is $\{0\}$. Thus, we have an embedding of linear spaces $B(H) \subset M_I(\mathbb{C})$.

(2) Our claim now is that this embedding is multiplicative. But this is clear too, because if we denote by $T \to M_T$ our correspondence, we have:

$$\begin{aligned}(M_{ST})_{ij} &= < STe_j, e_i > \\ &= \left\langle S \sum_k < Te_j, e_k > e_k, e_i \right\rangle \\ &= \sum_k < Se_k, e_i >< Te_j, e_k > \\ &= \sum_k (M_S)_{ik}(M_T)_{kj} \\ &= (M_S M_T)_{ij}\end{aligned}$$

(3) Finally, we must prove that the original operator $T : H \to H$ can be recovered from its matrix $M \in M_I(\mathbb{C})$ via the formula in the statement, namely $Tx = Mx$. But this latter formula holds for the vectors of the basis, $x = e_j$, because we have:

$$\begin{aligned}(Te_j)_i &= < Te_j, e_i > \\ &= M_{ij} \\ &= (Me_j)_i\end{aligned}$$

Now by linearity we obtain from this that the formula $Tx = Mx$ holds everywhere, on any vector $x \in H$, and this finishes the proof of the first assertion.

(4) In finite dimensions we obtain an isomorphism, because any matrix $M \in M_N(\mathbb{C})$ determines an operator $T : \mathbb{C}^N \to \mathbb{C}^N$, according to the formula $< Te_j, e_i >= M_{ij}$. In infinite dimensions, however, we do not have an isomorphism. For instance on $H = l^2(\mathbb{N})$ the following matrix does not define an operator:

$$M = \begin{pmatrix} 1 & 1 & \ldots \\ 1 & 1 & \ldots \\ \vdots & \vdots \end{pmatrix}$$

Indeed, $T(e_1)$ should be the all-one vector, which is not square-summable. □

In connection with our previous comments on bases, the above result is something quite theoretical, because for basic Hilbert spaces like $L^2[0,1]$, which do not have a simple orthonormal basis, the embedding $B(H) \subset M_\infty(\mathbb{C})$ that we obtain is not something very useful. In short, while the bounded operators $T : H \to H$ are basically some infinite matrices, it is better to think of these operators as being objects on their own.

As another comment, the construction $T \to M$ makes sense for any linear operator $T : H \to H$, but when $\dim H = \infty$, we do not obtain an embedding $\mathcal{L}(H) \subset M_I(\mathbb{C})$ in this way. Indeed, set $H = l^2(\mathbb{N})$, let $E = span(e_i)$ be the linear space spanned by the standard basis, and pick an algebraic complement F of this space E, so that we have $H = E \oplus F$, as an algebraic direct sum. Then any linear operator $S : F \to F$ gives rise to a linear operator $T : H \to H$, given by $T(e,f) = (0, S(f))$, whose associated matrix is 0. And, restrospectively speaking, it is in order to avoid such pathologies that we decided some time ago to restrict the attention to the bounded case, $T \in B(H)$.

As in the finite dimensional case, we can talk about adjoint operators, in this setting, the definition and main properties of the construction $T \to T^*$ being as follows:

THEOREM 2.16. *Given a bounded operator $T \in B(H)$, the following formula defines a bounded operator $T^* \in B(H)$, called adjoint of H:*

$$< Tx, y >=< x, T^*y >$$

The correspondence $T \to T^*$ is antilinear, antimultiplicative, and is an involution, and an isometry. In finite dimensions, we recover the usual adjoint operator.

PROOF. There are several things to be done here, the idea being as follows:

(1) We will need a standard functional analysis result, stating that the continuous linear forms $\varphi : H \to \mathbb{C}$ appear as scalar products, as follows, with $z \in H$:

$$\varphi(x) =< x, z >$$

Indeed, in one sense this is clear, because given $z \in H$, the application $\varphi(x) =< x, z >$ is linear, and continuous as well, because by Cauchy-Schwarz we have:

$$|\varphi(x)| \leq ||x|| \cdot ||z||$$

Conversely now, by using a basis we can assume $H = l^2(\mathbb{N})$, and our linear form $\varphi : H \to \mathbb{C}$ must be then, by linearity, given by a formula of the following type:

$$\varphi(x) = \sum_i x_i \bar{z}_i$$

But, again by Cauchy-Schwarz, in order for such a formula to define indeed a continuous linear form $\varphi : H \to \mathbb{C}$ we must have $z \in l^2(\mathbb{N})$, and so $z \in H$, as desired.

(2) With this in hand, we can now construct the adjoint T^*, by the formula in the statement. Indeed, given $y \in H$, the formula $\varphi(x) =< Tx, y >$ defines a linear map $H \to \mathbb{C}$. Thus, we must have a formula as follows, for a certain vector $T^*y \in H$:

$$\varphi(x) =< x, T^*y >$$

Moreover, this vector $T^*y \in H$ is unique with this property, and we conclude from this that the formula $y \to T^*y$ defines a certain map $T^* : H \to H$, which is unique with the property in the statement, namely $< Tx, y >=< x, T^*y >$ for any x, y.

(3) Let us prove that we have $T^*\in B(H)$. By using once again the uniqueness of T^*, we conclude that we have the following formulae, which show that T^* is linear:

$$T^*(x+y)=T^*x+T^*y\quad,\quad T^*(\lambda x)=\lambda T^*x$$

Observe also that T^* is bounded as well, because we have:

$$\begin{aligned}||T|| &= \sup_{||x||=1}\sup_{||y||=1}<Tx,y>\\ &= \sup_{||y||=1}\sup_{||x||=1}<x,T^*y>\\ &= ||T^*||\end{aligned}$$

(4) The fact that the correspondence $T\to T^*$ is antilinear, antimultiplicative, and is an involution comes from the following formulae, coming from uniqueness:

$$(S+T)^*=S^*+T^*\quad,\quad (\lambda T)^*=\bar{\lambda}T^*$$

$$(ST)^*=T^*S^*\quad,\quad (T^*)^*=T$$

As for the isometry property with respect to the operator norm, $||T||=||T^*||$, this is something that we already know, from the proof of (3) above.

(5) Regarding finite dimensions, let us first examine the general case where our Hilbert space comes with a basis, $H=l^2(I)$. We can compute the matrix $M^*\in M_I(\mathbb{C})$ associated to the operator $T^*\in B(H)$, by using $<Tx,y>=<x,T^*y>$, in the following way:

$$\begin{aligned}(M^*)_{ij} &= <T^*e_j,e_i>\\ &= \overline{<e_i,T^*e_j>}\\ &= \overline{<Te_i,e_j>}\\ &= \overline{M}_{ji}\end{aligned}$$

Thus, we have reached to the usual formula for the adjoints of matrices, and in the particular case $H=\mathbb{C}^N$, we conclude that T^* comes indeed from the usual M^*. □

As in finite dimensions, the operators T,T^* can be thought of as being "twin brothers", and there is a lot of interesting mathematics connecting them. We first have:

PROPOSITION 2.17. *Given a bounded operator $T\in B(H)$, the following happen:*
(1) $\ker T^*=(ImT)^\perp$.
(2) $\overline{ImT^*}=(\ker T)^\perp$.

PROOF. Both these assertions are elementary, as follows:

(1) Let us first prove "$\subset$". Assuming $T^*x=0$, we have indeed $x\perp ImT$, because:

$$<x,Ty>=<T^*x,y>=0$$

As for "$\supset$", assuming $< x, Ty >= 0$ for any y, we have $T^*x = 0$, because:

$$< T^*x, y >=< x, Ty >= 0$$

(2) This can be deduced from (1), applied to the operator T^*, as follows:

$$(\ker T)^\perp = (ImT^*)^{\perp\perp} = \overline{ImT^*}$$

Here we have used the formula $K^{\perp\perp} = \bar{K}$, valid for any linear subspace $K \subset H$ of a Hilbert space, which for K closed reads $K^{\perp\perp} = K$, and comes from $H = K \oplus K^\perp$, and which in general follows from $K^{\perp\perp} \subset \bar{K}^{\perp\perp} = \bar{K}$, the reverse inclusion being clear. □

Let us record as well the following useful formula, relating T and T^*:

THEOREM 2.18. *We have the following formula,*

$$||TT^*|| = ||T||^2$$

valid for any operator $T \in B(H)$.

PROOF. We recall from Theorem 2.16 that the correspondence $T \to T^*$ is an isometry with respect to the operator norm, in the sense that we have:

$$||T|| = ||T^*||$$

In order to prove now the formula in the statement, observe first that we have:

$$||TT^*|| \leq ||T|| \cdot ||T^*|| = ||T||^2$$

On the other hand, we have as well the following estimate:

$$\begin{aligned} ||T||^2 &= \sup_{||x||=1} | < Tx, Tx > | \\ &= \sup_{||x||=1} | < x, T^*Tx > | \\ &\leq ||T^*T|| \end{aligned}$$

By replacing $T \to T^*$ we obtain from this that we have:

$$||T||^2 \leq ||TT^*||$$

Thus, we have obtained the needed inequality, and we are done. □

2c. Unitaries, projections

Let us discuss now some explicit examples of operators, in analogy with what happens in finite dimensions. The most basic examples of linear transformations are the rotations, symmetries and projections. Then, we have certain remarkable classes of linear transformations, such as the positive, self-adjoint and normal ones. In what follows we will develop the basic theory of such transformations, in the present Hilbert space setting.

Let us begin with the rotations. The situation here is quite tricky in arbitrary dimensions, and we have several notions instead of one. We first have the following result:

THEOREM 2.19. *For a linear operator $U \in B(H)$ the following conditions are equivalent, and if they are satisfied, we say that U is an isometry:*

1. *U is a metric space isometry, $d(Ux, Uy) = d(x, y)$.*
2. *U is a normed space isometry, $||Ux|| = ||x||$.*
3. *U preserves the scalar product, $< Ux, Uy >=< x, y >$.*
4. *U satisfies the isometry condition $U^*U = 1$.*

In finite dimensions, we recover in this way the usual unitary transformations.

PROOF. The proofs are similar to those in finite dimensions, as follows:

(1) $\iff$ (2) This follows indeed from the formula of the distances, namely:

$$d(x, y) = ||x - y||$$

(2) $\iff$ (3) This is again standard, because we can pass from scalar products to distances, and vice versa, by using $||x|| = \sqrt{< x, x >}$, and the polarization formula.

(3) $\iff$ (4) We have indeed the following equivalences, by using the standard formula $< Tx, y >=< x, T^*y >$, which defines the adjoint operator:

$$\begin{aligned} < Ux, Uy >=< x, y > &\iff < x, U^*Uy >=< x, y > \\ &\iff U^*Uy = y \\ &\iff U^*U = 1 \end{aligned}$$

Thus, we are led to the conclusions in the statement. □

The point now is that the condition $U^*U = 1$ does not imply in general $UU^* = 1$, the simplest counterexample here being the shift operator on $l^2(\mathbb{N})$:

PROPOSITION 2.20. *The shift operator on the space $l^2(\mathbb{N})$, given by*

$$S(e_i) = e_{i+1}$$

is an isometry, $S^*S = 1$. However, we have $SS^* \neq 1$.

PROOF. The adjoint of the shift is given by the following formula:

$$S^*(e_i) = \begin{cases} e_{i-1} & \text{if } i > 0 \\ 0 & \text{if } i = 0 \end{cases}$$

When composing S, S^*, in one sense we obtain the following formula:

$$S^*S(e_i) = e_i$$

In other other sense now, we obtain the following formula:

$$SS^*(e_i) = \begin{cases} e_i & \text{if } i > 0 \\ 0 & \text{if } i = 0 \end{cases}$$

Summarizing, the compositions are given by the following formulae:

$$S^*S = 1 \quad , \quad SS^* = Proj(e_0^\perp)$$

Thus, we are led to the conclusions in the statement. □

As a conclusion, the notion of isometry is not the correct infinite dimensional analogue of the notion of unitary, and the unitary operators must be introduced as follows:

THEOREM 2.21. *For a linear operator $U \in B(H)$ the following conditions are equivalent, and if they are satisfied, we say that U is a unitary:*

(1) *U is an isometry, which is invertible.*
(2) *U, U^{-1} are both isometries.*
(3) *U, U^* are both isometries.*
(4) *$UU^* = U^*U = 1$.*
(5) *$U^* = U^{-1}$.*

Moreover, the unitary operators from a group $U(H) \subset B(H)$.

PROOF. There are several statements here, the idea being as follows:

(1) The various equivalences in the statement are all clear from definitions, and from Theorem 2.19 in what regards the various possible notions of isometries which can be used, by using the formula $(ST)^* = T^*S^*$ for the adjoints of the products of operators.

(2) The fact that the products and inverses of unitaries are unitaries is also clear, and we conclude that the unitary operators from a group $U(H) \subset B(H)$, as stated. □

Let us discuss now the projections. Modulo the fact that all the subspaces $K \subset H$ where these projections project must be assumed to be closed, in the present setting, here the result is perfectly similar to the one in finite dimensions, as follows:

THEOREM 2.22. *For a linear operator $P \in B(H)$ the following conditions are equivalent, and if they are satisfied, we say that P is a projection:*

(1) *P is the orthogonal projection on a closed subspace $K \subset H$.*
(2) *P satisfies the projection equations $P^2 = P^* = P$.*

PROOF. As in finite dimensions, P is an abstract projection, not necessarily orthogonal, when it is an idempotent, algebrically speaking, in the sense that we have:

$$P^2 = P$$

The point now is that this projection is orthogonal when:

$$\begin{aligned}
< Px - x, Py >= 0 &\iff < P^*Px - P^*x, y >= 0 \\
&\iff P^*Px - P^*x = 0 \\
&\iff P^*P - P^* = 0 \\
&\iff P^*P = P^*
\end{aligned}$$

Now observe that by conjugating, we obtain $P^*P = P$. Thus, we must have $P = P^*$, and so we have shown that any orthogonal projection must satisfy, as claimed:

$$P^2 = P^* = P$$

Conversely, if this condition is satisfied, $P^2 = P$ shows that P is a projection, and $P = P^*$ shows via the above computation that P is indeed orthogonal. □

There is a relation between the projections and the general isometries, such as the shift S that we met before, and we have the following result:

PROPOSITION 2.23. *Given an isometry $U \in B(H)$, the operator*

$$P = UU^*$$

is a projection, namely the orthogonal projection on $Im(U)$.

PROOF. Assume indeed that we have an isometry, $U^*U = 1$. The fact that $P = UU^*$ is indeed a projection can be checked abstractly, as follows:

$$(UU^*)^* = UU^*$$

$$UU^*UU^* = UU^*$$

As for the last assertion, this is something that we already met, for the shift, and the situation in general is similar, with the result itself being clear. □

More generally now, along the same lines, and clarifying the whole situation with the unitaries and isometries, we have the following result:

THEOREM 2.24. *An operator $U \in B(H)$ is a partial isometry, in the usual geometric sense, when the following two operators are projections:*

$$P = UU^* \quad , \quad Q = U^*U$$

Moreover, the isometries, adjoints of isometries and unitaries are respectively characterized by the conditions $Q = 1$, $P = 1$, $P = Q = 1$.

PROOF. The first assertion is a straightforward extension of Proposition 2.23, and the second assertion follows from various results regarding isometries established above. □

It is possible to talk as well about symmetries, in the following way:

DEFINITION 2.25. *An operator $S \in B(H)$ is called a symmetry when $S^2 = 1$, and a unitary symmetry when one of the following equivalent conditions is satisfied:*

1. *S is a unitary, $S^* = S^{-1}$, and a symmetry as well, $S^2 = 1$.*
2. *S satisfies the equations $S = S^* = S^{-1}$.*

Here the terminology is a bit non-standard, because even in finite dimensions, $S^2 = 1$ is not exactly what you would require for a "true" symmetry, as shown by the following transformation, which is a symmetry in our sense, but not a unitary symmetry:

$$\begin{pmatrix}0&2\\1/2&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2y\\x/2\end{pmatrix}$$

Let us study now some larger classes of operators, which are of particular importance, namely the self-adjoint, positive and normal ones. We first have:

THEOREM 2.26. *For an operator $T \in B(H)$, the following conditions are equivalent, and if they are satisfied, we call T self-adjoint:*

(1) *$T = T^*$.*
(2) *$< Tx, x >\in \mathbb{R}$.*

In finite dimensions, we recover in this way the usual self-adjointness notion.

PROOF. There are several assertions here, the idea being as follows:

(1) $\implies$ (2) This is clear, because we have:

$$\begin{aligned}\overline{< Tx, x >} &= < x, Tx >\\ &= < T^*x, x >\\ &= < Tx, x >\end{aligned}$$

(2) $\implies$ (1) In order to prove this, observe that the beginning of the above computation shows that, when assuming $< Tx, x >\in \mathbb{R}$, the following happens:

$$< Tx, x >=< T^*x, x >$$

Thus, in terms of the operator $S = T - T^*$, we have:

$$< Sx, x >= 0$$

In order to finish, we use a polarization trick. We have the following formula:

$$< S(x + y), x + y >=< Sx, x > + < Sy, y > + < Sx, y > + < Sy, x >$$

Since the first 3 terms vanish, the sum of the 2 last terms vanishes too. But, by using $S^* = -S$, coming from $S = T - T^*$, we can process this latter vanishing as follows:

$$\begin{aligned}< Sx, y > &= - < Sy, x >\\ &= < y, Sx >\\ &= \overline{< Sx, y >}\end{aligned}$$

Thus we must have $< Sx, y >\in \mathbb{R}$, and with $y \to iy$ we obtain $< Sx, y >\in i\mathbb{R}$ too, and so $< Sx, y >= 0$. Thus $S = 0$, which gives $T = T^*$, as desired.

(3) Finally, in what regards the finite dimensions, or more generally the case where our Hilbert space comes with a basis, $H = l^2(I)$, here the condition $T = T^*$ corresponds to the usual self-adjointness condition $M = M^*$ at the level of the associated matrices. □

At the level of the basic examples, the situation is as follows:

PROPOSITION 2.27. *The folowing operators are self-adjoint:*

1. *The projections, $P^2 = P^* = P$. In fact, an abstract, algebraic projection is an orthogonal projection precisely when it is self-adjoint.*
2. *The unitary symmetries, $S = S^* = S^{-1}$. In fact, a unitary is a unitary symmetry precisely when it is self-adjoint.*

PROOF. These assertions are indeed all clear from definitions. □

Next in line, we have the notion of positive operator. We have here:

THEOREM 2.28. *The positive operators, which are the operators $T \in B(H)$ satisfying $< Tx, x > \geq 0$, have the following properties:*

1. *They are self-adjoint, $T = T^*$.*
2. *As examples, we have the projections, $P^2 = P^* = P$.*
3. *More generally, $T = S^*S$ is positive, for any $S \in B(H)$.*
4. *In finite dimensions, we recover the usual positive operators.*

PROOF. All these assertions are elementary, the idea being as follows:

(1) This follows from Theorem 2.26, because $< Tx, x > \geq 0$ implies $< Tx, x > \in \mathbb{R}$.

(2) This is clear from $P^2 = P = P^*$, because we have:

$$\begin{aligned} < Px, x > &= < P^2x, x > \\ &= < Px, Px > \\ &= ||Px||^2 \end{aligned}$$

(3) This follows from a similar computation, namely:

$$< S^*Sx, x > = < Sx, Sx > = ||Sx||^2$$

(4) This is well-known, the idea being that the condition $< Tx, x > \geq 0$ corresponds to the usual positivity condition $A \geq 0$, at the level of the associated matrix. □

It is possible to talk as well about strictly positive operators, and we have here:

THEOREM 2.29. *The strictly positive operators, which are the operators $T \in B(H)$ satisfying $< Tx, x > > 0$, for any $x \neq 0$, have the following properties:*

1. *They are self-adjoint, $T = T^*$.*
2. *As examples, $T = S^*S$ is positive, for any $S \in B(H)$ injective.*
3. *In finite dimensions, we recover the usual strictly positive operators.*

PROOF. As before, all these assertions are elementary, the idea being as follows:

(1) This is something that we know, from Theorem 2.28.

(2) This follows from the injectivity of S, because for any $x \neq 0$ we have:

$$\begin{aligned} < S^*Sx, x > &= < Sx, Sx > \\ &= ||Sx||^2 \\ &> 0 \end{aligned}$$

(3) This is well-known, the idea being that the condition $< Tx, x >> 0$ corresponds to the usual strict positivity condition $A > 0$, at the level of the associated matrix. □

As a comment, while any strictly positive matrix $A > 0$ is well-known to be invertible, the analogue of this fact does not hold in infinite dimensions, a counterexample here being the following operator on $l^2(\mathbb{N})$, which is strictly positive but not invertible:

$$T = \begin{pmatrix} 1 &&& \\ & \frac{1}{2} && \\ && \frac{1}{3} & \\ &&& \ddots \end{pmatrix}$$

As a last remarkable class of operators, we have the normal ones. We have here:

Theorem 2.30. *For an operator $T \in B(H)$, the following conditions are equivalent, and if they are satisfied, we call T normal:*

(1) $TT^* = T^*T$.
(2) $||Tx|| = ||T^*x||$.

In finite dimensions, we recover in this way the usual normality notion.

Proof. There are several assertions here, the idea being as follows:

(1) $\implies$ (2) This is clear, due to the following computation:

$$\begin{aligned} ||Tx||^2 &= < Tx, Tx > \\ &= < T^*Tx, x > \\ &= < TT^*x, x > \\ &= < T^*x, T^*x > \\ &= ||T^*x||^2 \end{aligned}$$

(2) $\implies$ (1) This is clear as well, because the above computation shows that, when assuming $||Tx|| = ||T^*x||$, the following happens:

$$< TT^*x, x >=< T^*Tx, x >$$

Thus, in terms of the operator $S = TT^* - T^*T$, we have:

$$< Sx, x >= 0$$

In order to finish, we use a polarization trick. We have the following formula:

$$< S(x+y), x+y >=< Sx, x > + < Sy, y > + < Sx, y > + < Sy, x >$$

Since the first 3 terms vanish, the sum of the 2 last terms vanishes too. But, by using $S=S^*$, coming from $S=TT^*-T^*T$, we can process this latter vanishing as follows:

$$\begin{aligned}<Sx,y> &= -<Sy,x>\\ &= -<y,Sx>\\ &= -\overline{<Sx,y>}\end{aligned}$$

Thus we must have $<Sx,y>\in i\mathbb{R}$, and with $y\to iy$ we obtain $<Sx,y>\in\mathbb{R}$ too, and so $<Sx,y>=0$. Thus $S=0$, which gives $TT^*=T^*T$, as desired.

(3) Finally, in what regards finite dimensions, or more generally the case where our Hilbert space comes with a basis, $H=l^2(I)$, here the condition $TT^*=T^*T$ corresponds to the usual normality condition $MM^*=M^*M$ at the level of the associated matrices. □

Observe that the normal operators generalize both the self-adjoint operators, and the unitaries. We will be back to such operators, on many occassions, in what follows.

2d. Diagonal operators

Let us work out now what happens in the case that we are mostly interested in, namely $H=L^2(X)$, with X being a measured space. We first have:

THEOREM 2.31. *Given a measured space X, consider the Hilbert space $H=L^2(X)$. Associated to any function $f\in L^\infty(X)$ is then the multiplication operator*

$$T_f:H\to H\quad,\quad T_f(g)=fg$$

which is well-defined, linear and bounded, having norm as follows:

$$||T_f||=||f||_\infty$$

Moreover, the correspondence $f\to T_f$ is linear, multiplicative and involutive.

PROOF. There are several assertions here, the idea being as follows:

(1) We must first prove that the formula in the statement, $T_f(g)=fg$, defines indeed an operator $H\to H$, which amounts in saying that we have:

$$f\in L^\infty(X),\ g\in L^2(X)\implies fg\in L^2(X)$$

But this follows from the following explicit estimate:

$$\begin{aligned}||fg||_2 &= \sqrt{\int_X|f(x)|^2|g(x)|^2d\mu(x)}\\ &\leq \sup_{x\in X}|f(x)|^2\sqrt{\int_X|g(x)|^2d\mu(x)}\\ &= ||f||_\infty||g||_2\\ &< \infty\end{aligned}$$

(2) Next in line, we must prove that T is linear and bounded. We have:

$$T_f(g+h) = T_f(g) + T_f(h) \quad , \quad T_f(\lambda g) = \lambda T_f(g)$$

As for the boundedness condition, this follows from the estimate from the proof of (1), which gives, in terms of the operator norm of $B(H)$:

$$||T_f|| \leq ||f||_\infty$$

(3) Let us prove now that we have equality, $||T_f|| = ||f||_\infty$, in the above estimate. For this purpose, we use the well-known fact that the L^∞ functions can be approximated by L^2 functions. Indeed, with such an approximation $g_n \to f$ we obtain:

$$\begin{aligned} ||fg_n||_2 &= \sqrt{\int_X |f(x)|^2 |g_n(x)|^2 d\mu(x)} \\ &\simeq \sup_{x \in X} |f(x)|^2 \sqrt{\int_X |g_n(x)|^2 d\mu(x)} \\ &= ||f||_\infty ||g_n||_2 \end{aligned}$$

Thus, with $n \to \infty$ we obtain $||T_f|| \geq ||f||_\infty$, which is reverse to the inequality obtained in the proof of (2), and this leads to the conclusion in the statement.

(4) Regarding now the fact that the correspondence $f \to T_f$ is indeed linear and multiplicative, the corresponding formulae are as follows, both clear:

$$T_{f+h}(g) = T_f(g) + T_h(g) \quad , \quad T_{\lambda f}(g) = \lambda T_f(g)$$

(5) Finally, let us prove that the correspondence $f \to T_f$ is involutive, in the sense that it transforms the standard involution $f \to \bar{f}$ of the algebra $L^\infty(X)$ into the standard involution $T \to T^*$ of the algebra $B(H)$. We must prove that we have:

$$T_f^* = T_{\bar{f}}$$

But this follows from the following computation:

$$\begin{aligned} < T_f g, h > &= < fg, h > \\ &= \int_X f(x) g(x) \bar{h}(x) d\mu(x) \\ &= \int_X g(x) f(x) \bar{h}(x) d\mu(x) \\ &= < g, \bar{f} h > \\ &= < g, T_{\bar{f}} h > \end{aligned}$$

Indeed, since the adjoint is unique, we obtain from this $T_f^* = T_{\bar{f}}$. Thus the correspondence $f \to T_f$ is indeed involutive, as claimed. $\square$

In what regards now the basic classes of operators, the above construction provides us with many new examples, which are very explicit, and are complementary to the usual finite dimensional examples that we usually have in mind, as follows:

THEOREM 2.32. *The multiplication operators $T_f(g) = fg$ on the Hilbert space $H = L^2(X)$ associated to the functions $f \in L^\infty(X)$ are as follows:*

(1) *T_f is unitary when $f : X \to \mathbb{T}$.*
(2) *T_f is a symmetry when $f : X \to \{-1, 1\}$.*
(3) *T_f is a projection when $f = \chi_Y$ with $Y \in X$.*
(4) *There are no non-unitary isometries.*
(5) *There are no non-unitary symmetries.*
(6) *T_f is positive when $f : X \to \mathbb{R}_+$.*
(7) *T_f is self-adjoint when $f : X \to \mathbb{R}$.*
(8) *T_f is always normal, for any $f : X \to \mathbb{C}$.*

PROOF. All these assertions are clear from definitions, and from the various properties of the correspondence $f \to T_f$, established above, as follows:

(1) The unitarity condition $U^* = U^{-1}$ for the operator T_f reads $\bar{f} = f^{-1}$, which means that we must have $f : X \to \mathbb{T}$, as claimed.

(2) The symmetry condition $S^2 = 1$ for the operator T_f reads $f^2 = 1$, which means that we must have $f : X \to \{-1, 1\}$, as claimed.

(3) The projection condition $P^2 = P^* = P$ for the operator T_f reads $f^2 = f = \bar{f}$, which means that we must have $f : X \to \{0, 1\}$, or equivalently, $f = \chi_Y$ with $Y \subset X$.

(4) A non-unitary isometry must satisfy by definition $U^*U = 1, UU^* \neq 1$, and for the operator T_f this means that we must have $|f|^2 = 1, |f|^2 \neq 1$, which is impossible.

(5) This follows from (1) and (2), because the solutions found in (2) for the symmetry problem are included in the solutions found in (1) for the unitarity problem.

(6) The fact that T_f is positive amounts in saying that we must have $< fg, g > \geq 0$ for any $g \in L^2(X)$, and this is equivalent to the fact that we must have $f \geq 0$, as desired.

(7) The self-adjointness condition $T = T^*$ for the operator T_f reads $f = \bar{f}$, which means that we must have $f : X \to \mathbb{R}$, as claimed.

(8) The normality condition $TT^* = T^*T$ for the operator T_f reads $f\bar{f} = \bar{f}f$, which is automatic for any function $f : X \to \mathbb{C}$, as claimed. □

The above result might look quite puzzling, at a first glance, messing up our intuition with various classes of operators, coming from usual linear algebra. However, a bit of further thinking tells us that there is no contradiction, and that Theorem 2.32 in fact is very similar to what we know about the diagonal matrices. To be more precise, the diagonal matrices are unitaries precisely when their entries are in $\mathbb{T}$, there are no non-unitary isometries, all such matrices are normal, and so on. In order to understand all this, let us work out what happens with the correspondence $f \to T_f$, in finite dimensions. The situation here is in fact extremely simple, and illuminating, as follows:

THEOREM 2.33. *Assuming $X = \{1, \ldots, N\}$ with the counting measure, the embedding*

$$L^\infty(X) \subset B(L^2(X))$$

constructed via multiplication operators, $T_f(g) = fg$, corresponds to the embedding

$$\mathbb{C}^N \subset M_N(\mathbb{C})$$

given by the diagonal matrices, constructed as follows:

$$f \to diag(f_1, \ldots, f_N)$$

Thus, Theorem 2.32 generalizes what we know about the diagonal matrices.

PROOF. The idea is that all this is trivial, with not a single new computation needed, modulo some algebraic thinking, of quite soft type. Let us go back indeed to Theorem 2.31 above and its proof, with the abstract measured space X appearing there being now the following finite space, with its counting mesure:

$$X = \{1, \ldots, N\}$$

Regarding the functions $f \in L^\infty(X)$, these are now functions as follows:

$$f : \{1, \ldots, N\} \to \mathbb{C}$$

We can identify such a function with the corresponding vector $(f(i))_i \in \mathbb{C}^N$, and so we conclude that our input algebra $L^\infty(X)$ is the algebra $\mathbb{C}^N$:

$$L^\infty(X) = \mathbb{C}^N$$

Regarding now the Hilbert space $H = L^2(X)$, this is equal as well to $\mathbb{C}^N$, and for the same reasons, namely that $g \in L^2(X)$ can be identified with the vector $(g(i))_i \in \mathbb{C}^N$:

$$L^2(X) = \mathbb{C}^N$$

Observe that, due to our assumption that X comes with its counting measure, the scalar product that we obtain on $\mathbb{C}^N$ is the usual one, without weights. Now, let us identify the operators on $L^2(X) = \mathbb{C}^N$ with the square matrices, in the usual way:

$$B(L^2(X)) = M_N(\mathbb{C})$$

This was our final identification, in order to get started. Now by getting back to Theorem 2.31, the embedding $L^\infty(X) \subset B(L^2(X))$ constructed there reads:

$$\mathbb{C}^N \subset M_N(\mathbb{C})$$

But this can only be the embedding given by the diagonal matrices, so are basically done. In order to finish, however, let us understand what the operator associated to an arbitrary vector $f\in\mathbb C^N$ is. We can regard this vector as a function, $f(i)=f_i$, and so the action $T_f(g)=fg$ on the vectors of $L^2(X)=\mathbb C^N$ is by componentwise multiplication by the numbers $f_1,\ldots,f_N$. But this is exactly the action of the diagonal matrix $diag(f_1,\ldots,f_N)$, and so we are led to the conclusion in the statement. □

There are other things that can be said about the embedding $L^\infty(X)\subset B(L^2(X))$, a key observation here, which is elementary to prove, being the fact that the image of $L^\infty(X)$ is closed with respect to the weak topology, the one where $T_n\to T$ when $T_nx\to Tx$ for any $x\in H$. And with this meaning that $L^\infty(X)$ is a so-called von Neumann algebra on $L^2(X)$. We will be back to this, on numerous occasions, in what follows.

2e. Exercises

Exercises:

Exercise 2.34.

Exercise 2.35.

Exercise 2.36.

Exercise 2.37.

Exercise 2.38.

Exercise 2.39.

Exercise 2.40.

Exercise 2.41.

Bonus exercise.

CHAPTER 3

Spectral radius

3a. The spectrum

We would like now to discuss the diagonalization problem for the operators $T \in B(H)$, in analogy with the diagonalization problem for the usual matrices $A \in M_N(\mathbb{C})$. As a first observation, we can talk about eigenvalues and eigenvectors, as follows:

Definition 3.1. *Given an operator $T \in B(H)$, assuming that we have*

$$Tx = \lambda x$$

we say that $x \in H$ is an eigenvector of T, with eigenvalue $\lambda \in \mathbb{C}$.

We know many things about eigenvalues and eigenvectors, in the finite dimensional case. However, most of these will not extend to the infinite dimensional case, or at least not extend in a straightforward way, due to a number of reasons:

(1) Most of basic linear algebra is based on the fact that $Tx = \lambda x$ is equivalent to $(T - \lambda)x = 0$, so that λ is an eigenvalue when $T - \lambda$ is not invertible. In the infinite dimensional setting $T - \lambda$ might be injective and not surjective, or vice versa, or invertible with $(T - \lambda)^{-1}$ not bounded, and so on.

(2) Also, in linear algebra $T - \lambda$ is not invertible when $\det(T - \lambda) = 0$, and with this leading to most of the advanced results about eigenvalues and eigenvectors. In infinite dimensions, however, it is impossible to construct a determinant function $\det : B(H) \to \mathbb{C}$, and this even for the diagonal operators on $l^2(\mathbb{N})$.

Summarizing, we are in trouble with our extension program, and this right from the beginning. In order to have some theory started, however, let us forget about (2), which obviously leads nowhere, and focus on the difficulties in (1).

In order to cut short the discussion there, regarding the various properties of $T - \lambda$, we can just say that $T - \lambda$ is either invertible with bounded inverse, the "good case", or not. We are led in this way to the following definition:

Definition 3.2. *The spectrum of an operator $T \in B(H)$ is the set*

$$\sigma(T) = \left\{\lambda \in \mathbb{C}\Big|T - \lambda \notin B(H)^{-1}\right\}$$

where $B(H)^{-1} \subset B(H)$ is the set of invertible operators.

As a basic example, in the finite dimensional case, $H=\mathbb{C}^N$, the spectrum of a usual matrix $A\in M_N(\mathbb{C})$ is the collection of its eigenvalues, taken without multiplicities. We will see many other examples. In general, the spectrum has the following properties:

PROPOSITION 3.3. *The spectrum of $T\in B(H)$ contains the eigenvalue set*

$$\varepsilon(T)=\left\{\lambda\in\mathbb{C}\Big|\ker(T-\lambda)\neq\{0\}\right\}$$

and $\varepsilon(T)\subset\sigma(T)$ is an equality in finite dimensions, but not in infinite dimensions.

PROOF. We have several assertions here, the idea being as follows:

(1) First of all, the eigenvalue set is indeed the one in the statement, because $Tx=\lambda x$ tells us precisely that $T-\lambda$ must be not injective. The fact that we have $\varepsilon(T)\subset\sigma(T)$ is clear as well, because if $T-\lambda$ is not injective, it is not bijective.

(2) In finite dimensions we have $\varepsilon(T)=\sigma(T)$, because $T-\lambda$ is injective if and only if it is bijective, with the boundedness of the inverse being automatic.

(3) In infinite dimensions we can assume $H=l^2(\mathbb{N})$, and the shift operator $S(e_i)=e_{i+1}$ is injective but not surjective. Thus $0\in\sigma(T)-\varepsilon(T)$. □

We will see more examples and counterexamples, and some general theory, in a moment. Philosophically speaking, the best way of thinking at all this is as follows:

– The numbers $\lambda\notin\sigma(T)$ are good, because we can invert $T-\lambda$.

– The numbers $\lambda\in\sigma(T)-\varepsilon(T)$ are bad.

– The eigenvalues $\lambda\in\varepsilon(T)$ are evil.

Note that this is somewhat contrary to what happens in linear algebra, where the eigenvalues are highly valued, and cherished, and regarded as being the source of all good things on Earth. Welcome to operator theory, where some things are upside down.

Let us develop now some general theory for the spectrum, or perhaps for its complement, with the promise to come back to eigenvalues later. As a first result, we would like to prove that the spectra are non-empty. This is something tricky, and we will need:

PROPOSITION 3.4. *The following happen:*

(1) $||T||<1\implies(1-T)^{-1}=1+T+T^2+\ldots$
(2) *The set $B(H)^{-1}$ is open.*
(3) *The map $T\to T^{-1}$ is differentiable.*

PROOF. All these assertions are elementary, as follows:

(1) This follows as in the scalar case, the computation being as follows, provided that everything converges under the norm, which amounts in saying that $||T||<1$:

$$\begin{aligned}(1-T)(1+T+T^2+\ldots) &= 1-T+T-T^2+T^2-T^3+\ldots\\ &= 1\end{aligned}$$

(2) Assuming $T\in B(H)^{-1}$, let us pick $S\in B(H)$ such that:

$$||T-S||<\frac{1}{||T^{-1}||}$$

We have then the following estimate:

$$\begin{aligned}||1-T^{-1}S|| &= ||T^{-1}(T-S)||\\ &\leq ||T^{-1}||\cdot||T-S||\\ &< 1\end{aligned}$$

Thus we have $T^{-1}S\in B(H)^{-1}$, and so $S\in B(H)^{-1}$, as desired.

(3) In the scalar case, the derivative of $f(t)=t^{-1}$ is $f'(t)=-t^{-2}$. In the present normed space setting the derivative is no longer a number, but rather a linear transformation, which can be found by developing $f(T)=T^{-1}$ at order 1, as follows:

$$\begin{aligned}(T+S)^{-1} &= ((1+ST^{-1})T)^{-1}\\ &= T^{-1}(1+ST^{-1})^{-1}\\ &= T^{-1}(1-ST^{-1}+(ST^{-1})^2-\ldots)\\ &\simeq T^{-1}(1-ST^{-1})\\ &= T^{-1}-T^{-1}ST^{-1}\end{aligned}$$

Thus $f(T)=T^{-1}$ is indeed differentiable, with derivative $f'(T)S=-T^{-1}ST^{-1}$. $\square$

We can now formulate our first theorem about spectra, as follows:

THEOREM 3.5. *The spectrum of a bounded operator $T\in B(H)$ is:*

(1) *Compact.*
(2) *Contained in the disc $D_0(||T||)$.*
(3) *Non-empty.*

PROOF. This can be proved by using Proposition 3.4, along with a bit of complex and functional analysis, for which we refer to Rudin [81] or Lax [68], as follows:

(1) In view of (2) below, it is enough to prove that $\sigma(T)$ is closed. But this follows from the following computation, with $|\varepsilon|$ being small:

$$\begin{aligned}\lambda\notin\sigma(T) &\implies T-\lambda\in B(H)^{-1}\\ &\implies T-\lambda-\varepsilon\in B(H)^{-1}\\ &\implies \lambda+\varepsilon\notin\sigma(T)\end{aligned}$$

(2) This follows from the following computation:

$$\begin{aligned}\lambda > \|T\| &\implies \left\|\frac{T}{\lambda}\right\| < 1\\ &\implies 1 - \frac{T}{\lambda} \in B(H)^{-1}\\ &\implies \lambda - T \in B(H)^{-1}\\ &\implies \lambda \notin \sigma(T)\end{aligned}$$

(3) Assume by contradiction $\sigma(T) = \emptyset$. Given a linear form $f \in B(H)^*$, consider the following map, which is well-defined, due to our assumption $\sigma(T) = \emptyset$:

$$\varphi : \mathbb{C} \to \mathbb{C} \quad , \quad \lambda \to f((T - \lambda)^{-1})$$

By using the fact that $T \to T^{-1}$ is differentiable, that we know from Proposition 3.4, we conclude that this map is differentiable, and so holomorphic. Also, we have:

$$\begin{aligned}\lambda \to \infty &\implies T - \lambda \to \infty\\ &\implies (T - \lambda)^{-1} \to 0\\ &\implies f((T - \lambda))^{-1} \to 0\end{aligned}$$

Thus by the Liouville theorem we obtain $\varphi = 0$. But, in view of the definition of φ, this gives $(T - \lambda)^{-1} = 0$, which is a contradiction, as desired. □

Here is now a second basic result regarding the spectra, inspired from what happens in finite dimensions, for the usual complex matrices, and which shows that things do not necessarily extend without troubles to the infinite dimensional setting:

Theorem 3.6. *We have the following formula, valid for any operators S, T:*

$$\sigma(ST) \cup \{0\} = \sigma(TS) \cup \{0\}$$

In finite dimensions we have $\sigma(ST) = \sigma(TS)$, but this fails in infinite dimensions.

Proof. There are several assertions here, the idea being as follows:

(1) This is something that we know in finite dimensions, coming from the fact that the characteristic polynomials of the associated matrices A, B coincide:

$$P_{AB} = P_{BA}$$

Thus we obtain $\sigma(ST) = \sigma(TS)$ in this case, as claimed. Observe that this improves twice the general formula in the statement, first because we have no issues at 0, and second because what we obtain is actually an equality of sets with mutiplicities.

(2) In general now, let us first prove the main assertion, stating that $\sigma(ST), \sigma(TS)$ coincide outside 0. We first prove that we have the following implication:

$$1 \notin \sigma(ST) \implies 1 \notin \sigma(TS)$$

Assume indeed that $1-ST$ is invertible, with inverse denoted R:

$$R=(1-ST)^{-1}$$

We have then the following formulae, relating our variables R,S,T:

$$RST=STR=R-1$$

By using $RST=R-1$, we have the following computation:

$$\begin{aligned}(1+TRS)(1-TS) &= 1+TRS-TS-TRSTS\\ &= 1+TRS-TS-TRS+TS\\ &= 1\end{aligned}$$

A similar computation, using $STR=R-1$, shows that we have:

$$(1-TS)(1+TRS)=1$$

Thus $1-TS$ is invertible, with inverse $1+TRS$, which proves our claim. Now by multiplying by scalars, we deduce from this that for any $\lambda\in\mathbb{C}-\{0\}$ we have:

$$\lambda\notin\sigma(ST)\implies\lambda\notin\sigma(TS)$$

But this leads to the conclusion in the statement.

(3) Regarding now the counterexample to the formula $\sigma(ST)=\sigma(TS)$, in general, let us take S to be the shift on $H=L^2(\mathbb{N})$, given by the following formula:

$$S(e_i)=e_{i+1}$$

As for T, we can take it to be the adjoint of S, which is the following operator:

$$S^*(e_i)=\begin{cases}e_{i-1} & \text{if } i>0\\ 0 & \text{if } i=0\end{cases}$$

Let us compose now these two operators. In one sense, we have:

$$S^*S=1\implies 0\notin\sigma(S^*S)$$

In the other sense, however, the situation is different, as follows:

$$SS^*=Proj(e_0^\perp)\implies 0\in\sigma(SS^*)$$

Thus, the spectra do not match on 0, and we have our counterexample, as desired. □

3b. Functional calculus

Let us develop now some systematic theory for the computation of the spectra, based on what we know about the eigenvalues of the usual complex matrices. As a first result, which is well-known for the usual matrices, and extends well, we have:

THEOREM 3.7. *We have the "polynomial functional calculus" formula*

$$\sigma(P(T)) = P(\sigma(T))$$

valid for any polynomial $P \in \mathbb{C}[X]$, and any operator $T \in B(H)$.

PROOF. We pick a scalar $\lambda \in \mathbb{C}$, and we decompose the polynomial $P - \lambda$:

$$P(X) - \lambda = c(X - r_1)\ldots(X - r_n)$$

We have then the following equivalences:

$$\begin{aligned}
\lambda \notin \sigma(P(T)) &\iff P(T) - \lambda \in B(H)^{-1}\\
&\iff c(T - r_1)\ldots(T - r_n) \in B(H)^{-1}\\
&\iff T - r_1, \ldots, T - r_n \in B(H)^{-1}\\
&\iff r_1, \ldots, r_n \notin \sigma(T)\\
&\iff \lambda \notin P(\sigma(T))
\end{aligned}$$

Thus, we are led to the formula in the statement. □

The above result is something very useful, and generalizing it will be our next task. As a first ingredient here, assuming that $A \in M_N(\mathbb{C})$ is invertible, we have:

$$\sigma(A^{-1}) = \sigma(A)^{-1}$$

It is possible to extend this formula to the arbitrary operators, and we will do this in a moment. Before starting, however, we have to think in advance on how to unify this potential result, that we have in mind, with Theorem 3.7 itself.

What we have to do here is to find a class of functions generalizing both the polynomials $P \in \mathbb{C}[X]$ and the inverse function $x \to x^{-1}$, and the answer to this question is provided by the rational functions, which are as follows:

DEFINITION 3.8. *A rational function $f \in \mathbb{C}(X)$ is a quotient of polynomials:*

$$f = \frac{P}{Q}$$

Assuming that P, Q are prime to each other, we can regard f as a usual function,

$$f : \mathbb{C} - X \to \mathbb{C}$$

with X being the set of zeros of Q, also called poles of f.

We should mention here that the term "poles" comes from the fact that, if you want to imagine the graph of such a rational function f, in two complex dimensions, what you get is some sort of tent, supported by poles of infinite height, situated at the zeros of Q. For more on all this, and on complex analysis in general, we refer as usual to Rudin [**81**]. Although a look at an abstract algebra book can be interesting as well.

Now that we have our class of functions, the next step consists in applying them to operators. Here we cannot expect $f(T)$ to make sense for any f and any T, for instance because T^{-1} is defined only when T is invertible. We are led in this way to:

DEFINITION 3.9. *Given an operator $T \in B(H)$, and a rational function $f = P/Q$ having poles outside $\sigma(T)$, we can construct the following operator,*

$$f(T) = P(T)Q(T)^{-1}$$

that we can denote as a usual fraction, as follows,

$$f(T) = \frac{P(T)}{Q(T)}$$

due to the fact that $P(T), Q(T)$ commute, so that the order is irrelevant.

To be more precise, $f(T)$ is indeed well-defined, and the fraction notation is justified too. In more formal terms, we can say that we have a morphism of complex algebras as follows, with $\mathbb{C}(X)^T$ standing for the rational functions having poles outside $\sigma(T)$:

$$\mathbb{C}(X)^T \to B(H) \quad , \quad f \to f(T)$$

Summarizing, we have now a good class of functions, generalizing both the polynomials and the inverse map $x \to x^{-1}$. We can now extend Theorem 3.7, as follows:

THEOREM 3.10. *We have the "rational functional calculus" formula*

$$\sigma(f(T)) = f(\sigma(T))$$

valid for any rational function $f \in \mathbb{C}(X)$ having poles outside $\sigma(T)$.

PROOF. We pick a scalar $\lambda \in \mathbb{C}$, we write $f = P/Q$, and we set:

$$F = P - \lambda Q$$

By using now Theorem 3.7, for this polynomial, we obtain:

$$\begin{aligned}
\lambda \in \sigma(f(T)) &\iff F(T) \notin B(H)^{-1} \\
&\iff 0 \in \sigma(F(T)) \\
&\iff 0 \in F(\sigma(T)) \\
&\iff \exists \mu \in \sigma(T), F(\mu) = 0 \\
&\iff \lambda \in f(\sigma(T))
\end{aligned}$$

Thus, we are led to the formula in the statement. □

As an application of the above methods, we can investigate certain special classes of operators, such as the self-adjoint ones, and the unitary ones. Let us start with:

PROPOSITION 3.11. *The following happen:*

(1) *We have $\sigma(T^*)=\overline{\sigma(T)}$, for any $T\in B(H)$.*
(2) *If $T=T^*$ then $X=\sigma(T)$ satisfies $X=\overline{X}$.*
(3) *If $U^*=U^{-1}$ then $X=\sigma(U)$ satisfies $X^{-1}=\overline{X}$.*

PROOF. We have several assertions here, the idea being as follows:

(1) The spectrum of the adjoint operator T^* can be computed as follows:

$$\begin{aligned}\sigma(T^*)&=\left\{\lambda\in\mathbb C\Big|T^*-\lambda\notin B(H)^{-1}\right\}\\&=\left\{\lambda\in\mathbb C\Big|T-\bar\lambda\notin B(H)^{-1}\right\}\\&=\overline{\sigma(T)}\end{aligned}$$

(2) This is clear indeed from (1).

(3) For a unitary operator, $U^*=U^{-1}$, Theorem 3.10 and (1) give:

$$\sigma(U)^{-1}=\sigma(U^{-1})=\sigma(U^*)=\overline{\sigma(U)}$$

Thus, we are led to the conclusion in the statement. □

In analogy with what happens for the usual matrices, we would like to improve now (2,3) above, with results stating that the spectrum $X=\sigma(T)$ satisfies $X\subset\mathbb R$ for self-adjoints, and $X\subset\mathbb T$ for unitaries. This will be tricky. Let us start with:

THEOREM 3.12. *The spectrum of a unitary operator*

$$U^*=U^{-1}$$

is on the unit circle, $\sigma(U)\subset\mathbb T$.

PROOF. Assuming $U^*=U^{-1}$, we have the following norm computation:

$$||U||=\sqrt{||UU^*||}=\sqrt1=1$$

Now if we denote by D the unit disk, we obtain from this:

$$\sigma(U)\subset D$$

On the other hand, once again by using $U^*=U^{-1}$, we have as well:

$$||U^{-1}||=||U^*||=||U||=1$$

Thus, as before with D being the unit disk in the complex plane, we have:

$$\sigma(U^{-1})\subset D$$

Now by using Theorem 3.10, we obtain $\sigma(U)\subset D\cap D^{-1}=\mathbb T$, as desired. □

We have as well a similar result for self-adjoints, as follows:

THEOREM 3.13. *The spectrum of a self-adjoint operator*

$$T = T^*$$

consists of real numbers, $\sigma(T) \subset \mathbb{R}$.

PROOF. The idea is that we can deduce the result from Theorem 3.12, by using the following remarkable rational function, depending on a parameter $r \in \mathbb{R}$:

$$f(z) = \frac{z + ir}{z - ir}$$

Indeed, for $r >> 0$ the operator $f(T)$ is well-defined, and we have:

$$\left(\frac{T + ir}{T - ir}\right)^* = \frac{T - ir}{T + ir} = \left(\frac{T + ir}{T - ir}\right)^{-1}$$

Thus $f(T)$ is unitary, and by using Theorem 3.12 we obtain:

$$\begin{aligned} \sigma(T) &\subset f^{-1}(f(\sigma(T))) \\ &= f^{-1}(\sigma(f(T))) \\ &\subset f^{-1}(\mathbb{T}) \\ &= \mathbb{R} \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

As a theoretical remark, it is possible to deduce as well Theorem 3.12 from Theorem 3.13, by performing the above computation in the other sense. Indeed, by assuming that Theorem 3.13 holds indeed, and starting with a unitary $U \in B(H)$, we obtain:

$$\begin{aligned} \sigma(U) &\subset f(f^{-1}(\sigma(U))) \\ &= f(\sigma(f^{-1}(U))) \\ &\subset f(\mathbb{R}) \\ &= \mathbb{T} \end{aligned}$$

As a conclusion now, we have so far a beginning of spectral theory, with results allowing us to investigate the unitaries and the self-adjoints, and with the remark that these two classes of operators are related by a certain wizarding rational function, namely:

$$f(z) = \frac{z + ir}{z - ir}$$

Let us keep now building on this, with some more complex analysis involved. One key thing that we know about matrices, and which follows for instance by using the fact that the diagonalizable matrices are dense, is the following formula:

$$\sigma(e^A) = e^{\sigma(A)}$$

We would like to have such formulae for the general operators $T\in B(H)$, but this is something quite technical. Consider the rational calculus morphism from Definition 3.9, which is as follows, with the exponent standing for "having poles outside $\sigma(T)$":

$$\mathbb{C}(X)^T\to B(H)\quad,\quad f\to f(T)$$

As mentioned before, the rational functions are holomorphic outside their poles, and this raises the question of extending this morphism, as follows:

$$Hol(\sigma(T))\to B(H)\quad,\quad f\to f(T)$$

Normally this can be done in several steps. Let us start with:

PROPOSITION 3.14. *We can exponentiate any operator $T\in B(H)$, by setting:*

$$e^T=\sum_{k=0}^\infty\frac{T^k}{k!}$$

Similarly, we can define $f(T)$, for any holomorphic function $f:\mathbb{C}\to\mathbb{C}$.

PROOF. We must prove that the series defining e^T converges, and this follows from:

$$||e^T||\leq\sum_{k=0}^\infty\frac{||T||^k}{k!}=e^{||T||}$$

The case of the arbitrary holomorphic functions $f:\mathbb{C}\to\mathbb{C}$ is similar. □

In general, the holomorphic functions are not entire, and the above method won't cover the rational functions $f\in\mathbb{C}(X)^T$ that we want to generalize. Thus, we must use something else. And the answer here comes from the Cauchy formula:

$$f(t)=\frac{1}{2\pi i}\int_\gamma\frac{f(z)}{z-t}\,dz$$

Indeed, given a rational function $f\in\mathbb{C}(X)^T$, the operator $f(T)\in B(H)$, constructed in Definition 3.9, can be recaptured in an analytic way, as follows:

$$f(T)=\frac{1}{2\pi i}\int_\gamma\frac{f(z)}{z-T}\,dz$$

Now given an arbitrary function $f\in Hol(\sigma(T))$, we can define $f(T)\in B(H)$ by the exactly same formula, and we obtain in this way the desired correspondence:

$$Hol(\sigma(T))\to B(H)\quad,\quad f\to f(T)$$

This was for the plan. In practice now, all this needs a bit of care, with many verifications needed, and with the technical remark that a winding number must be added to the above Cauchy formulae, for things to be correct. Let us start with:

Definition 3.15. *If γ is a loop in $\mathbb{C}$ the number of times γ goes around a point $z \in \mathbb{C} - \gamma$ is computed by the following integral, called winding number:*

$$Ind(\gamma, z) = \frac{1}{2\pi i} \int_\gamma \frac{d\xi}{\xi - z}$$

We say that γ turns around z if $Ind(\gamma, z) = 1$, and that it does not turn if $Ind(\gamma, z) = 0$. Otherwise, we say that γ turns around z many times, or in the bad sense, or both.

Let $f : U \to \mathbb{C}$ be an holomorphic function defined on an open subset of $\mathbb{C}$, and γ be a loop in U. If $Ind(\gamma, z) \neq 0$ for $z \in \mathbb{C} - U$ then $f(z)$ is given by the Cauchy formula:

$$Ind(\gamma, z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z} \, d\xi$$

Also, if $Ind(\gamma, z) = 0$ for $z \in \mathbb{C} - U$ then the integral of f on γ is zero:

$$\int_\gamma f(\xi) \, d\xi = 0$$

It is convenient to use formal combinations of loops, called cycles:

$$\Sigma = n_1\gamma_1 + \ldots + n_r\gamma_r$$

The winding number for Σ is by definition the corresponding linear combination of winding numbers of its loop components, and the Cauchy formula holds for arbitrary cycles. Now by getting back to operators, we can formulate:

Definition 3.16. *Let $T \in B(H)$ and let $f : U \to \mathbb{C}$ be an holomorphic function defined on an open set containing $\sigma(T)$. Define an element $f(T)$ by the formula*

$$f(T) = \frac{1}{2\pi i} \int_\Sigma \frac{f(\xi)}{\xi - T} \, d\xi$$

where Σ is a cycle in $U - \sigma(T)$ which turns around $\sigma(T)$ and doesn't turn around $\mathbb{C} - U$.

The formula makes sense because Σ is in $U - \sigma(T)$. Also, $f(T)$ is independent of the choice of Σ. Indeed, let Σ_1 and Σ_2 be two cycles. Their difference $\Sigma_1 - \Sigma_2$ is a cycle which doesn't turn around $\sigma(a)$, neither around $\mathbb{C} - U$. The function $z \to f(z)/(z - T)$ being holomorphic $U - \sigma(T) \to B(H)$, its integral on $\Sigma_1 - \Sigma_2$ must be zero:

$$\int_{\Sigma_1 - \Sigma_2} \frac{f(\xi)}{\xi - T} \, d\xi = 0$$

Thus $f(T)$ is the same with respect to Σ_1 and to Σ_2, and so Definition 3.16 is fully justified. Now with this definition in hand, we first have the following result:

Proposition 3.17. *We have the formula*

$$f(T)g(T) = (fg)(T)$$

whenever the equality makes sense.

PROOF. Let Σ_1 be a cycle in $U - \sigma(T)$ around $\sigma(T)$ and consider the following set:

$$Int(\Sigma_1) = \left\{z \in \mathbb{C} - \Sigma_1 \Big| Ind(\Sigma_1, z) \neq 0\right\} \cup \Sigma_1$$

This is a compact set, included in U and containing the spectrum of T:

$$\sigma(T) \subset Int(\Sigma_1) \subset U$$

Let Σ_2 be a cycle in $U - Int(\Sigma_1)$ turning around $Int(\Sigma_1)$. Consider two holomorphic functions f, g defined around $\sigma(T)$, so that the statement make sense. We have:

$$\begin{aligned} f(T)g(T) &= \left(\frac{1}{2\pi i}\right)^2 \left(\int_{\Sigma_1} \frac{f(\xi)}{\xi - T}\, d\xi\right) \left(\int_{\Sigma_2} \frac{g(\eta)}{\eta - T}\, d\eta\right) \\ &= \left(\frac{1}{2\pi i}\right)^2 \int_{\Sigma_1} \int_{\Sigma_2} \frac{f(\xi)g(\eta)}{(\xi - T)(\eta - T)}\, d\eta d\xi \end{aligned}$$

In order to integrate, we can use the following identity:

$$\frac{1}{(\xi - T)(\eta - T)} = \frac{1}{(\eta - \xi)(\xi - T)} + \frac{1}{(\xi - \eta)(\eta - T)}$$

Thus our integral, and so our formula for $f(T)g(T)$, splits into two terms. The first term can be computed by integrating first over Σ_2, and we obtain:

$$\frac{1}{2\pi i} \int_{\Sigma_1} \frac{f(\xi)g(\xi)}{\xi - T}\, d\xi = (fg)(T)$$

As for the second term, here we can integrate first over Σ_1, and we get:

$$\frac{1}{2\pi i} \int_{\Sigma_2} \frac{g(\eta)}{\eta - T} \left(\frac{1}{2\pi i} \int_{\Sigma_1} \frac{f(\xi)}{\xi - \eta} d\xi\right) d\eta = 0$$

It follows that $f(T)g(T)$ is equal to $(fg)(T)$, as claimed. $\square$

We can now formulate our extension of Theorem 3.10, as follows:

THEOREM 3.18. *Given $T \in B(H)$, we have a morphism of algebras as follows, where $Hol(\sigma(T))$ is the algebra of functions which are holomorphic around $\sigma(T)$,*

$$Hol(\sigma(T)) \to B(H) \quad , \quad f \to f(T)$$

which extends the previous rational functional calculus $f \to f(T)$. We have:

$$\sigma(f(T)) = f(\sigma(T))$$

Moreover, if $\sigma(T)$ is contained in an open set U and $f_n, f : U \to \mathbb{C}$ are holomorphic functions such that $f_n \to f$ uniformly on compact subsets of U then $f_n(T) \to f(T)$.

PROOF. There are several things to be proved here, as follows:

(1) Consider indeed the algebra $Hol(\sigma(T))$, with the convention that two functions are identified if they coincide on an open set containing $\sigma(T)$. We have then a construction $f \to f(T)$ as in the statement, provided by Definition 3.16 and Proposition 3.17.

(2) Let us prove now that our construction extends the one for rational functions. Since $1, z$ generate $\mathbb{C}(X)$, it is enough to show that $f(z) = 1$ implies $f(T) = 1$, and that $f(z) = z$ implies $f(T) = T$. For this purpose, we prove that $f(z) = z^n$ implies $f(T) = T^n$ for any n. But this follows by integrating over a circle γ of big radius, as follows:

$$\begin{aligned} f(T) &= \frac{1}{2\pi i}\int_\gamma \frac{\xi^n}{\xi - T}\,d\xi \\ &= \frac{1}{2\pi i}\int_\gamma \xi^{n-1}\left(1-\frac{T}{\xi}\right)^{-1} d\xi \\ &= \frac{1}{2\pi i}\int_\gamma \xi^{n-1}\left(\sum_{k=0}^\infty \xi^{-k}T^k\right) d\xi \\ &= \sum_{k=0}^\infty \left(\frac{1}{2\pi i}\int_\gamma \xi^{n-k-1}d\xi\right) T^k \\ &= T^n \end{aligned}$$

(3) Regarding $\sigma(f(T)) = f(\sigma(T))$, it is enough to prove that this equality holds on the point 0, and we can do this by double inclusion, as follows:

"$\supset$". Assume that $f(\sigma(T))$ contains 0, and let $z_0 \in \sigma(T)$ be such that $f(z_0) = 0$. Consider the function $g(z) = f(z)/(z - z_0)$. We have $g(T)(T - z_0) = f(T)$ by using the morphism property. Since $T - z_0$ is not invertible, $f(T)$ is not invertible either.

"$\subset$". Assume now that $f(\sigma(T))$ does not contain 0. With the holomorphic function $g(z) = 1/f(z)$ we get $g(T) = f(T)^{-1}$, so $f(T)$ is invertible, and we are done.

(4) Finally, regarding the last assertion, this is clear from definitions. And with the remark that this can be applied to holomorphic functions written as series:

$$f(z) = \sum_{n=0}^\infty a_n(z - z_0)^n$$

Indeed, if this is the expansion of f around z_0, with convergence radius r, and if $\sigma(T)$ is contained in the disc centered at z_0 of radius r, then $f(T)$ is given by:

$$f(T) = \sum_{n=0}^\infty a_n(T - z_0)^n$$

Summarizing, we have proved the result, and fully extended Theorem 3.10. □

3c. Spectral radius

In order to formulate now our next result, which will be a key step towards a theory of diagonalization, for the normal operators, we will need the following notion:

Definition 3.19. *Given an operator $T\in B(H)$, its spectral radius*

$$\rho(T)\in\big[0,||T||\big]$$

is the radius of the smallest disk centered at 0 containing $\sigma(T)$.

Here we have included for convenience a number of basic results from Theorem 3.5, namely the fact that the spectrum is non-empty, and is contained in the disk $D_0(||T||)$, which provide us respectively with the inequalities $\rho(T)\geq 0$, with the usual convention $\sup\emptyset=-\infty$, and $\rho(T)\leq||T||$. Now with this notion in hand, we have the following key result, improving our key result so far, namely $\sigma(T)\neq\emptyset$, from Theorem 3.5:

Theorem 3.20. *The spectral radius of an operator $T\in B(H)$ is given by*

$$\rho(T)=\lim_{n\to\infty}||T^n||^{1/n}$$

and in this formula, we can replace the limit by an inf.

Proof. We have several things to be proved, the idea being as follows:

(1) Our first claim is that the numbers $u_n=||T^n||^{1/n}$ satisfy:

$$(n+m)u_{n+m}\leq nu_n+mu_m$$

Indeed, we have the following estimate, using the Young inequality $ab\leq a^p/p+b^q/q$, with exponents $p=(n+m)/n$ and $q=(n+m)/m$:

$$\begin{aligned}
u_{n+m}&=&&||T^{n+m}||^{1/(n+m)}\\
&\leq&&||T^n||^{1/(n+m)}||T^m||^{1/(n+m)}\\
&\leq&&||T^n||^{1/n}\cdot\frac{n}{n+m}+||T^m||^{1/m}\cdot\frac{m}{n+m}\\
&=&&\frac{nu_n+mu_m}{n+m}
\end{aligned}$$

(2) Our second claim is that the second assertion holds, namely:

$$\lim_{n\to\infty}||T^n||^{1/n}=\inf_n||T^n||^{1/n}$$

For this purpose, we just need the inequality found in (1). Indeed, fix $m \geq 1$, let $n \geq 1$, and write $n = lm + r$ with $0 \leq r \leq m - 1$. By using twice $u_{ab} \leq u_b$, we get:

$$\begin{aligned} u_n &\leq \frac{1}{n}(lmu_{lm} + ru_r) \\ &\leq \frac{1}{n}(lmu_m + ru_1) \\ &\leq u_m + \frac{r}{n}\, u_1 \end{aligned}$$

It follows that we have $\limsup_n u_n \leq u_m$, which proves our claim.

(3) Summarizing, we are left with proving the main formula, which is as follows, and with the remark that we already know that the sequence on the right converges:

$$\rho(T) = \lim_{n\to\infty} ||T^n||^{1/n}$$

In one sense, we can use the polynomial calculus formula $\sigma(T^n) = \sigma(T)^n$. Indeed, this gives the following estimate, valid for any n, as desired:

$$\begin{aligned} \rho(T) &= \sup_{\lambda\in\sigma(T)} |\lambda| \\ &= \sup_{\rho\in\sigma(T)^n} |\rho|^{1/n} \\ &= \sup_{\rho\in\sigma(T^n)} |\rho|^{1/n} \\ &= \rho(T^n)^{1/n} \\ &\leq ||T^n||^{1/n} \end{aligned}$$

(4) For the reverse inequality, we fix a number $\rho > \rho(T)$, and we want to prove that we have $\rho \geq \lim_{n\to\infty} ||T^n||^{1/n}$. By using the Cauchy formula, we have:

$$\begin{aligned} \frac{1}{2\pi i}\int_{|z|=\rho} \frac{z^n}{z-T}\, dz &= \frac{1}{2\pi i}\int_{|z|=\rho} \sum_{k=0}^\infty z^{n-k-1}T^k\, dz \\ &= \sum_{k=0}^\infty \frac{1}{2\pi i}\left(\int_{|z|=\rho} z^{n-k-1}dz\right) T^k \\ &= \sum_{k=0}^\infty \delta_{n,k+1}T^k \\ &= T^{n-1} \end{aligned}$$

By applying the norm we obtain from this formula:

$$\begin{aligned} ||T^{n-1}|| &\leq \frac{1}{2\pi}\int_{|z|=\rho}\left|\left|\frac{z^n}{z-T}\right|\right| dz \\ &\leq \rho^n \cdot \sup_{|z|=\rho}\left|\left|\frac{1}{z-T}\right|\right| \end{aligned}$$

Since the sup does not depend on n, by taking n-th roots, we obtain in the limit:

$$\rho \geq \lim_{n\to\infty} ||T^n||^{1/n}$$

Now recall that ρ was by definition an arbitrary number satisfying $\rho > \rho(T)$. Thus, we have obtained the following estimate, valid for any $T \in B(H)$:

$$\rho(T) \geq \lim_{n\to\infty} ||T^n||^{1/n}$$

Thus, we are led to the conclusion in the statement. □

In the case of the normal elements, we have the following finer result:

THEOREM 3.21. *The spectral radius of a normal element,*

$$TT^* = T^*T$$

is equal to its norm.

PROOF. We can proceed in two steps, as follows:

Step 1. In the case $T = T^*$ we have $||T^n|| = ||T||^n$ for any exponent of the form $n = 2^k$, by using the formula $||TT^*|| = ||T||^2$, and by taking n-th roots we get:

$$\rho(T) \geq ||T||$$

Thus, we are done with the self-adjoint case, with the result $\rho(T) = ||T||$.

Step 2. In the general normal case $TT^* = T^*T$ we have $T^n(T^n)^* = (TT^*)^n$, and by using this, along with the result from Step 1, applied to TT^*, we obtain:

$$\begin{aligned} \rho(T) &= \lim_{n\to\infty} ||T^n||^{1/n} \\ &= \sqrt{\lim_{n\to\infty} ||T^n(T^n)^*||^{1/n}} \\ &= \sqrt{\lim_{n\to\infty} ||(TT^*)^n||^{1/n}} \\ &= \sqrt{\rho(TT^*)} \\ &= \sqrt{||T||^2} \\ &= ||T|| \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

As a first comment, the spectral radius formula $\rho(T) = ||T||$ does not hold in general, the simplest counterexample being the following non-normal matrix:

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

As another comment, we can combine the formula $\rho(T) = ||T||$ for normal operators with the formula $||TT^*|| = ||T||^2$, and we are led to the following statement:

THEOREM 3.22. *The norm of $B(H)$ is given by*

$$||T|| = \sqrt{\sup\left\{\lambda \in \mathbb{C}\Big|TT^* - \lambda \notin B(H)^{-1}\right\}}$$

and so is a purely algebraic quantity.

PROOF. We have the following computation, using the formula $||TT^*|| = ||T||^2$, then the spectral radius formula for TT^*, and finally the definition of the spectral radius:

$$\begin{aligned} ||T|| &= \sqrt{||TT^*||} \\ &= \sqrt{\rho(TT^*)} \\ &= \sqrt{\sup\left\{\lambda \in \mathbb{C}\Big|\lambda \in \sigma(TT^*)\right\}} \\ &= \sqrt{\sup\left\{\lambda \in \mathbb{C}\Big|TT^* - \lambda \notin B(H)^{-1}\right\}} \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

The above result is quite interesting, philosophically speaking. We will be back to this later, with some further results on $B(H)$, and other algebras of the same type.

3d. Normal operators

By using Theorem 3.21 we can say a number of non-trivial things about the normal operators, which are commonly known as "spectral theorem for normal operators". As a first result here, we can improve the polynomial functional calculus formula:

THEOREM 3.23. *Given $T \in B(H)$ normal, we have a morphism of algebras*

$$\mathbb{C}[X] \to B(H) \quad , \quad P \to P(T)$$

having the properties $||P(T)|| = ||P_{|\sigma(T)}||$, and $\sigma(P(T)) = P(\sigma(T))$.

PROOF. This is an improvement of Theorem 3.7 in the normal case, with the extra assertion being the norm estimate. But the element $P(T)$ being normal, we can apply to it the spectral radius formula for normal elements, and we obtain:

$$\begin{aligned}||P(T)|| &= \rho(P(T)) \\ &= \sup_{\lambda\in\sigma(P(T))}|\lambda| \\ &= \sup_{\lambda\in P(\sigma(T))}|\lambda| \\ &= ||P_{|\sigma(T)}||\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

We can improve as well the rational calculus formula, as follows:

THEOREM 3.24. *Given $T\in B(H)$ normal, we have a morphism of algebras*

$$\mathbb{C}(X)^T\to B(H)\quad,\quad f\to f(T)$$

having the properties $||f(T)||=||f_{|\sigma(T)}||$, and $\sigma(f(T))=f(\sigma(T))$.

PROOF. This is an improvement of Theorem 3.10 in the normal case, with all the details of the proof being identical to those of the proof of Theorem 3.23. □

It is possible to improve as well the holomorphic calculus formula, as follows:

THEOREM 3.25. *Given $T\in B(H)$ normal, we have a morphism of algebras*

$$Hol(\sigma(T))\to B(H)\quad,\quad f\to f(T)$$

having the properties $||f(T)||=||f_{|\sigma(T)}||$, and $\sigma(f(T))=f(\sigma(T))$.

PROOF. This is an improvement of Theorem 3.18 in the normal case, with all the details of the proof being again identical to those of the proof of Theorem 3.23. □

Summarizing, by using the spectral radius formula, we have now improvements of all our previous functional calculus theorems, in the case of the normal operators.

Importantly, in the case of the normal operators we have as well some new functional calculus results, using more general functions than those used before. There is a long story here, which is quite technical, and we will start our study here in this chapter, and be back to this in chapter 5, with shaper results. So, here is our first result:

THEOREM 3.26. *Given $T\in B(H)$ normal, we have a morphism of algebras*

$$C(\sigma(T))\to B(H)\quad,\quad f\to f(T)$$

which is isometric, $||f(T)||=||f||$, and has the property $\sigma(f(T))=f(\sigma(T))$.

PROOF. The idea here is to "complete" the morphism in Theorem 3.23, namely:

$$\mathbb{C}[X] \to B(H) \quad , \quad P \to P(T)$$

Indeed, we know from Theorem 3.23 that this morphism is continuous, and is in fact isometric, when regarding the polynomials $P \in \mathbb{C}[X]$ as functions on $\sigma(T)$:

$$||P(T)|| = ||P_{|\sigma(T)}||$$

We conclude from this that we have a unique isometric extension, as follows:

$$C(\sigma(T)) \to B(H) \quad , \quad f \to f(T)$$

It remains to prove $\sigma(f(T)) = f(\sigma(T))$, and we can do this by double inclusion:

"$\subset$" Given a continuous function $f \in C(\sigma(T))$, we must prove that we have:

$$\lambda \notin f(\sigma(T)) \implies \lambda \notin \sigma(f(T))$$

For this purpose, consider the following function, which is well-defined:

$$\frac{1}{f - \lambda} \in C(\sigma(T))$$

We can therefore apply this function to T, and we obtain:

$$\left(\frac{1}{f - \lambda}\right) T = \frac{1}{f(T) - \lambda}$$

In particular $f(T) - \lambda$ is invertible, so $\lambda \notin \sigma(f(T))$, as desired.

"$\supset$" Given a continuous function $f \in C(\sigma(T))$, we must prove that we have:

$$\lambda \in f(\sigma(T)) \implies \lambda \in \sigma(f(T))$$

But this is the same as proving that we have:

$$\mu \in \sigma(T) \implies f(\mu) \in \sigma(f(T))$$

For this purpose, we approximate our function by polynomials, $P_n \to f$, and we examine the following convergence, which follows from $P_n \to f$:

$$P_n(T) - P_n(\mu) \to f(T) - f(\mu)$$

We know from polynomial functional calculus that we have:

$$P_n(\mu) \in P_n(\sigma(T)) = \sigma(P_n(T))$$

Thus, the operators $P_n(T) - P_n(\mu)$ are not invertible. On the other hand, we know that the set formed by the invertible operators is open, so its complement is closed. Thus the limit $f(T) - f(\mu)$ is not invertible either, and so $f(\mu) \in \sigma(f(T))$, as desired. $\square$

As an important comment, Theorem 3.26 is not exactly in final form, because it misses an important point, namely that our correspondence maps:

$$\bar{z} \to T^*$$

However, this is something non-trivial, and we will be back to this later. Observe however that Theorem 3.26 is fully powerful for the self-adjoint operators, $T = T^*$.

With this discussed, we can now review the theory of positive operators, as follows:

THEOREM 3.27. *For an operator $T \in B(H)$, the following are equivalent:*
(1) $< Tx, x > \geq 0$, *for any $x \in H$.*
(2) T *is normal, and* $\sigma(T) \subset [0, \infty)$.
(3) $T = S^2$, *for some $S \in B(H)$ satisfying $S = S^*$.*
(4) $T = R^* R$, *for some $R \in B(H)$.*

If these conditions are satisfied, we call T positive, and write $T \geq 0$.

PROOF. We have already seen some implications in chapter 2, but the best is to forget the few partial results that we know, and prove everything, as follows:

(1) $\implies$ (2) Assuming $< Tx, x > \geq 0$, with $S = T - T^*$ we have:

$$\begin{aligned} < Sx, x > &= < Tx, x > - < T^* x, x > \\ &= < Tx, x > - < x, Tx > \\ &= < Tx, x > - \overline{< Tx, x >} \\ &= 0 \end{aligned}$$

The next step is to use a polarization trick, as follows:

$$\begin{aligned} < Sx, y > &= < S(x+y), x+y > - < Sx, x > - < Sy, y > - < Sy, x > \\ &= - < Sy, x > \\ &= < y, Sx > \\ &= \overline{< Sx, y >} \end{aligned}$$

Thus we must have $< Sx, y > \in \mathbb{R}$, and with $y \to iy$ we obtain $< Sx, y > \in i\mathbb{R}$ too, and so $< Sx, y > = 0$. Thus $S = 0$, which gives $T = T^*$. Now since T is self-adjoint, it is normal as claimed. Moreover, by self-adjointness, we have:

$$\sigma(T) \subset \mathbb{R}$$

In order to prove now that we have indeed $\sigma(T) \subset [0, \infty)$, as claimed, we must invert $T + \lambda$, for any $\lambda > 0$. For this purpose, observe that we have:

$$\begin{aligned} < (T+\lambda)x, x > &= < Tx, x > + < \lambda x, x > \\ &\geq < \lambda x, x > \\ &= \lambda ||x||^2 \end{aligned}$$

But this shows that $T+\lambda$ is injective. In order to prove now the surjectivity, and the boundedness of the inverse, observe first that we have:

$$\begin{aligned} Im(T+\lambda)^\perp &= \ker(T+\lambda)^* \\ &= \ker(T+\lambda) \\ &= \{0\} \end{aligned}$$

Thus $Im(T+\lambda)$ is dense. On the other hand, observe that we have:

$$\begin{aligned} ||(T+\lambda)x||^2 &= < Tx+\lambda x, Tx+\lambda x > \\ &= ||Tx||^2 + 2\lambda < Tx, x > +\lambda^2||x||^2 \\ &\geq \lambda^2||x||^2 \end{aligned}$$

Thus for any vector in the image $y \in Im(T+\lambda)$ we have:

$$||y|| \geq \lambda \big|\big|(T+\lambda)^{-1}y\big|\big|$$

As a conclusion to what we have so far, $T+\lambda$ is bijective and invertible as a bounded operator from H onto its image, with the following norm bound:

$$||(T+\lambda)^{-1}|| \leq \lambda^{-1}$$

But this shows that $Im(T+\lambda)$ is complete, hence closed, and since we already knew that $Im(T+\lambda)$ is dense, our operator $T+\lambda$ is surjective, and we are done.

(2) $\implies$ (3) Since T is normal, and with spectrum contained in $[0,\infty)$, we can use the continuous functional calculus formula for the normal operators from Theorem 3.26, with the function $f(x)=\sqrt{x}$, as to construct a square root $S=\sqrt{T}$.

(3) $\implies$ (4) This is trivial, because we can set $R=S$.

(4) $\implies$ (1) This is clear, because we have the following computation:

$$< R^*Rx, x >=< Rx, Rx >= ||Rx||^2$$

Thus, we have the equivalences in the statement. □

In analogy with what happens in finite dimensions, where among the positive matrices $A \geq 0$ we have the strictly positive ones, $A > 0$, given by the fact that the eigenvalues are strictly positive, we have as well a "strict" version of the above result, as follows:

THEOREM 3.28. *For an operator $T \in B(H)$, the following are equivalent:*

(1) *T is positive and invertible.*
(2) *T is normal, and $\sigma(T) \subset (0,\infty)$.*
(3) *$T = S^2$, for some $S \in B(H)$ invertible, satisfying $S = S^*$.*
(4) *$T = R^*R$, for some $R \in B(H)$ invertible.*

If these conditions are satisfied, we call T strictly positive, and write $T > 0$.

PROOF. Our claim is that the above conditions (1-4) are precisely the conditions (1-4) in Theorem 3.27, with the assumption "T is invertible" added. Indeed:

(1) This is clear by definition.

(2) In the context of Theorem 3.27 (2), namely when T is normal, and $\sigma(T) \subset [0, \infty)$, the invertibility of T, which means $0 \notin \sigma(T)$, gives $\sigma(T) \subset (0, \infty)$, as desired.

(3) In the context of Theorem 3.27 (3), namely when $T = S^2$, with $S = S^*$, by using the basic properties of the functional calculus for normal operators, the invertibility of T is equivalent to the invertibility of its square root $S = \sqrt{T}$, as desired.

(4) In the context of Theorem 3.27 (4), namely when $T = RR^*$, the invertibility of T is equivalent to the invertibility of R. This can be either checked directly, or deduced via the equivalence (3) $\iff$ (4) from Theorem 3.27, by using the above argument (3). □

As a subtlety now, we have the following complement to the above result:

PROPOSITION 3.29. *For a strictly positive operator, $T > 0$, we have*

$$< Tx, x >> 0 \quad , \quad \forall x \neq 0$$

but the converse of this fact is not true, unless we are in finite dimensions.

PROOF. We have several things to be proved, the idea being as follows:

(1) Regarding the main assertion, the inequality can be deduced as follows, by using the fact that the operator $S = \sqrt{T}$ is invertible, and in particular injective:

$$\begin{aligned} < Tx, x > &= < S^2 x, x > \\ &= < Sx, S^* x > \\ &= < Sx, Sx > \\ &= ||Sx||^2 \\ &> 0 \end{aligned}$$

(2) In finite dimensions, assuming $< Tx, x >> 0$ for any $x \neq 0$, we know from Theorem 3.27 that we have $T \geq 0$. Thus we have $\sigma(T) \subset [0, \infty)$, and assuming by contradiction $0 \in \sigma(T)$, we obtain that T has $\lambda = 0$ as eigenvalue, and the corresponding eigenvector $x \neq 0$ has the property $< Tx, x >= 0$, contradiction. Thus $T > 0$, as claimed.

(3) Regarding now the counterexample, consider the following operator on $l^2(\mathbb{N})$:

$$T = \begin{pmatrix} 1 &&& \\ & \frac{1}{2} && \\ && \frac{1}{3} & \\ &&& \ddots \end{pmatrix}$$

This operator T is well-defined and bounded, and we have $< Tx, x >> 0$ for any $x \neq 0$. However T is not invertible, and so the converse does not hold, as stated. □

Good news, we can now discuss the polar decomposition. Let us start with:

THEOREM 3.30. *Given an operator $T \in B(H)$, we can construct a positive operator $|T| \in B(H)$ as follows, by using the fact that T^*T is positive:*

$$|T| = \sqrt{T^*T}$$

The square of this operator is then $|T|^2 = T^*T$. In the case $H = \mathbb{C}$, we obtain in this way the usual absolute value of the complex numbers:

$$|z| = \sqrt{z\bar{z}}$$

More generally, in the case where $H = \mathbb{C}^N$ is finite dimensional, we obtain in this way the usual moduli of the complex matrices $A \in M_N(\mathbb{C})$.

PROOF. We have several things to be proved, the idea being as follows:

(1) The first assertion follows from Theorem 3.27. Indeed, according to (4) there the operator T^*T is indeed positive, and then according to (2) there we can extract the square root of this latter positive operator, by applying to it the function $\sqrt{z}$.

(2) By functional calculus we have then $|T|^2 = T^*T$, as desired.

(3) In the case $H = \mathbb{C}$, we obtain indeed the absolute value of complex numbers.

(4) In the case where the space H is finite dimensional, $H = \mathbb{C}^N$, we obtain indeed the usual moduli of the complex matrices $A \in M_N(\mathbb{C})$. □

As a comment here, it is possible to talk as well about $\sqrt{TT^*}$, which is in general different from $\sqrt{T^*T}$. Note that when T is normal, no issue, because we have:

$$TT^* = T^*T \implies \sqrt{TT^*} = \sqrt{T^*T}$$

Regarding now the polar decomposition formula, let us start with a weak version of this statement, regarding the invertible operators, as follows:

THEOREM 3.31. *We have the polar decomposition formula*

$$T = U\sqrt{T^*T}$$

with U being a unitary, for any $T \in B(H)$ invertible.

PROOF. According to our definition of the modulus, $|T| = \sqrt{T^*T}$, we have:

$$\begin{aligned} < |T|x, |T|y > &= < x, |T|^2 y > \\ &= < x, T^*Ty > \\ &= < Tx, Ty > \end{aligned}$$

Thus we can define a unitary operator $U \in B(H)$ by the following formula:

$$U(|T|x) = Tx$$

But this formula shows that we have $T = U|T|$, as desired. □

Observe that we have uniqueness in the above result, in what regards the choice of the unitary $U\in B(H)$, due to the fact that we can write this unitary as follows:

$$U=T(\sqrt{T^*T})^{-1}$$

More generally now, we have the following result:

THEOREM 3.32. *We have the polar decomposition formula*

$$T=U\sqrt{T^*T}$$

with U being a partial isometry, for any $T\in B(H)$.

PROOF. As before, we have the following equality, for any two vectors $x,y\in H$:

$$<|T|x,|T|y>=<Tx,Ty>$$

We conclude that the following linear application is well-defined, and isometric:

$$U:Im|T|\to Im(T)\quad,\quad |T|x\to Tx$$

Now by continuity we can extend this isometry U into an isometry between certain Hilbert subspaces of H, as follows:

$$U:\overline{Im|T|}\to\overline{Im(T)}\quad,\quad |T|x\to Tx$$

Moreover, we can further extend U into a partial isometry $U:H\to H$, by setting $Ux=0$, for any $x\in\overline{Im|T|}^\perp$, and with this convention, the result follows. □

3e. Exercises

Exercises:

EXERCISE 3.33.

EXERCISE 3.34.

EXERCISE 3.35.

EXERCISE 3.36.

EXERCISE 3.37.

EXERCISE 3.38.

EXERCISE 3.39.

EXERCISE 3.40.

Bonus exercise.

CHAPTER 4

Basic examples

4a.

4b.

4c.

4d.

4e. Exercises

Exercises:

Exercise 4.1.

Exercise 4.2.

Exercise 4.3.

Exercise 4.4.

Exercise 4.5.

Exercise 4.6.

Exercise 4.7.

Exercise 4.8.

Bonus exercise.

Part II

Spectral theorems

All right, we're jamming
I wanna jam it with you
We're jamming, jamming
And I hope you like jamming, too

CHAPTER 5

Spectral theorems

5a. Measurable calculus

Welcome to advanced operator theory. Our purpose in this chapter, and in this whole Part II of the present book, will be to develop spectral theorems, mostly for the normal operators, and their applications, going beyond what we know from chapter 3.

As a starting point for our study, we have the material from chapter 3, with the main result there, along with a bit more, being summarized as follows:

THEOREM 5.1. *Given $T \in B(H)$ normal, we have a unique morphism of algebras*

$$C(\sigma(T)) \to B(H) \quad , \quad f \to f(T)$$

given by $X \to T$, which has the following properties:

(1) $\sigma(f(T)) = f(\sigma(T))$.
(2) $||f(T)|| = ||f||$.
(3) $Tx = \lambda x \implies f(T)x = f(\lambda)x$.
(4) $[S, T] = 0 \implies [S, f(T)] = 0$.

PROOF. This is a slight improvement of what we know from chapter 3, with all the extra assertions, which are good to know, in practice, being clear from definitions. □

As a first new result now, along the same lines, but better, we can further extend Theorem 5.1 into a measurable functional calculus theorem, as follows:

THEOREM 5.2. *Given $T \in B(H)$ normal, we have a unique continuous morphism of algebras as follows, with L^∞ standing for abstract measurable functions*

$$L^\infty(\sigma(T)) \to B(H) \quad , \quad f \to f(T)$$

given by $z \to T$, which has the following properties:

(1) $||f(T)|| = ||f||$.
(2) $\sigma(f(T)) = f(\sigma(T))$.
(3) $Tx = \lambda x \implies f(T)x = f(\lambda)x$.
(4) $[S, T] = 0 \implies [S, f(T)] = 0$.

PROOF. As before, the idea will be that of "completing" what we have. To be more precise, we can use the Riesz theorem and a polarization trick, as follows:

(1) Given a vector $x \in H$, consider the following functional:

$$C(\sigma(T)) \to \mathbb{C} \quad , \quad g \to < g(T)x, x >$$

By the Riesz theorem, this functional must be the integration with respect to a certain measure μ on the space $\sigma(T)$. Thus, we have a formula as follows:

$$< g(T)x, x > = \int_{\sigma(T)} g(z) d\mu(z)$$

Now given an arbitrary Borel function $f \in L^\infty(\sigma(T))$, as in the statement, we can define a number $< f(T)x, x > \in \mathbb{C}$, by using exactly the same formula, namely:

$$< f(T)x, x > = \int_{\sigma(T)} f(z) d\mu(z)$$

Thus, we have managed to define numbers $< f(T)x, x > \in \mathbb{C}$, for all vectors $x \in H$, and in addition we can recover these numbers as follows, with $g_n \in C(\sigma(T))$:

$$< f(T)x, x > = \lim_{g_n \to f} < g_n(T)x, x >$$

(2) In order to define now numbers $< f(T)x, y > \in \mathbb{C}$, for all vectors $x, y \in H$, we can use a polarization trick. Indeed, for any operator $S \in B(H)$ we have:

$$\begin{aligned} < S(x+y), x+y > &= < Sx, x > + < Sy, y > \\ &\quad + < Sx, y > + < Sy, x > \end{aligned}$$

By replacing $y \to iy$, we have as well the following formula:

$$\begin{aligned} < S(x+iy), x+iy > &= < Sx, x > + < Sy, y > \\ &\quad -i < Sx, y > +i < Sy, x > \end{aligned}$$

By multiplying this latter formula by i, we obtain the following formula:

$$\begin{aligned} i < S(x+iy), x+iy > &= i < Sx, x > +i < Sy, y > \\ &\quad + < Sx, y > - < Sy, x > \end{aligned}$$

Now by summing this latter formula with the first one, we obtain:

$$\begin{aligned} < S(x+y), x+y > +i < S(x+iy), x+iy > &= (1+i)[< Sx, x > + < Sy, y >] \\ &\quad +2 < Sx, y > \end{aligned}$$

(3) But with this, we can now finish. Indeed, by combining (1,2), given a Borel function $f \in L^\infty(\sigma(T))$, we can define numbers $< f(T)x, y > \in \mathbb{C}$ for any $x, y \in H$, and it is routine to check, by using approximation by continuous functions $g_n \to f$ as in (1), that we obtain in this way an operator $f(T) \in B(H)$, having all the desired properties. $\square$

Very nice all this, we are learning new things, but for the rest, exactly as for Theorem 5.1, the same comments as in chapter 3 apply. Indeed, Theorem 5.2 is not exactly in final form, because it misses an important point, namely that our correspondence maps:

$$\bar{z} \to T^*$$

However, this is something non-trivial, and we will be back to this later. Observe however that Theorem 5.2 is fully powerful for the self-adjoint operators, $T = T^*$, where the spectrum is real, and so where $z = \bar{z}$ on the spectrum. We will be back to this.

As another comment, the above result and its proof provide us with more than a Borel functional calculus, because what we got is a certain measure on the spectrum $\sigma(T)$, along with a functional calculus for the L^∞ functions with respect to this measure.

Again, this is something quite subtle, and we will be back to it later. For the moment, in view of some applications, to be developed next, we will only need Theorem 5.2 as formulated, with $L^\infty(\sigma(T))$ standing, a bit abusively, for the Borel functions on $\sigma(T)$.

5b. Basic applications

With this done, let us discuss now some useful decomposition results for the bounded operators $T \in B(H)$, that we can now establish, by using the above measurable calculus technology. We know that any $z \in \mathbb{C}$ can be written as follows, with $a, b \in \mathbb{R}$:

$$z = a + ib$$

Also, we know that both the real and imaginary parts $a, b \in \mathbb{R}$, and more generally any real number $c \in \mathbb{R}$, can be written as follows, with $r, s \geq 0$:

$$c = r - s$$

In order to discuss now the operator theoretic generalizations of these results, which by the way covers the usual matrix case too, let us start with the following basic fact:

THEOREM 5.3. *Any operator $T \in B(H)$ can be written as*

$$T = Re(T) + iIm(T)$$

with $Re(T), Im(T) \in B(H)$ being self-adjoint, and this decomposition is unique.

PROOF. This is something elementary, the idea being as follows:

(1) As a first observation, in the case $H = \mathbb{C}$ our operators are usual complex numbers, and the formula in the statement corresponds to the following basic fact:

$$z = Re(z) + iIm(z)$$

(2) In general now, we can use the same formulae for the real and imaginary part as in the complex number case, the decomposition formula being as follows:

$$T=\frac{T+T^*}{2}+i\cdot\frac{T-T^*}{2i}$$

To be more precise, both the operators on the right are self-adjoint, and the summing formula holds indeed, and so we have our decomposition result, as desired.

(3) Regarding now the uniqueness, by linearity it is enough to show that $R+iS=0$ with R,S both self-adjoint implies $R=S=0$. But this follows by applying the adjoint to $R+iS=0$, which gives $R-iS=0$, and so $R=S=0$, as desired. □

More generally now, as a continuation of this, and as an answer to some of the questions raised in the beginning of this section, we have the following result:

THEOREM 5.4. *Given an operator $T\in B(H)$, the following happen:*
1. *We can write $T=A+iB$, with $A,B\in B(H)$ being self-adjoint.*
2. *When $T=T^*$, we can write $T=R-S$, with $R,S\in B(H)$ being positive.*
3. *Thus, we can write any T as a linear combination of 4 positive elements.*

PROOF. All this follows from basic spectral theory, as follows:

(1) This is something that we already know, from Theorem 5.3, with the decomposition formula there being something straightforward, as follows:

$$T=\frac{T+T^*}{2}+i\cdot\frac{T-T^*}{2i}$$

(2) This follows from the measurable functional calculus. Indeed, assuming $T=T^*$ we have $\sigma(T)\subset\mathbb{R}$, so we can use the following decomposition formula on $\mathbb{R}$:

$$1=\chi_{[0,\infty)}+\chi_{(-\infty,0)}$$

To be more precise, let us multiply by z, and rewrite this formula as follows:

$$z=\chi_{[0,\infty)}z-\chi_{(-\infty,0)}(-z)$$

Now by applying these measurable functions to T, we obtain as formula as follows, with both the operators $T_+,T_-\in B(H)$ being positive, as desired:

$$T=T_+-T_-$$

(3) This follows indeed by combining the results in (1) and (2) above. □

Going ahead with our decomposition results, another basic thing that we know about complex numbers is that any $z\in\mathbb{C}$ appears as a real multiple of a unitary:

$$z=re^{it}$$

Finding the correct operator theoretic analogue of this is quite tricky, and this even for the usual matrices $A\in M_N(\mathbb{C})$. As a basic result here, we have:

THEOREM 5.5. *Given an operator $T\in B(H)$, the following happen:*

(1) *When $T=T^*$ and $||T||\leq 1$, we can write T as an average of 2 unitaries:*

$$T=\frac{U+V}{2}$$

(2) *In the general $T=T^*$ case, we can write T as a rescaled sum of unitaries:*

$$T=\lambda(U+V)$$

(3) *Thus, in general, we can write T as a rescaled sum of 4 unitaries.*

PROOF. This follows from the results that we have, as follows:

(1) Assuming $T=T^*$ and $||T||\leq 1$ we have $1-T^2\geq 0$, and the decomposition that we are looking for is as follows, with both the components being unitaries:

$$T=\frac{T+i\sqrt{1-T^2}}{2}+\frac{T-i\sqrt{1-T^2}}{2}$$

To be more precise, the square root can be extracted as explained in chapter 3, and the check of the unitarity of the components goes as follows:

$$\begin{aligned}(T+i\sqrt{1-T^2})(T-i\sqrt{1-T^2}) &= T^2+(1-T^2)\\ &= 1\end{aligned}$$

(2) This simply follows by applying (1) to the operator $T/||T||$.

(3) Assuming first that we have $||T||\leq 1$, we know from Theorem 5.4 (1) that we can write $T=A+iB$, with A,B being self-adjoint, and satisfying $||A||,||B||\leq 1$. Now by applying (1) to both A and B, we obtain a decomposition of T as follows:

$$T=\frac{U+V+W+X}{2}$$

In general, we can apply this to the operator $T/||T||$, and we obtain the result. □

So long for decomposition results for the linear operators. Needless to say, what we learned in the above, coming as a complement to the polar decomposition result from chapter 3, was just the tip of the iceberg, and this because the comparison of what we have with the usual decomposition theory of the complex numbers shows that we are still quite far away from that, with many natural questions remaining still open.

In answer, such questions are in fact non-trivial even for the usual matrices, and we will be back to more such decomposition results later on, in Part III of the present book, when discussing the compact operators, which are quite close to the usual matrices.

5c. Diagonalization

Good news, we are now in position to diagonalize the normal operators. Indeed, in order to diagonalize the normal operators, we can do this in 3 steps, first for the self-adjoint operators, then for the families of commuting self-adjoint operators, and finally for the general normal operators, by using a trick of the following type:

$$T = Re(T) + iIm(T)$$

So, we have a good plan here, just waiting to be developed. However, technically speaking now, and coming somewhat as bad news, the diagonalization in infinite dimensions is more tricky than in finite dimensions, and instead of writing a formula of type $T = UDU^*$, with $U, D \in B(H)$ being respectively unitary and diagonal, we will express our operator as $T = U^*MU$, with $U : H \to K$ being a certain unitary, and with $M \in B(K)$ being a certain diagonal operator. Which will be something a bit abstract.

However, there is no escape from this, because this is indeed how the spectral theorem is best formulated, in view of applications. That is, in practice, the explicit construction of U, M, which will be actually rather part of the proof, is also needed.

But probably too much talking, let us get to work. For the self-adjoint operators, the statement and proof of the spectral theorem, in its most general form, are as follows:

Theorem 5.6. *Any self-adjoint operator $T \in B(H)$ can be diagonalized,*

$$T = U^*M_fU$$

with $U : H \to L^2(X)$ being a unitary operator from H to a certain L^2 space associated to T, with $f : X \to \mathbb{R}$ being a certain function, once again associated to T, and with

$$M_f(g) = fg$$

being the usual multiplication operator by f, on the Hilbert space $L^2(X)$.

Proof. The construction of U, f can be done in several steps, as follows:

(1) We first prove the result in the special case where our operator T has a cyclic vector $x \in H$, with this meaning that the following holds:

$$\overline{span\left(T^kx\Big|n \in \mathbb{N}\right)} = H$$

For this purpose, let us go back to the proof of Theorem 5.2. We will use the following formula from there, with μ being the measure on $X = \sigma(T)$ associated to x:

$$< g(T)x, x >= \int_{\sigma(T)} g(z)d\mu(z)$$

Our claim is that we can define a unitary $U : H \to L^2(X)$, first on the dense part spanned by the vectors $T^k x$, by the following formula, and then by continuity:

$$U[g(T)x] = g$$

Indeed, the following computation shows that U is well-defined, and isometric:

$$\begin{aligned} ||g(T)x||^2 &= < g(T)x, g(T)x > \\ &= < g(T)^* g(T)x, x > \\ &= < |g|^2(T)x, x > \\ &= \int_{\sigma(T)} |g(z)|^2 d\mu(z) \\ &= ||g||_2^2 \end{aligned}$$

We can then extend U by continuity into a unitary $U : H \to L^2(X)$, as claimed. Now observe that we have the following formula:

$$\begin{aligned} UTU^* g &= U[Tg(T)x] \\ &= U[(zg)(T)x] \\ &= zg \end{aligned}$$

Thus our result is proved in the present case, with U as above, and with $f(z) = z$.

(2) We discuss now the general case. Our first claim is that H has a decomposition as follows, with each H_i being invariant under T, and admitting a cyclic vector x_i:

$$H = \bigoplus_i H_i$$

Indeed, this is something elementary, the construction being by recurrence in finite dimensions, in the obvious way, and by using the Zorn lemma in general. Now with this decomposition in hand, we can make a direct sum of the diagonalizations obtained in (1), for each of the restrictions $T_{|H_i}$, and we obtain the formula in the statement. □

Next, we have the following technical generalization of the above result:

THEOREM 5.7. *Any family of commuting self-adjoint operators $T_i \in B(H)$ can be jointly diagonalized,*

$$T_i = U^* M_{f_i} U$$

with $U : H \to L^2(X)$ being a unitary operator from H to a certain L^2 space associated to $\{T_i\}$, with $f_i : X \to \mathbb{R}$ being certain functions, once again associated to T_i, and with

$$M_{f_i}(g) = f_i g$$

being the usual multiplication operator by f_i, on the Hilbert space $L^2(X)$.

PROOF. This is similar to the proof of Theorem 5.6, by suitably modifying the measurable calculus formula, and the measure μ itself, as to have this formula working for all the operators T_i. With this modification done, everything extends. □

Good news, after all these preliminaries, that you enjoyed I hope, as much as I did, we can eventually discuss the case of arbitrary normal operators. We have here the following result, generalizing what we know from chapter 1 about the normal matrices:

THEOREM 5.8. *Any normal operator $T \in B(H)$ can be diagonalized,*

$$T = U^* M_f U$$

with $U : H \to L^2(X)$ being a unitary operator from H to a certain L^2 space associated to T, with $f : X \to \mathbb{C}$ being a certain function, once again associated to T, and with

$$M_f(g) = fg$$

being the usual multiplication operator by f, on the Hilbert space $L^2(X)$.

PROOF. This is our main diagonalization theorem, the idea being as follows:

(1) Consider the decomposition of T into its real and imaginary parts, as constructed in the proof of Theorem 5.3, namely:

$$T = \frac{T + T^*}{2} + i \cdot \frac{T - T^*}{2i}$$

We know that the real and imaginary parts are self-adjoint operators. Now since T was assumed to be normal, $TT^* = T^*T$, these real and imaginary parts commute:

$$\left[\frac{T + T^*}{2} , \frac{T - T^*}{2i} \right] = 0$$

Thus Theorem 5.7 applies to these real and imaginary parts, and gives the result.

(2) Alternatively, we can use methods similar to those that we used in chapter 1, in order to deal with the usual normal matrices, involving the special relation between T and the operator TT^*, which is self-adjoint. We will leave this as an instructive exercise. □

This was for our series of diagonalization theorems. There is of course one more result here, regarding the families of commuting normal operators, as follows:

THEOREM 5.9. *Any family of commuting normal operators $T_i \in B(H)$ can be jointly diagonalized,*

$$T_i = U^* M_{f_i} U$$

with $U : H \to L^2(X)$ being a unitary operator from H to a certain L^2 space associated to $\{T_i\}$, with $f_i : X \to \mathbb{C}$ being certain functions, once again associated to T_i, and with

$$M_{f_i}(g) = f_i g$$

being the usual multiplication operator by f_i, on the Hilbert space $L^2(X)$.

PROOF. This is similar to the proof of Theorem 5.6 and Theorem 5.8, by combining the arguments there. To be more precise, this follows as Theorem 5.6, by using the decomposition trick from the proof of Theorem 5.8. □

5d. Further results

With the above diagonalization results in hand, we can now "fix" the continuous and measurable functional calculus theorems, with a key complement, as follows:

THEOREM 5.10. *Given a normal operator $T \in B(H)$, the following hold, for both the functional calculus and the measurable calculus morphisms:*

(1) *These morphisms are $*$-morphisms.*
(2) *The function $\bar{z}$ gets mapped to T^*.*
(3) *The functions $Re(z), Im(z)$ get mapped to $Re(T), Im(T)$.*
(4) *The function $|z|^2$ gets mapped to $TT^* = T^*T$.*
(5) *If f is real, then $f(T)$ is self-adjoint.*

PROOF. These assertions are more or less equivalent, with (1) being the main one, which obviously implies everything else. But this assertion (1) follows from the diagonalization result for normal operators, from Theorem 5.6. □

This was for the spectral theory of the arbitrary and normal operators, or at least for the basics of this theory. As a conclusion here, our main results are as follows:

(1) Regarding the arbitrary operators, the main results here, or rather the most advanced results that we have, are the holomorphic calculus formula from chapter 3, and the spectral radius estimate, from chapter 3 too.

(2) For the self-adjoint operators, the main results that we have are the spectral radius formula from chapter 3, the measurable calculus formula from Theorem 5.2, and the diagonalization result from Theorem 5.6.

(3) For general normal operators, the main results are the spectral radius formula from chapter 3, the measurable calculus formula from Theorem 5.2, complemented by Theorem 5.10, and the diagonalization result in Theorem 5.8.

(4) Finally, we have as well some joint diagonalization results, for the commuting families of self-adjoint or normal operators, namely Theorem 5.7 and Theorem 5.9. These results are something very useful too, for various applications.

There are of course many other things that can be said about the spectral theory of the bounded operators $T \in B(H)$, and on that of the unbounded operators too. We will be back to these questions, on numerous occasions, in what follows.

5e. Exercises

Exercises:

Exercise 5.11.

Exercise 5.12.

Exercise 5.13.

Exercise 5.14.

Exercise 5.15.

Exercise 5.16.

Exercise 5.17.

Exercise 5.18.

Bonus exercise.

CHAPTER 6

Random matrices

6a. Random matrices

Beyond the usual matrices, the simplest examples of operators are the random matrices. In order to talk about such random matrices, and their laws, we will need:

THEOREM 6.1. *Given an operator algebra $A \subset B(H)$ with a faithful trace $tr : A \to \mathbb{C}$, any normal element $T \in A$ has a law, namely a probability measure μ satisfying*

$$tr(T^k) = \int_{\mathbb{C}} z^k d\mu(z)$$

with the powers being with respect to colored exponents $k = \circ \bullet \bullet \circ \ldots$, defined via

$$a^{\emptyset} = 1 \quad , \quad a^{\circ} = a \quad , \quad a^{\bullet} = a^*$$

and multiplicativity. This law is unique, and is supported by the spectrum $\sigma(T) \subset \mathbb{C}$. In the non-normal case, $TT^* \neq T^*T$, such a law does not exist.

PROOF. We have two assertions here, the idea being as follows:

(1) In the normal operator case, where $TT^* = T^*T$, we know from the continuous functional calculus theorem that we have a formula as follows:

$$< T >= C(\sigma(T))$$

Thus the functional $f(T) \to tr(f(T))$ can be regarded as an integration functional on the algebra $C(\sigma(T))$, and by the Riesz theorem, this latter functional must come from a probability measure μ on the spectrum $\sigma(T)$, in the sense that we must have:

$$tr(f(T)) = \int_{\sigma(T)} f(z) d\mu(z)$$

We are therefore led to the conclusions in the statement, with the uniqueness assertion coming from the fact that the operators T^k, taken as usual with respect to colored integer exponents, $k = \circ \bullet \bullet \circ \ldots$, generate the whole operator algebra $C(\sigma(T))$.

(2) In the non-normal case now, $TT^* \neq T^*T$, we must show that such a law does not exist. For this purpose, we can use a positivity trick, as follows:

$$\begin{aligned}
TT^*-T^*T\neq 0 &\implies (TT^*-T^*T)^2>0\\
&\implies TT^*TT^*-TT^*T^*T-T^*TTT^*+T^*TT^*T>0\\
&\implies tr(TT^*TT^*-TT^*T^*T-T^*TTT^*+T^*TT^*T)>0\\
&\implies tr(TT^*TT^*+T^*TT^*T)>tr(TT^*T^*T+T^*TTT^*)\\
&\implies tr(TT^*TT^*)>tr(TTT^*T^*)
\end{aligned}$$

Now assuming that T has a law $\mu \in \mathcal{P}(\mathbb{C})$, in the sense that the moment formula in the statement holds, the above two different numbers would have to both appear by integrating $|z|^2$ with respect to this law μ, which is contradictory, as desired. □

Back now to the random matrices, as a basic example, assume $X = \{.\}$, so that we are dealing with a usual scalar matrix, $T \in M_N(\mathbb{C})$. By changing the basis of $\mathbb{C}^N$, which won't affect our trace computations, we can assume that T is diagonal:

$$T \sim \begin{pmatrix}\lambda_1 && \\ &\ddots& \\ &&\lambda_N\end{pmatrix}$$

But for such a diagonal matrix, we have the following formula:

$$tr(T^k) = \frac{1}{N}(\lambda_1^k + \ldots + \lambda_N^k)$$

Thus, the law of T is the average of the Dirac masses at the eigenvalues:

$$\mu = \frac{1}{N}\left(\delta_{\lambda_1} + \ldots + \delta_{\lambda_N}\right)$$

As a second example now, assume $N = 1$, and so $T \in L^\infty(X)$. In this case we obtain the usual law of T, because the equation to be satisfied by μ is:

$$\int_X \varphi(T) = \int_{\mathbb{C}} \varphi(x)d\mu(x)$$

Let us record these simple observations, in the form of a theorem, as follows:

THEOREM 6.2. *The laws of basic random matrices $T \in M_N(L^\infty(X))$ are as follows:*

(1) *In the case $N = 1$ the random matrix is a usual random variable, $T \in L^\infty(X)$, automatically normal, and its law as defined above is the usual law.*
(2) *In the case $X = \{.\}$ the random matrix is a usual scalar matrix, $T \in M_N(\mathbb{C})$, and in the diagonalizable case, the law is $\mu = \frac{1}{N}\left(\delta_{\lambda_1} + \ldots + \delta_{\lambda_N}\right)$.*

PROOF. This is something that we know from the above, and which is elementary. Indeed, the first assertion follows from definitions, and the above discussion. As for the second assertion, this follows by diagonalizing the matrix. □

In general, what we have can only be a mixture of (1) and (2) above. Our plan will be that of discussing more in detail (1), and then getting into the general case, or rather into the case of the most interesting random matrices, with inspiration from (2).

At a more advanced level now, the main problem regarding the random matrices is that of computing the law of various classes of such matrices, coming in series:

QUESTION 6.3. *What is the law of random matrices coming in series*

$$T_N \in M_N(L^\infty(X))$$

in the $N >> 0$ regime?

The general strategy here, coming from physicists, is that of computing first the asymptotic law μ^0, in the $N \to \infty$ limit, and then looking for the higher order terms as well, as to finally reach to a series in N^{-1} giving the law of T_N, as follows:

$$\mu_N = \mu^0 + N^{-1}\mu^1 + N^{-2}\mu^2 + \ldots$$

As a basic example here, of particular interest are the matrices having i.i.d. complex normal entries, under the constraint $T = T^*$. Here the asymptotic law μ^0 is the Wigner semicircle law on $[-2, 2]$. We will discuss this in a moment, after some preliminaries.

6b. Probability theory

Let us set $N = 1$. Here our algebra is $A = L^\infty(X)$, an arbitrary commutative von Neumann algebra. The most interesting linear operators $T \in A$, that we will rather denote as complex functions $f : X \to \mathbb{C}$, and call random variables, as it is customary, are the normal, or Gaussian variables, which are defined as follows:

DEFINITION 6.4. *A variable $f : X \to \mathbb{R}$ is called standard normal when its law is:*

$$g_1 = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

More generally, the normal law of parameter $t > 0$ is the following measure:

$$g_t = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$$

These are also called Gaussian distributions, with "g" standing for Gauss.

Observe that these normal laws have indeed mass 1, as they should, as shown by a quick change of variable, and the Gauss formula, namely:

$$\begin{aligned}\left(\int_{\mathbb{R}} e^{-x^2}dx\right)^2 &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-x^2-y^2}dxdy\\ &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2}rdrdt\\ &= 2\pi\times\frac{1}{2}\\ &= \pi\end{aligned}$$

Let us start with some basic results regarding the normal laws. We first have:

PROPOSITION 6.5. *The normal law g_t with $t>0$ has the following properties:*

(1) *The variance is $V=t$.*
(2) *The density is even, so the odd moments vanish.*
(3) *The even moments are $M_k=t^{k/2}\times k!!$, with $k!!=(k-1)(k-3)(k-5)\ldots$.*
(4) *Equivalently, the moments are $M_k=\sum_{\pi\in P_2(k)}t^{|\pi|}$, for any $k\in\mathbb{N}$.*
(5) *The Fourier transform $F_f(x)=E(e^{ixf})$ is given by $F(x)=e^{-tx^2/2}$.*
(6) *We have the convolution semigroup formula $g_s*g_t=g_{s+t}$, for any $s,t>0$.*

PROOF. All this is very standard, with the various notations used in the statement being explained below, the idea being as follows:

(1) The normal law g_t being centered, its variance is the second moment, $V=M_2$. Thus the result follows from (3), proved below, which gives in particular:

$$M_2=t^{2/2}\times 2!!=t$$

(2) This is indeed something self-explanatory.

(3) We have indeed the following computation, by partial integration:

$$\begin{aligned}M_k &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} x^k e^{-x^2/2t}dx\\ &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}(tx^{k-1})\left(-e^{-x^2/2t}\right)'dx\\ &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} t(k-1)x^{k-2}e^{-x^2/2t}dx\\ &= t(k-1)\times\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} x^{k-2}e^{-x^2/2t}dx\\ &= t(k-1)M_{k-2}\end{aligned}$$

The initial value being $M_0=1$, we obtain the result.

(4) We know from (2,3) that the moments of the normal law g_t satisfy the following recurrence formula, with the initial data $M_0 = 1, M_1 = 0$:

$$M_k = t(k-1)M_{k-2}$$

Now let us look at $P_2(k)$, the set of pairings of $\{1, \ldots, k\}$. In order to have such a pairing, we must pair 1 with a number chosen among $2, \ldots, k$, and then come up with a pairing of the remaining $k-2$ numbers. Thus, the number $N_k = |P_2(k)|$ of such pairings is subject to the following recurrence formula, with initial data $N_0 = 1, N_1 = 0$:

$$N_k = (k-1)N_{k-2}$$

But this solves our problem at $t = 1$, because in this case we obtain the following formula, with $|.|$ standing as usual for the number of blocks of a partition:

$$M_k = N_k = |P_2(k)| = \sum_{\pi \in P_2(k)} 1 = \sum_{\pi \in P_2(k)} 1^{|\pi|}$$

Now back to the general case, $t > 0$, our problem here is solved in fact too, because the number of blocks of a pairing $\pi \in P_2(k)$ being constant, $|\pi| = k/2$, we obtain:

$$M_k = t^{k/2} N_k = \sum_{\pi \in P_2(k)} t^{k/2} = \sum_{\pi \in P_2(k)} t^{|\pi|}$$

(5) The Fourier transform formula can be established as follows:

$$\begin{aligned}
F(x) &= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-y^2/2t + ixy} dy \\
&= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-(y/\sqrt{2t} - \sqrt{t/2}ix)^2 - tx^2/2} dy \\
&= \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-z^2 - tx^2/2} \sqrt{2t} dz \\
&= \frac{1}{\sqrt{\pi}} e^{-tx^2/2} \int_{\mathbb{R}} e^{-z^2} dz \\
&= e^{-tx^2/2}
\end{aligned}$$

(6) This follows indeed from (5), because $\log F_{g_t}$ is linear in t. □

We are now ready to establish the Central Limit Theorem (CLT), which is a key result, telling us why the normal laws appear a bit everywhere, in the real life:

THEOREM 6.6. *Given a sequence of real random variables $f_1, f_2, f_3, \ldots \in L^\infty(X)$, which are i.i.d., centered, and with variance $t > 0$, we have*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} f_i \sim g_t$$

with $n \to \infty$, in moments.

Proof. In terms of moments, the Fourier transform $F_f(x)=E(e^{ixf})$ is given by:

$$F_f(x)=E\left(\sum_{k=0}^\infty\frac{(ixf)^k}{k!}\right)=\sum_{k=0}^\infty\frac{i^kM_k(f)}{k!}\,x^k$$

Thus, the Fourier transform of the variable in the statement is:

$$\begin{aligned}F(x)&=\left[F_f\left(\frac{x}{\sqrt{n}}\right)\right]^n\\&=\left[1-\frac{tx^2}{2n}+O(n^{-2})\right]^n\\&\simeq\left[1-\frac{tx^2}{2n}\right]^n\\&\simeq e^{-tx^2/2}\end{aligned}$$

But this latter function being the Fourier transform of g_t, we obtain the result. □

Let us discuss as well the "discrete" counterpart of the above results, that we will need too a bit later, in relation with the random matrices. We have:

Definition 6.7. *The Poisson law of parameter* 1 *is the following measure,*

$$p_1=\frac{1}{e}\sum_k\frac{\delta_k}{k!}$$

and the Poisson law of parameter $t>0$ *is the following measure,*

$$p_t=e^{-t}\sum_k\frac{t^k}{k!}\,\delta_k$$

with the letter "p" standing for Poisson.

We will see in a moment why these laws appear everywhere, in discrete probability, the reasons behind this coming from the Poisson Limit Theorem (PLT). Getting started now, in analogy with the normal laws, the Poisson laws have the following properties:

Proposition 6.8. *The Poisson law* p_t *with* $t>0$ *has the following properties:*

(1) *The variance is* $V=t$.
(2) *The moments are* $M_k=\sum_{\pi\in P(k)}t^{|\pi|}$.
(3) *The Fourier transform is* $F(x)=\exp\left((e^{ix}-1)t\right)$.
(4) *We have the semigroup formula* $p_s*p_t=p_{s+t}$, *for any* $s,t>0$.

Proof. We have four formulae to be proved, the idea being as follows:

(1) The variance is $V=M_2-M_1^2$, and by using the formulae $M_1=t$ and $M_2=t+t^2$, coming from (2), proved below, we obtain as desired, $V=t$.

(2) This is something more tricky. Consider indeed the set $P(k)$ of all partitions of $\{1,\ldots,k\}$. At $t=1$, to start with, the formula that we want to prove is:

$$M_k = |P(k)|$$

We have the following recurrence formula for the moments of p_1:

$$\begin{aligned}
M_{k+1} &= \frac{1}{e}\sum_s \frac{(s+1)^{k+1}}{(s+1)!}\\
&= \frac{1}{e}\sum_s \frac{s^k}{s!}\left(1+\frac{1}{s}\right)^k\\
&= \frac{1}{e}\sum_s \frac{s^k}{s!}\sum_r \binom{k}{r} s^{-r}\\
&= \sum_r \binom{k}{r}\cdot\frac{1}{e}\sum_s \frac{s^{k-r}}{s!}\\
&= \sum_r \binom{k}{r} M_{k-r}
\end{aligned}$$

Our claim is that the numbers $B_k = |P(k)|$ satisfy the same recurrence formula. Indeed, since a partition of $\{1,\ldots,k+1\}$ appears by choosing r neighbors for 1, among the k numbers available, and then partitioning the $k-r$ elements left, we have:

$$B_{k+1} = \sum_r \binom{k}{r} B_{k-r}$$

Thus we obtain by recurrence $M_k = B_k$, as desired. Regarding now the general case, $t>0$, we can use here a similar method. We have the following recurrence formula for the moments of p_t, obtained by using the binomial formula:

$$\begin{aligned}
M_{k+1} &= e^{-t}\sum_s \frac{t^{s+1}(s+1)^{k+1}}{(s+1)!}\\
&= e^{-t}\sum_s \frac{t^{s+1}s^k}{s!}\left(1+\frac{1}{s}\right)^k\\
&= e^{-t}\sum_s \frac{t^{s+1}s^k}{s!}\sum_r \binom{k}{r} s^{-r}\\
&= \sum_r \binom{k}{r}\cdot e^{-t}\sum_s \frac{t^{s+1}s^{k-r}}{s!}\\
&= t\sum_r \binom{k}{r} M_{k-r}
\end{aligned}$$

On the other hand, consider the numbers in the statement, $S_k=\sum_{\pi\in P(k)}t^{|\pi|}$. As before, since a partition of $\{1,\ldots,k+1\}$ appears by choosing r neighbors for 1, among the k numbers available, and then partitioning the $k-r$ elements left, we have:

$$S_{k+1}=t\sum_r\binom{k}{r}S_{k-r}$$

Thus we obtain by recurrence $M_k=B_k$, as desired.

(3) The Fourier transform formula can be established as follows:

$$\begin{aligned}
F_{p_t}(x) &= e^{-t}\sum_k\frac{t^k}{k!}F_{\delta_k}(x)\\
&= e^{-t}\sum_k\frac{t^k}{k!}e^{ikx}\\
&= e^{-t}\sum_k\frac{(e^{ix}t)^k}{k!}\\
&= \exp(-t)\exp(e^{ix}t)\\
&= \exp\left((e^{ix}-1)t\right)
\end{aligned}$$

(4) This follows from (3), because $\log F_{p_t}$ is linear in t. □

We are now ready to establish the Poisson Limit Theorem (PLT), as follows:

THEOREM 6.9. *We have the following convergence, in moments,*

$$\left(\left(1-\frac{t}{n}\right)\delta_0+\frac{t}{n}\delta_1\right)^{*n}\to p_t$$

for any $t>0$.

PROOF. Let us denote by μ_n the Bernoulli measure appearing under the convolution sign. We have then the following computation:

$$\begin{aligned}
F_{\delta_r}(x)=e^{irx} &\implies F_{\mu_n}(x)=\left(1-\frac{t}{n}\right)+\frac{t}{n}e^{ix}\\
&\implies F_{\mu_n^{*n}}(x)=\left(\left(1-\frac{t}{n}\right)+\frac{t}{n}e^{ix}\right)^n\\
&\implies F_{\mu_n^{*n}}(x)=\left(1+\frac{(e^{ix}-1)t}{n}\right)^n\\
&\implies F(x)=\exp\left((e^{ix}-1)t\right)
\end{aligned}$$

Thus, we obtain the Fourier transform of p_t, as desired. □

As a third and last topic from classical probability, let us discuss now the complex normal laws, that we will need too. To start with, we have the following definition:

DEFINITION 6.10. *The complex Gaussian law of parameter $t > 0$ is*

$$G_t = law\left(\frac{1}{\sqrt{2}}(a+ib)\right)$$

where a, b are independent, each following the law g_t.

As in the real case, these measures form convolution semigroups:

PROPOSITION 6.11. *The complex Gaussian laws have the property*

$$G_s * G_t = G_{s+t}$$

for any $s, t > 0$, and so they form a convolution semigroup.

PROOF. This follows indeed from the real result, namely $g_s * g_t = g_{s+t}$, established above, simply by taking real and imaginary parts. □

We have the following complex analogue of the CLT:

THEOREM 6.12 (CCLT). *Given complex random variables $f_1, f_2, f_3, \ldots \in L^\infty(X)$ which are i.i.d., centered, and with variance $t > 0$, we have, with $n \to \infty$, in moments,*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n f_i \sim G_t$$

where G_t is the complex Gaussian law of parameter t.

PROOF. This follows indeed from the real CLT, established above, simply by taking the real and imaginary parts of all the variables involved. □

Regarding now the moments, we have to use here a more general formalism, involving colored integer exponents $k = \circ\bullet\bullet\circ\ldots$ We say that a pairing $\pi \in P_2(k)$ is matching when it pairs $\circ - \bullet$ symbols. With this convention, we have the following result:

THEOREM 6.13. *The moments of the complex normal law are the numbers*

$$M_k(G_t) = \sum_{\pi\in\mathcal{P}_2(k)} t^{|\pi|}$$

where $\mathcal{P}_2(k)$ are the matching pairings of $\{1,\ldots,k\}$, and $|.|$ is the number of blocks.

PROOF. This is something well-known, which can be established as follows:

(1) As a first observation, by using a standard dilation argument, it is enough to do this at $t = 1$. So, let us first recall from the above that the moments of the real Gaussian law g_1, with respect to integer exponents $k \in \mathbb{N}$, are the following numbers:

$$m_k = |P_2(k)|$$

Numerically, we have the following formula, explained as well in the above:

$$m_k=\begin{cases}k!! & (k\text{ even})\\ 0 & (k\text{ odd})\end{cases}$$

(2) We will show here that in what concerns the complex Gaussian law G_1, similar results hold. Numerically, we will prove that we have the following formula, where a colored integer $k=\circ\bullet\bullet\circ\ldots$ is called uniform when it contains the same number of $\circ$ and $\bullet$, and where $|k|\in\mathbb{N}$ is the length of such a colored integer:

$$M_k=\begin{cases}(|k|/2)! & (k\text{ uniform})\\ 0 & (k\text{ not uniform})\end{cases}$$

Now since the matching partitions $\pi\in\mathcal{P}_2(k)$ are counted by exactly the same numbers, and this for trivial reasons, we will obtain the formula in the statement, namely:

$$M_k=|\mathcal{P}_2(k)|$$

(3) This was for the plan. In practice now, we must compute the moments, with respect to colored integer exponents $k=\circ\bullet\bullet\circ\ldots$, of the variable in the statement:

$$c=\frac{1}{\sqrt{2}}(a+ib)$$

As a first observation, in the case where such an exponent $k=\circ\bullet\bullet\circ\ldots$ is not uniform in $\circ,\bullet$, a rotation argument shows that the corresponding moment of c vanishes. To be more precise, the variable $c'=wc$ can be shown to be complex Gaussian too, for any $w\in\mathbb{C}$, and from $M_k(c)=M_k(c')$ we obtain $M_k(c)=0$, in this case.

(4) In the uniform case now, where $k=\circ\bullet\bullet\circ\ldots$ consists of p copies of $\circ$ and p copies of $\bullet$, the corresponding moment can be computed as follows:

$$\begin{aligned}M_k&=\frac{1}{2^p}\int(a^2+b^2)^p\\&=\frac{1}{2^p}\sum_s\binom{p}{s}\int a^{2s}\int b^{2p-2s}\\&=\frac{1}{2^p}\sum_s\binom{p}{s}(2s)!!(2p-2s)!!\\&=\frac{1}{2^p}\sum_s\frac{p!}{s!(p-s)!}\cdot\frac{(2s)!}{2^s s!}\cdot\frac{(2p-2s)!}{2^{p-s}(p-s)!}\\&=\frac{p!}{4^p}\sum_s\binom{2s}{s}\binom{2p-2s}{p-s}\end{aligned}$$

(5) In order to finish now the computation, let us recall that we have the following formula, coming from the generalized binomial formula, or from the Taylor formula:

$$\frac{1}{\sqrt{1+t}}=\sum_{k=0}^\infty\binom{2k}{k}\left(\frac{-t}{4}\right)^k$$

By taking the square of this series, we obtain the following formula:

$$\begin{aligned}\frac{1}{1+t} &= \sum_{ks}\binom{2k}{k}\binom{2s}{s}\left(\frac{-t}{4}\right)^{k+s}\\ &= \sum_p\left(\frac{-t}{4}\right)^p\sum_s\binom{2s}{s}\binom{2p-2s}{p-s}\end{aligned}$$

Now by looking at the coefficient of t^p on both sides, we conclude that the sum on the right equals 4^p. Thus, we can finish the moment computation in (4), as follows:

$$M_p=\frac{p!}{4^p}\times 4^p=p!$$

(6) As a conclusion, if we denote by $|k|$ the length of a colored integer $k=\circ\bullet\bullet\circ\ldots$, the moments of the variable c in the statement are given by:

$$M_k=\begin{cases}(|k|/2)! & (k\ \text{uniform})\\ 0 & (k\ \text{not uniform})\end{cases}$$

On the other hand, the numbers $|\mathcal{P}_2(k)|$ are given by exactly the same formula. Indeed, in order to have matching pairings of k, our exponent $k=\circ\bullet\bullet\circ\ldots$ must be uniform, consisting of p copies of $\circ$ and p copies of $\bullet$, with $p=|k|/2$. But then the matching pairings of k correspond to the permutations of the $\bullet$ symbols, as to be matched with $\circ$ symbols, and so we have $p!$ such matching pairings. Thus, we have the same formula as for the moments of c, and we are led to the conclusion in the statement. □

This was for the basic probability theory, which is in a certain sense advanced operator theory, inside the commutative von Neumann algebras, $A=L^\infty(X)$. We will be back to this, with some further limiting theorems, in chapter 15 below.

6c. Wigner matrices

Let us exit now the classical world, that of the commutative von Neumann algebras $A=L^\infty(X)$, and do as promised some random matrix theory. We recall that a random matrix algebra is a von Neumann algebra of type $A=M_N(L^\infty(X))$, and that we are interested in the computation of the laws of the operators $T\in A$, called random matrices. Regarding the precise classes of random matrices that we are interested in, first we have the complex Gaussian matrices, which are constructed as follows:

Definition 6.14. *A complex Gaussian matrix is a random matrix of type*

$$Z\in M_N(L^\infty(X))$$

which has i.i.d. complex normal entries.

We will see that the above matrices have an interesting, and "central" combinatorics, among all kinds of random matrices, with the study of the other random matrices being usually obtained as a modification of the study of the Gaussian matrices.

As a somewhat surprising remark, using real normal variables in Definition 6.14, instead of the complex ones appearing there, leads nowhere. The correct real versions of the Gaussian matrices are the Wigner random matrices, constructed as follows:

Definition 6.15. *A Wigner matrix is a random matrix of type*

$$Z\in M_N(L^\infty(X))$$

which has i.i.d. complex normal entries, up to the constraint $Z=Z^*$.

In other words, a Wigner matrix must be as follows, with the diagonal entries being real normal variables, $a_i\sim g_t$, for some $t>0$, the upper diagonal entries being complex normal variables, $b_{ij}\sim G_t$, the lower diagonal entries being the conjugates of the upper diagonal entries, as indicated, and with all the variables a_i,b_{ij} being independent:

$$Z=\begin{pmatrix}a_1&b_{12}&\ldots&\ldots&b_{1N}\\ \bar{b}_{12}&a_2&\ddots&&\vdots\\ \vdots&\ddots&\ddots&\ddots&\vdots\\ \vdots&&\ddots&a_{N-1}&b_{N-1,N}\\ \bar{b}_{1N}&\ldots&\ldots&\bar{b}_{N-1,N}&a_N\end{pmatrix}$$

As a comment here, for many concrete applications the Wigner matrices are in fact the central objects in random matrix theory, and in particular, they are often more important than the Gaussian matrices. In fact, these are the random matrices which were first considered and investigated, a long time ago, by Wigner himself [**99**].

Finally, we will be interested as well in the complex Wishart matrices, which are the positive versions of the above random matrices, constructed as follows:

Definition 6.16. *A complex Wishart matrix is a random matrix of type*

$$Z=YY^*\in M_N(L^\infty(X))$$

with Y being a complex Gaussian matrix.

As before with the Gaussian and Wigner matrices, there are many possible comments that can be made here, of technical or historical nature. First, using real Gaussian variables instead of complex ones leads to a less interesting combinatorics. Also, these matrices were introduced and studied by Marchenko-Pastur not long after Wigner, in [70], and so historically came second. Finally, in what regards their combinatorics and applications, these matrices quite often come first, before both the Gaussian and the Wigner ones, with all this being of course a matter of knowledge and taste.

Summarizing, we have three main types of random matrices, which can be somehow designated as "complex", "real" and "positive", and that we will study in what follows. Let us also mention that there are many other interesting classes of random matrices, usually appearing as modifications of the above. More on these later.

In order to compute the asymptotic laws of the above matrices, we will use the moment method. We have the following result, which will be our main tool here:

Theorem 6.17. *Given independent variables X_i, each following the complex normal law G_t, with $t>0$ being a fixed parameter, we have the Wick formula*

$$E\left(X_{i_1}^{k_1}\ldots X_{i_s}^{k_s}\right)=t^{s/2}\#\left\{\pi\in\mathcal{P}_2(k)\Big|\pi\leq\ker i\right\}$$

where $k=k_1\ldots k_s$ and $i=i_1\ldots i_s$, for the joint moments of these variables.

Proof. This is something well-known, and the basis for all possible computations with complex normal variables, which can be proved in two steps, as follows:

(1) Let us first discuss the case where we have a single complex normal variable X, which amounts in taking $X_i=X$ for any i in the formula in the statement. What we have to compute here are the moments of X, with respect to colored integer exponents $k=\circ\bullet\bullet\circ\ldots$, and the formula in the statement tells us that these moments must be:

$$E(X^k)=t^{|k|/2}|\mathcal{P}_2(k)|$$

But this is something that we know well from the above, the idea being that at $t=1$ this follows by doing some combinatorics and calculus, in analogy with the combinatorics and calculus from the real case, where the moment formula is identical, save for the matching pairings $\mathcal{P}_2$ being replaced by the usual pairings P_2, and then that the general case $t>0$ follows from this, by rescaling. Thus, we are done with this case.

(2) In general now, the point is that we obtain the formula in the statement. Indeed, when expanding the product $X_{i_1}^{k_1}\ldots X_{i_s}^{k_s}$ and rearranging the terms, we are left with doing a number of computations as in (1), and then making the product of the expectations that we found. But this amounts precisely in counting the partitions in the statement, with the condition $\pi\leq\ker i$ there standing precisely for the fact that we are doing the various type (1) computations independently, and then making the product. $\square$

Now by getting back to the Gaussian matrices, we have the following result, with $\mathcal{NC}_2(k)=\mathcal{P}_2(k)\cap NC(k)$ standing for the noncrossing pairings of a colored integer k:

THEOREM 6.18. *Given a sequence of Gaussian random matrices*

$$Z_N\in M_N(L^\infty(X))$$

having independent G_t variables as entries, for some fixed $t>0$, we have

$$M_k\left(\frac{Z_N}{\sqrt{N}}\right)\simeq t^{|k|/2}|\mathcal{NC}_2(k)|$$

for any colored integer $k=\circ\bullet\bullet\circ\ldots$, in the $N\to\infty$ limit.

PROOF. This is something standard, which can be done as follows:

(1) We fix $N\in\mathbb{N}$, and we let $Z=Z_N$. Let us first compute the trace of Z^k. With $k=k_1\ldots k_s$, and with the convention $(ij)^\circ=ij,(ij)^\bullet=ji$, we have:

$$\begin{aligned}
Tr(Z^k) &= Tr(Z^{k_1}\ldots Z^{k_s})\\
&= \sum_{i_1=1}^N\ldots\sum_{i_s=1}^N(Z^{k_1})_{i_1i_2}(Z^{k_2})_{i_2i_3}\ldots(Z^{k_s})_{i_si_1}\\
&= \sum_{i_1=1}^N\ldots\sum_{i_s=1}^N(Z_{(i_1i_2)^{k_1}})^{k_1}(Z_{(i_2i_3)^{k_2}})^{k_2}\ldots(Z_{(i_si_1)^{k_s}})^{k_s}
\end{aligned}$$

(2) Next, we rescale our variable Z by a $\sqrt{N}$ factor, as in the statement, and we also replace the usual trace by its normalized version, $tr=Tr/N$. Our formula becomes:

$$tr\left(\left(\frac{Z}{\sqrt{N}}\right)^k\right)=\frac{1}{N^{s/2+1}}\sum_{i_1=1}^N\ldots\sum_{i_s=1}^N(Z_{(i_1i_2)^{k_1}})^{k_1}(Z_{(i_2i_3)^{k_2}})^{k_2}\ldots(Z_{(i_si_1)^{k_s}})^{k_s}$$

Thus, the moment that we are interested in is given by:

$$M_k\left(\frac{Z}{\sqrt{N}}\right)=\frac{1}{N^{s/2+1}}\sum_{i_1=1}^N\ldots\sum_{i_s=1}^N\int_X(Z_{(i_1i_2)^{k_1}})^{k_1}(Z_{(i_2i_3)^{k_2}})^{k_2}\ldots(Z_{(i_si_1)^{k_s}})^{k_s}$$

(3) Let us apply now the Wick formula, from Theorem 6.17. We conclude that the moment that we are interested in is given by the following formula:

$$\begin{aligned}
&M_k\left(\frac{Z}{\sqrt{N}}\right)\\
&= \frac{t^{s/2}}{N^{s/2+1}}\sum_{i_1=1}^N\ldots\sum_{i_s=1}^N \#\left\{\pi\in\mathcal{P}_2(k)\Big|\pi\leq\ker\left((i_1i_2)^{k_1},(i_2i_3)^{k_2},\ldots,(i_si_1)^{k_s}\right)\right\}\\
&= t^{s/2}\sum_{\pi\in\mathcal{P}_2(k)}\frac{1}{N^{s/2+1}}\#\left\{i\in\{1,\ldots,N\}^s\Big|\pi\leq\ker\left((i_1i_2)^{k_1},(i_2i_3)^{k_2},\ldots,(i_si_1)^{k_s}\right)\right\}
\end{aligned}$$

(4) Our claim now is that in the $N\to\infty$ limit the combinatorics of the above sum simplifies, with only the noncrossing partitions contributing to the sum, and with each of them contributing precisely with a 1 factor, so that we will have, as desired:

$$\begin{aligned}
M_k\left(\frac{Z}{\sqrt{N}}\right) &= t^{s/2}\sum_{\pi\in\mathcal{P}_2(k)}\left(\delta_{\pi\in NC_2(k)}+O(N^{-1})\right)\\
&\simeq t^{s/2}\sum_{\pi\in\mathcal{P}_2(k)}\delta_{\pi\in NC_2(k)}\\
&= t^{s/2}|\mathcal{NC}_2(k)|
\end{aligned}$$

(5) In order to prove this, the first observation is that when k is not uniform, in the sense that it contains a different number of $\circ,\bullet$ symbols, we have $\mathcal{P}_2(k)=\emptyset$, and so:

$$M_k\left(\frac{Z}{\sqrt{N}}\right)=t^{s/2}|\mathcal{NC}_2(k)|=0$$

(6) Thus, we are left with the case where k is uniform. Let us examine first the case where k consists of an alternating sequence of $\circ$ and $\bullet$ symbols, as follows:

$$k=\underbrace{\circ\bullet\circ\bullet\ldots\ldots\circ\bullet}_{2p}$$

In this case it is convenient to relabel our multi-index $i=(i_1,\ldots,i_s)$, with $s=2p$, in the form $(j_1,l_1,j_2,l_2,\ldots,j_p,l_p)$. With this done, our moment formula becomes:

$$M_k\left(\frac{Z}{\sqrt{N}}\right)=t^p\sum_{\pi\in\mathcal{P}_2(k)}\frac{1}{N^{p+1}}\#\left\{j,l\in\{1,\ldots,N\}^p\Big|\pi\leq\ker\left(j_1l_1,j_2l_1,j_2l_2,\ldots,j_1l_p\right)\right\}$$

Now observe that, with k being as above, we have an identification $\mathcal{P}_2(k)\simeq S_p$, obtained in the obvious way. With this done too, our moment formula becomes:

$$M_k\left(\frac{Z}{\sqrt{N}}\right)=t^p\sum_{\pi\in S_p}\frac{1}{N^{p+1}}\#\left\{j,l\in\{1,\ldots,N\}^p\Big|j_r=j_{\pi(r)+1},l_r=l_{\pi(r)},\forall r\right\}$$

(7) We are now ready to do our asymptotic study, and prove the claim in (4). Let indeed $\gamma \in S_p$ be the full cycle, which is by definition the following permutation:

$$\gamma = (1\,2\,\ldots\,p)$$

In terms of γ, the conditions $j_r = j_{\pi(r)+1}$ and $l_r = l_{\pi(r)}$ found above read:

$$\gamma\pi \leq \ker j \quad , \quad \pi \leq \ker l$$

Counting the number of free parameters in our moment formula, we obtain:

$$M_k\left(\frac{Z}{\sqrt{N}}\right) = \frac{t^p}{N^{p+1}}\sum_{\pi\in S_p} N^{|\pi|+|\gamma\pi|} = t^p \sum_{\pi\in S_p} N^{|\pi|+|\gamma\pi|-p-1}$$

(8) The point now is that the last exponent is well-known to be ≤ 0, with equality precisely when the permutation $\pi \in S_p$ is geodesic, which in practice means that π must come from a noncrossing partition. Thus we obtain, in the $N \to \infty$ limit, as desired:

$$M_k\left(\frac{Z}{\sqrt{N}}\right) \simeq t^p|\mathcal{NC}_2(k)|$$

This finishes the proof in the case of the exponents k which are alternating, and the case where k is an arbitrary uniform exponent is similar, by permuting everything. $\square$

As a conclusion to this, we have obtained as asymptotic law for the Gaussian matrices a certain mysterious distribution, having as moments some numbers which are similar to the moments of the usual normal laws, but with the "underlying matching pairings being now replaced by underlying matching noncrossing pairings". More on this later.

Regarding now the Wigner matrices, we have here the following result, coming as a consequence of Theorem 6.18, via some simple algebraic manipulations:

THEOREM 6.19. *Given a sequence of Wigner random matrices*

$$Z_N \in M_N(L^\infty(X))$$

having independent G_t variables as entries, with $t > 0$, up to $Z_N = Z_N^*$, we have

$$M_k\left(\frac{Z_N}{\sqrt{N}}\right) \simeq t^{k/2}|NC_2(k)|$$

for any integer $k \in \mathbb{N}$, in the $N \to \infty$ limit.

PROOF. This can be deduced from a direct computation based on the Wick formula, similar to that from the proof of Theorem 6.18, but the best is to deduce this result from Theorem 6.18 itself. Indeed, we know from there that for Gaussian matrices $Y_N \in M_N(L^\infty(X))$ we have the following formula, valid for any colored integer $K = \circ\bullet\bullet\circ\ldots$, in the $N \to \infty$ limit, with $\mathcal{NC}_2$ standing for noncrossing matching pairings:

$$M_K\left(\frac{Y_N}{\sqrt{N}}\right) \simeq t^{|K|/2}|\mathcal{NC}_2(K)|$$

By doing some combinatorics, we deduce from this that we have the following formula for the moments of the matrices $Re(Y_N)$, with respect to usual exponents, $k \in \mathbb{N}$:

$$\begin{aligned} M_k\left(\frac{Re(Y_N)}{\sqrt{N}}\right) &= 2^{-k} \cdot M_k\left(\frac{Y_N}{\sqrt{N}} + \frac{Y_N^*}{\sqrt{N}}\right) \\ &= 2^{-k} \sum_{|K|=k} M_K\left(\frac{Y_N}{\sqrt{N}}\right) \\ &\simeq 2^{-k} \sum_{|K|=k} t^{k/2} |\mathcal{NC}_2(K)| \\ &= 2^{-k} \cdot t^{k/2} \cdot 2^{k/2} |\mathcal{NC}_2(k)| \\ &= 2^{-k/2} \cdot t^{k/2} |NC_2(k)| \end{aligned}$$

Now since the matrices $Z_N = \sqrt{2} Re(Y_N)$ are of Wigner type, this gives the result. □

Summarizing, all this brings us into counting noncrossing pairings. So, let us start with some preliminaries here. We first have the following well-known result:

Theorem 6.20. *The Catalan numbers, which are by definition given by*

$$C_k = |NC_2(2k)|$$

satisfy the following recurrence formula, with initial data $C_0 = C_1 = 1$,

$$C_{k+1} = \sum_{a+b=k} C_a C_b$$

their generating series $f(z) = \sum_{k \geq 0} C_k z^k$ satisfies the equation

$$zf^2 - f + 1 = 0$$

and is given by the following explicit formula,

$$f(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$

and we have the following explicit formula for these numbers:

$$C_k = \frac{1}{k+1} \binom{2k}{k}$$

Numerically, these numbers are $1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \ldots$

Proof. We must count the noncrossing pairings of $\{1, \ldots, 2k\}$. Now observe that such a pairing appears by pairing 1 to an odd number, $2a + 1$, and then inserting a noncrossing pairing of $\{2, \ldots, 2a\}$, and a noncrossing pairing of $\{2a + 2, \ldots, 2l\}$. We conclude that we have the following recurrence formula for the Catalan numbers:

$$C_k = \sum_{a+b=k-1} C_a C_b$$

In terms of the generating series $f(z)=\sum_{k\geq 0}C_kz^k$, this recurrence formula reads:

$$\begin{aligned} zf^2 &= \sum_{a,b\geq 0} C_aC_bz^{a+b+1}\\ &= \sum_{k\geq 1}\sum_{a+b=k-1} C_aC_bz^k\\ &= \sum_{k\geq 1} C_kz^k\\ &= f-1 \end{aligned}$$

Thus f satisfies $zf^2-f+1=0$, and by solving this equation, and choosing the solution which is bounded at $z=0$, we obtain the following formula:

$$f(z)=\frac{1-\sqrt{1-4z}}{2z}$$

In order to finish, we use the generalized binomial formula, which gives:

$$\sqrt{1+t}=1-2\sum_{k=1}^\infty\frac{1}{k}\binom{2k-2}{k-1}\left(\frac{-t}{4}\right)^k$$

Now back to our series f, we obtain the following formula for it:

$$\begin{aligned} f(z) &= \frac{1-\sqrt{1-4z}}{2z}\\ &= \sum_{k=1}^\infty\frac{1}{k}\binom{2k-2}{k-1}z^{k-1}\\ &= \sum_{k=0}^\infty\frac{1}{k+1}\binom{2k}{k}z^k \end{aligned}$$

It follows that the Catalan numbers are given by:

$$C_k=\frac{1}{k+1}\binom{2k}{k}$$

Thus, we are led to the conclusion in the statement. $\square$

In order to recapture now the Wigner measure from its moments, we can use:

PROPOSITION 6.21. *The Catalan numbers are the even moments of*

$$\gamma_1=\frac{1}{2\pi}\sqrt{4-x^2}dx$$

called standard semicircle law. As for the odd moments of γ_1, these all vanish.

PROOF. The even moments of the semicircle law in the statement can be computed with the change of variable $x = 2\cos t$, and we are led to the following formula:

$$\begin{aligned} M_{2k} &= \frac{1}{\pi}\int_0^2 \sqrt{4-x^2}x^{2k}dx \\ &= \frac{1}{\pi}\int_0^{\pi/2} \sqrt{4-4\cos^2 t}\,(2\cos t)^{2k} 2\sin t\,dt \\ &= \frac{4^{k+1}}{\pi}\int_0^{\pi/2} \cos^{2k} t\sin^2 t\,dt \\ &= \frac{4^{k+1}}{\pi}\cdot\frac{\pi}{2}\cdot\frac{(2k)!!2!!}{(2k+3)!!} \\ &= 2\cdot 4^k\cdot\frac{(2k)!/2^k k!}{2^{k+1}(k+1)!} \\ &= C_k \end{aligned}$$

As for the odd moments, these all vanish, because the density of γ_1 is an even function. Thus, we are led to the conclusion in the statement. □

More generally, we have the following result, involving a parameter $t > 0$:

PROPOSITION 6.22. *Given $t > 0$, the real measure having as even moments the numbers $M_{2k} = t^k C_k$ and having all odd moments 0 is the measure*

$$\gamma_t = \frac{1}{2\pi t}\sqrt{4t - x^2}dx$$

called Wigner semicircle law on $[-2\sqrt{t}, 2\sqrt{t}]$.

PROOF. This follows indeed from Proposition 6.21, via a change of variables. □

Now by putting everything together, we obtain the Wigner theorem, as follows:

THEOREM 6.23. *Given a sequence of Wigner random matrices*

$$Z_N \in M_N(L^\infty(X))$$

which by definition have i.i.d. complex normal entries, up to $Z_N = Z_N^*$, we have

$$Z_N \sim \gamma_t$$

in the $N \to \infty$ limit, where $\gamma_t = \frac{1}{2\pi t}\sqrt{4t - x^2}dx$ is the Wigner semicircle law.

PROOF. This follows indeed from all the above, and more specifically, by combining Theorem 6.19, Theorem 6.20 and Proposition 6.22. □

Regarding now the complex Gaussian matrices, in view of this result, it is natural to think at the law found in Theorem 6.18 as being "circular". But this is just a thought, and more on this later, in chapter 15 below, when doing free probability.

6d. Wishart matrices

Let us discuss now the Wishart matrices, which are the positive analogues of the Wigner matrices. Quite surprisingly, the computation here leads to the Catalan numbers, but not in the same way as for the Wigner matrices, the result being as follows:

THEOREM 6.24. *Given a sequence of complex Wishart matrices*

$$W_N = Y_N Y_N^* \in M_N(L^\infty(X))$$

with Y_N being $N \times N$ complex Gaussian of parameter $t > 0$, we have

$$M_k\left(\frac{W_N}{N}\right) \simeq t^k C_k$$

for any exponent $k \in \mathbb{N}$, in the $N \to \infty$ limit.

PROOF. There are several possible proofs for this result, as follows:

(1) A first method is by using the formula that we have in Theorem 6.18, for the Gaussian matrices Y_N. Indeed, we know from there that we have the following formula, valid for any colored integer $K = \circ\bullet\bullet\circ\ldots$, in the $N \to \infty$ limit:

$$M_K\left(\frac{Y_N}{\sqrt{N}}\right) \simeq t^{|K|/2}|\mathcal{NC}_2(K)|$$

With $K = \circ\bullet\circ\bullet\ldots$, alternating word of length $2k$, with $k \in \mathbb{N}$, this gives:

$$M_k\left(\frac{Y_N Y_N^*}{N}\right) \simeq t^k|\mathcal{NC}_2(K)|$$

Thus, in terms of the Wishart matrix $W_N = Y_N Y_N^*$ we have, for any $k \in \mathbb{N}$:

$$M_k\left(\frac{W_N}{N}\right) \simeq t^k|\mathcal{NC}_2(K)|$$

The point now is that, by doing some combinatorics, we have:

$$|\mathcal{NC}_2(K)| = |NC_2(2k)| = C_k$$

Thus, we are led to the formula in the statement.

(2) A second method, that we will explain now as well, is by proving the result directly, starting from definitions. The matrix entries of our matrix $W = W_N$ are given by:

$$W_{ij} = \sum_{r=1}^N Y_{ir}\bar{Y}_{jr}$$

Thus, the normalized traces of powers of W are given by the following formula:

$$\begin{aligned} tr(W^k) &= \frac{1}{N}\sum_{i_1=1}^N\ldots\sum_{i_k=1}^N W_{i_1i_2}W_{i_2i_3}\ldots W_{i_ki_1} \\ &= \frac{1}{N}\sum_{i_1=1}^N\ldots\sum_{i_k=1}^N\sum_{r_1=1}^N\ldots\sum_{r_k=1}^N Y_{i_1r_1}\bar{Y}_{i_2r_1}Y_{i_2r_2}\bar{Y}_{i_3r_2}\ldots Y_{i_kr_k}\bar{Y}_{i_1r_k}\end{aligned}$$

By rescaling now W by a $1/N$ factor, as in the statement, we obtain:

$$tr\left(\left(\frac{W}{N}\right)^k\right) = \frac{1}{N^{k+1}}\sum_{i_1=1}^N\ldots\sum_{i_k=1}^N\sum_{r_1=1}^N\ldots\sum_{r_k=1}^N Y_{i_1r_1}\bar{Y}_{i_2r_1}Y_{i_2r_2}\bar{Y}_{i_3r_2}\ldots Y_{i_kr_k}\bar{Y}_{i_1r_k}$$

By using now the Wick rule, we obtain the following formula for the moments, with $K = \circ\bullet\circ\bullet\ldots$, alternating word of lenght $2k$, and with $I = (i_1r_1, i_2r_1, \ldots, i_kr_k, i_1r_k)$:

$$\begin{aligned} M_k\left(\frac{W}{N}\right) &= \frac{t^k}{N^{k+1}}\sum_{i_1=1}^N\ldots\sum_{i_k=1}^N\sum_{r_1=1}^N\ldots\sum_{r_k=1}^N \#\left\{\pi\in\mathcal{P}_2(K)\Big|\pi\leq\ker(I)\right\} \\ &= \frac{t^k}{N^{k+1}}\sum_{\pi\in\mathcal{P}_2(K)}\#\left\{i,r\in\{1,\ldots,N\}^k\Big|\pi\leq\ker(I)\right\}\end{aligned}$$

In order to compute this quantity, we use the standard bijection $\mathcal{P}_2(K)\simeq S_k$. By identifying the pairings $\pi\in\mathcal{P}_2(K)$ with their counterparts $\pi\in S_k$, we obtain:

$$M_k\left(\frac{W}{N}\right) = \frac{t^k}{N^{k+1}}\sum_{\pi\in S_k}\#\left\{i,r\in\{1,\ldots,N\}^k\Big|i_s=i_{\pi(s)+1}, r_s=r_{\pi(s)},\forall s\right\}$$

Now let $\gamma\in S_k$ be the full cycle, which is by definition the following permutation:

$$\gamma = (1\,2\,\ldots\,k)$$

The general factor in the product computed above is then 1 precisely when following two conditions are simultaneously satisfied:

$$\gamma\pi\leq\ker i\quad,\quad\pi\leq\ker r$$

Counting the number of free parameters in our moment formula, we obtain:

$$M_k\left(\frac{W}{N}\right) = t^k\sum_{\pi\in S_k}N^{|\pi|+|\gamma\pi|-k-1}$$

The point now is that the last exponent is well-known to be ≤ 0, with equality precisely when the permutation $\pi\in S_k$ is geodesic, which in practice means that π must come from a noncrossing partition. Thus we obtain, in the $N \to \infty$ limit:

$$M_k\left(\frac{W}{N}\right) \simeq t^k C_k$$

Thus, we are led to the conclusion in the statement. □

As a consequence of the above result, we have a new look on the Catalan numbers, which is more adapted to our present Wishart matrix considerations, as follows:

PROPOSITION 6.25. *The Catalan numbers $C_k = |NC_2(2k)|$ appear as well as*

$$C_k = |NC(k)|$$

where $NC(k)$ is the set of all noncrossing partitions of $\{1, \ldots, k\}$.

PROOF. This follows indeed from the proof of Theorem 6.24. Observe that we obtain as well a formula in terms of matching pairings of alternating colored integers. □

The direct explanation for the above formula, relating noncrossing partitions and pairings, comes form the following result, which is very useful, and good to know:

PROPOSITION 6.26. *We have a bijection between noncrossing partitions and pairings*

$$NC(k) \simeq NC_2(2k)$$

which is constructed as follows:

1. *The application $NC(k) \to NC_2(2k)$ is the "fattening" one, obtained by doubling all the legs, and doubling all the strings as well.*
2. *Its inverse $NC_2(2k) \to NC(k)$ is the "shrinking" application, obtained by collapsing pairs of consecutive neighbors.*

PROOF. The fact that the two operations in the statement are indeed inverse to each other is clear, by computing the corresponding two compositions, with the remark that the construction of the fattening operation requires the partitions to be noncrossing. □

Getting back now to probability, we are led to the question of finding the law having the Catalan numbers as moments, in the above way. The result here is as follows:

PROPOSITION 6.27. *The real measure having the Catalan numbers as moments is*

$$\pi_1 = \frac{1}{2\pi}\sqrt{4x^{-1} - 1}\, dx$$

called Marchenko-Pastur law of parameter 1.

PROOF. The moments of the law π_1 in the statement can be computed with the change of variable $x = 4\cos^2 t$, as follows:

$$\begin{aligned} M_k &= \frac{1}{2\pi}\int_0^4 \sqrt{4x^{-1}-1}\,x^k dx \\ &= \frac{1}{2\pi}\int_0^{\pi/2}\frac{\sin t}{\cos t}\cdot(4\cos^2 t)^k\cdot 2\cos t\sin t\,dt \\ &= \frac{4^{k+1}}{\pi}\int_0^{\pi/2}\cos^{2k}t\sin^2 t\,dt \\ &= \frac{4^{k+1}}{\pi}\cdot\frac{\pi}{2}\cdot\frac{(2k)!!2!!}{(2k+3)!!} \\ &= 2\cdot 4^k\cdot\frac{(2k)!/2^k k!}{2^{k+1}(k+1)!} \\ &= C_k \end{aligned}$$

Thus, we are led to the conclusion in the statement. ☐

Now back to the Wishart matrices, we are led to the following result:

THEOREM 6.28. *Given a sequence of complex Wishart matrices*

$$W_N = Y_N Y_N^* \in M_N(L^\infty(X))$$

with Y_N being $N\times N$ complex Gaussian of parameter $t>0$, we have

$$\frac{W_N}{tN}\sim\frac{1}{2\pi}\sqrt{4x^{-1}-1}\,dx$$

with $N\to\infty$, with the limiting measure being the Marchenko-Pastur law π_1.

PROOF. This follows indeed from Theorem 6.24 and Proposition 6.27. ☐

As a comment now, while the above result is definitely something interesting at $t=1$, at general $t>0$ this looks more like a "fake" generalization of the $t=1$ result, because the law π_1 stays the same, modulo a trivial rescaling. The reasons behind this phenomenon are quite subtle, and skipping some discussion, the point is that Theorem 6.28 is indeed something "fake" at general $t>0$, and the correct generalization of the $t=1$ computation, involving more general classes of complex Wishart matrices, is as follows:

THEOREM 6.29. *Given a sequence of general complex Wishart matrices*

$$W_N = Y_N Y_N^* \in M_N(L^\infty(X))$$

with Y_N being $N\times M$ complex Gaussian of parameter 1, we have

$$\frac{W_N}{N}\sim\max(1-t,0)\delta_0+\frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx$$

with $M=tN\to\infty$, with the limiting measure being the Marchenko-Pastur law π_t.

PROOF. This follows once again by using the moment method, the limiting moments in the $M = tN \to \infty$ regime being as follows, after doing the combinatorics:

$$M_k\left(\frac{W_N}{N}\right) \simeq \sum_{\pi \in NC(k)} t^{|\pi|}$$

But these numbers are the moments of the Marchenko-Pastur law π_t, which in addition has the density given by the formula in the statement, and this gives the result. $\square$

As a philosophical conclusion now, we have 4 main laws in what we have been doing so far, namely the Gaussian laws g_t, the Poisson laws p_t, the Wigner laws γ_t and the Marchenko-Pastur laws π_t. These laws naturally form a diagram, as follows:

$$\begin{array}{ccc}
\pi_t & \text{---} & \gamma_t \\
| & & | \\
p_t & \text{---} & g_t
\end{array}$$

We will see in chapter 15 that π_t, γ_t appear as "free analogues" of p_t, g_t, and that a full theory can be developed, with central limiting theorems for all 4 laws, convolution semigroup results for all 4 laws too, and Lie group type results for all 4 laws too. And also, we will be back to the random matrices as well, with further results about them.

6e. Exercises

Exercises:

EXERCISE 6.30.

EXERCISE 6.31.

EXERCISE 6.32.

EXERCISE 6.33.

EXERCISE 6.34.

EXERCISE 6.35.

EXERCISE 6.36.

EXERCISE 6.37.

Bonus exercise.

CHAPTER 7

Unbounded operators

7a.

7b.

7c.

7d.

7e. Exercises

Exercises:

Exercise 7.1.

Exercise 7.2.

Exercise 7.3.

Exercise 7.4.

Exercise 7.5.

Exercise 7.6.

Exercise 7.7.

Exercise 7.8.

Bonus exercise.

CHAPTER 8

Some applications

8a.

8b.

8c.

8d.

8e. Exercises

Exercises:

EXERCISE 8.1.

EXERCISE 8.2.

EXERCISE 8.3.

EXERCISE 8.4.

EXERCISE 8.5.

EXERCISE 8.6.

EXERCISE 8.7.

EXERCISE 8.8.

Bonus exercise.

Part III

Compact operators

Shut up and let me go
This hurts, what I can't show
For the last time you have me in bits
Now shut up and let me go

CHAPTER 9

Functional analysis

9a. Normed spaces

We have seen so far the basic theory of bounded operators, in the arbitrary, normal and self-adjoint cases, and in a few other cases of interest. In this Part III we discuss a number of more specialized questions, for the most dealing with the compact operators, which are particularly close, conceptually speaking, to the usual complex matrices.

We have in fact considerably many interesting things that we can talk about, in this present Part III, and our choices will be as follows:

(1) We will first need a number of preliminaries, namely some basic functional analysis, to be explained in the present chapter, and some advanced linear algebra too, namely the singular value theorem for matrices, to be explained in the beginning of chapter 10.

(2) Motivated by this advanced linear algebra, we will first go on a lengthy discussion on the algebra of compact operators $K(H) \subset B(H)$, which for many advanced operator theory purposes is the correct generalization of the matrix algebra $M_N(\mathbb{C})$.

(3) Our discussion on the compact operators will feature as well some more specialized types of operators, $F(H) \subset B_1(H) \subset B_2(H) \subset K(H)$, with $F(H)$ being the finite rank ones, $B_1(H)$ being the trace class ones, and $B_2(H)$ being the Hilbert-Schmidt ones.

Getting started now, for a more advanced study of the linear operators we will need some further functional analysis knowledge, going beyond what we got away with, so far. Things here will be a bit abstract, but do not worry, all this will be quality mathematics, which is good to know, and which will have applications. Let us start with:

Definition 9.1. *A normed space is a complex vector space V, which can be finite or infinite dimensional, together with a map*

$$||.|| : V \to \mathbb{R}_+$$

called norm, subject to the following conditions:

(1) $||x|| = 0$ *implies* $x = 0$.
(2) $||\lambda x|| = |\lambda| \cdot ||x||$, *for any* $x \in V$, *and* $\lambda \in \mathbb{C}$.
(3) $||x + y|| \leq ||x|| + ||y||$, *for any* $x, y \in V$.

As a basic example here, which is finite dimensional, we have the space $V = \mathbb{C}^N$, with the norm on it being the usual length of the vectors, namely:

$$||x|| = \sqrt{\sum_i |x_i|^2}$$

Indeed, for this space (1) is clear, (2) is clear too, and (3) is something well-known, which is equivalent to the triangle inequality in $\mathbb{C}^N$, and which can be deduced from the Cauchy-Schwarz inequality. More on this, with some generalizations, in a moment.

Getting back now to the general case, we have the following result:

PROPOSITION 9.2. *Any normed vector space V is a metric space, with*

$$d(x, y) = ||x - y||$$

as distance. If this metric space is complete, we say that V is a Banach space.

PROOF. This follows from the definition of the metric spaces, as follows:

(1) The first distance axiom, $d(x, y) \geq 0$, and $d(x, y) = 0$ precisely when $x = y$, follows from the fact that the norm takes values in $\mathbb{R}_+$, and from $||x|| = 0 \implies x = 0$.

(2) The second distance axiom, which is the symmetry one, $d(x, y) = d(y, x)$, follows from our condition $||\lambda x|| = |\lambda| \cdot ||x||$, with $\lambda = -1$.

(3) As for the third distance axiom, which is the triangle inequality $d(x, y) \leq d(x, z) + d(y, z)$, this follows from our third norm axiom, namely $||x + y|| \leq ||x|| + ||y||$. □

Very nice all this, and it is possible to develop some general theory here, but before everything, however, we need more examples, besides $\mathbb{C}^N$ with its usual norm.

However, these further examples are actually quite tricky to construct, needing some inequality know-how. Let us start with a very basic result, as follows:

THEOREM 9.3 (Jensen). *Given a convex function $f : \mathbb{R} \to \mathbb{R}$, we have the following inequality, for any $x_1, \ldots, x_N \in \mathbb{R}$, and any $\lambda_1, \ldots, \lambda_N > 0$ summing up to 1,*

$$f(\lambda_1 x_1 + \ldots + \lambda_N x_N) \leq \lambda_1 f(x_1) + \ldots + \lambda_N x_N$$

with equality when $x_1 = \ldots = x_N$. In particular, by taking the weights λ_i to be all equal, we obtain the following inequality, valid for any $x_1, \ldots, x_N \in \mathbb{R}$,

$$f\left(\frac{x_1 + \ldots + x_N}{N}\right) \leq \frac{f(x_1) + \ldots + f(x_N)}{N}$$

and once again with equality when $x_1 = \ldots = x_N$. We have a similar statement holds for the concave functions, with all the inequalities being reversed.

Proof. This is indeed something quite routine, the idea being as follows:

(1) First, we can talk about convex functions in a usual, intuitive way, with this meaning by definition that the following inequality must be satisfied:

$$f\left(\frac{x+y}{2}\right)\leq\frac{f(x)+f(y)}{2}$$

(2) But this means, via a simple argument, by approximating numbers $t\in[0,1]$ by sums of powers 2^{-k}, that for any $t\in[0,1]$ we must have:

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)$$

Alternatively, via yet another simple argument, this time by doing some geometry with triangles, this means that we must have:

$$f\left(\frac{x_1+\ldots+x_N}{N}\right)\leq\frac{f(x_1)+\ldots+f(x_N)}{N}$$

But then, again alternatively, by combining the above two simple arguments, the following must happen, for any $\lambda_1,\ldots,\lambda_N>0$ summing up to 1:

$$f(\lambda_1x_1+\ldots+\lambda_Nx_N)\leq\lambda_1f(x_1)+\ldots+\lambda_Nx_N$$

(3) Summarizing, all our Jensen inequalities, at $N=2$ and at $N\in\mathbb{N}$ arbitrary, are equivalent. The point now is that, if we look at what the first Jensen inequality, that we took as definition for the convexity, means, this is simply equivalent to:

$$f''(x)\geq 0$$

(4) Thus, we are led to the conclusions in the statement, regarding the convex functions. As for the concave functions, the proof here is similar. Alternatively, we can say that f is concave precisely when $-f$ is convex, and get the results from what we have. □

As a basic application, that we actually already used in chapter 3, we have:

Theorem 9.4 (Young). *We have the following inequality,*

$$ab\leq\frac{a^p}{p}+\frac{b^q}{q}$$

valid for any $a,b\geq 0$, and any exponents $p,q>1$ satisfying $\frac{1}{p}+\frac{1}{q}=1$.

Proof. We use the logarithm function, which is concave on $(0,\infty)$, due to:

$$(\log x)''=\left(-\frac{1}{x}\right)'=-\frac{1}{x^2}$$

Thus we can apply the Jensen inequality, and we obtain in this way:

$$\begin{aligned}\log\left(\frac{a^p}{p}+\frac{b^q}{q}\right) &\geq \frac{\log(a^p)}{p}+\frac{\log(b^q)}{q}\\ &= \log(a)+\log(b)\\ &= \log(ab)\end{aligned}$$

Now by exponentiating, we obtain the Young inequality. □

Moving forward now, as a consequence of the Young inequality, we have:

THEOREM 9.5 (Hölder). *Assuming that $p,q\geq 1$ are conjugate, in the sense that*

$$\frac{1}{p}+\frac{1}{q}=1$$

we have the following inequality, valid for any two vectors $x,y\in\mathbb C^N$,

$$\sum_i|x_iy_i|\leq\left(\sum_i|x_i|^p\right)^{1/p}\left(\sum_i|y_i|^q\right)^{1/q}$$

with the convention that an ∞ exponent produces a $\max|x_i|$ quantity.

PROOF. This is something very standard, the idea being as follows:

(1) Assume first that we are dealing with finite exponents, $p,q\in(1,\infty)$. By linearity we can assume that x,y are normalized, in the following way:

$$\sum_i|x_i|^p=\sum_i|y_i|^q=1$$

In this case, we want to prove that the following inequality holds:

$$\sum_i|x_iy_i|\leq 1$$

For this purpose, we use the Young inequality, which gives, for any i:

$$|x_iy_i|\leq\frac{|x_i|^p}{p}+\frac{|y_i|^q}{q}$$

By summing now over $i=1,\ldots,N$, we obtain from this, as desired:

$$\begin{aligned}\sum_i|x_iy_i| &\leq \sum_i\frac{|x_i|^p}{p}+\sum_i\frac{|y_i|^q}{q}\\ &= \frac{1}{p}+\frac{1}{q}\\ &= 1\end{aligned}$$

(2) In the case $p=1$ and $q=\infty$, or vice versa, the inequality holds too, trivially, with the convention that an ∞ exponent produces a max quantity, according to:

$$\lim_{p\to\infty}\left(\sum_i|x_i|^p\right)^{1/p}=\max|x_i|$$

Thus, we are led to the conclusion in the statement. $\square$

As a consequence now of the Hölder inequality, we have:

THEOREM 9.6 (Minkowski). *Assuming $p\in[1,\infty]$, we have the inequality*

$$\left(\sum_i|x_i+y_i|^p\right)^{1/p}\leq\left(\sum_i|x_i|^p\right)^{1/p}+\left(\sum_i|y_i|^p\right)^{1/p}$$

for any two vectors $x,y\in\mathbb{C}^N$, with our usual conventions at $p=\infty$.

PROOF. We have indeed the following estimate, using the Hölder inequality, and the conjugate exponent $q\in[1,\infty]$, given by $1/p+1/q=1$:

$$\begin{aligned}\sum_i|x_i+y_i|^p&=\sum_i|x_i+y_i|\cdot|x_i+y_i|^{p-1}\\&\leq\sum_i|x_i|\cdot|x_i+y_i|^{p-1}+\sum_i|y_i|\cdot|x_i+y_i|^{p-1}\\&\leq\left(\sum_i|x_i|^p\right)^{1/p}\left(\sum_i|x_i+y_i|^{(p-1)q}\right)^{1/q}\\&+\left(\sum_i|y_i|^p\right)^{1/p}\left(\sum_i|x_i+y_i|^{(p-1)q}\right)^{1/q}\\&=\left[\left(\sum_i|x_i|^p\right)^{1/p}+\left(\sum_i|y_i|^p\right)^{1/p}\right]\left(\sum_i|x_i+y_i|^p\right)^{1-1/p}\end{aligned}$$

Here we have used the following fact, at the end:

$$\frac{1}{p}+\frac{1}{q}=1\implies\frac{1}{q}=\frac{p-1}{p}\implies(p-1)q=p$$

Now by dividing both sides by the last quantity at the end, we obtain:

$$\left(\sum_i|x_i+y_i|^p\right)^{1/p}\leq\left(\sum_i|x_i|^p\right)^{1/p}+\left(\sum_i|y_i|^p\right)^{1/p}$$

Thus, we are led to the conclusion in the statement. $\square$

Good news, done with inequalities, and as a consequence of the above results, and more specifically of the Minkowski inequality obtained above, we can formulate:

THEOREM 9.7. *Given an exponent $p \in [1,\infty]$, the formula*

$$||x||_p = \left(\sum_i |x_i|^p\right)^{1/p}$$

with usual conventions at $p=\infty$, defines a norm on $\mathbb{C}^N$, making it a Banach space.

PROOF. Here the normed space assertion follows from the Minkowski inequality, established above, and the Banach space assertion is trivial, because our space being finite dimensional, by standard linear algebra all the Cauchy sequences converge. □

Very nice all this, but you might wonder at this point, what is the relation of all this with functions. In answer, Theorem 9.7 can be reformulated as follows:

THEOREM 9.8. *Given an exponent $p \in [1,\infty]$, the formula*

$$||f||_p = \left(\int |f(x)|^p\right)^{1/p}$$

with usual conventions at $p=\infty$, defines a norm on the space of functions

$$f : \{1,\ldots,N\} \to \mathbb{C}$$

making it a Banach space.

PROOF. This is a just fancy reformulation of Theorem 9.7, by using the fact that the space formed by the functions $f : \{1,\ldots,N\} \to \mathbb{C}$ is canonically isomorphic to $\mathbb{C}^N$, in the obvious way, and by replacing the sums from the $\mathbb{C}^N$ context with integrals with respect to the counting measure on $\{1,\ldots,N\}$, in the function context. □

9b. Banach spaces

Moving now towards infinite dimensions and more standard analysis, the idea will be that of extending Theorem 9.8 to the arbitrary measured spaces. Let us start with:

THEOREM 9.9. *Given an exponent $p \in [1,\infty]$, the formula*

$$||x||_p = \left(\sum_i |x_i|^p\right)^{1/p}$$

with usual conventions at $p=\infty$, defines a norm on the space of sequences

$$l^p = \left\{ (x_i)_{i\in\mathbb{N}} \Big| \sum_i |x_i|^p < \infty \right\}$$

making it a Banach space.

Proof. As before with the finite sequences, the normed space assertion follows from the Minkowski inequality, established above, which extends without problems to the case of the infinite sequences, and with the Banach space assertion being clear too. □

We can unify and generalize what we have, in the following way:

Theorem 9.10. *Given a discrete measured space X, and an exponent $p\in[1,\infty]$,*

$$||f||_p=\left(\int_X|f(x)|^p\right)^{1/p}$$

with usual conventions at $p=\infty$, defines a norm on the space of functions

$$l^p(X)=\left\{f:X\to\mathbb C\Big|\int_X|f(x)|^p<\infty\right\}$$

making it a Banach space.

Proof. This is just a fancy reformulation of what we have:

(1) The case where X is finite corresponds to Theorem 9.8.

(2) The case where X is countable corresponds to Theorem 9.9.

(3) Finally, the case where X is uncountable is easy to deal with too, by using the same arguments as in the countable case. □

In order to further extend the above result, to the case of the arbitrary measured spaces X, which are not necessarily discrete, let us start with:

Theorem 9.11. *Given two functions $f,g:X\to\mathbb C$ and an exponent $p\geq 1$, we have*

$$\left(\int_X|f+g|^p\right)^{1/p}\leq\left(\int_X|f|^p\right)^{1/p}+\left(\int_X|g|^p\right)^{1/p}$$

called Minkowski inequality. Also, assuming that $p,q\geq 1$ satisfy $1/p+1/q=1$, we have

$$\int_X|fg|\leq\left(\int_X|f|^p\right)^{1/p}\left(\int_X|g|^q\right)^{1/q}$$

called Hölder inequality. These inequalities hold as well for ∞ values of the exponents.

Proof. This is very standard, exactly as in the case of sequences, finite or not, but since the above inequalities are really very general and final, here are the details:

(1) Let us first prove Hölder, in the case of finite exponents, $p,q\in(1,\infty)$. By linearity we can assume that f,g are normalized, in the following way:

$$\int_X|f|^p=\int_X|g|^q=1$$

In this case, we want to prove that the following inequality holds:

$$\int_X |fg| \leq 1$$

For this purpose, we use the Young inequality, which gives, for any $x \in X$:

$$|f(x)g(x)| \leq \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q}$$

By integrating now over $x \in X$, we obtain from this, as desired:

$$\begin{aligned}\int_X |fg| &\leq \int_X \frac{|f(x)|^p}{p} + \int_X \frac{|g(x)|^q}{q}\\ &= \frac{1}{p} + \frac{1}{q}\\ &= 1\end{aligned}$$

(2) Let us prove now Minkowski, again in the finite exponent case, $p \in (1, \infty)$. We have the following estimate, using the Hölder inequality, and the conjugate exponent:

$$\begin{aligned}\int_X |f+g|^p &= \int_X |f+g| \cdot |f+g|^{p-1}\\ &\leq \int_X |f| \cdot |f+g|^{p-1} + \int_X |g| \cdot |f+g|^{p-1}\\ &\leq \left(\int_X |f|^p\right)^{1/p} \left(\int_X |f+g|^{(p-1)q}\right)^{1/q}\\ &+ \left(\int_X |g|^p\right)^{1/p} \left(\int_X |f+g|^{(p-1)q}\right)^{1/q}\\ &= \left[\left(\int |f|^p\right)^{1/p} + \left(\int_X |g|^p\right)^{1/p}\right] \left(\int_X |f+g|^p\right)^{1-1/p}\end{aligned}$$

Thus, we are led to the Minkowski inequality in the statement.

(3) Finally, in the infinite exponent cases we have similar results, which are trivial this time, with the convention that an ∞ exponent produces an essential supremum, according to the following formula, which follows from the measure theory that we know:

$$\lim_{p \to \infty} \left(\int_X |f|^p\right)^{1/p} = \text{ess sup}|f|$$

Thus, we are led to the conclusion in the statement. □

We can now extend Theorem 9.10, into something very general, as follows:

THEOREM 9.12. *Given a measured space X, and $p \in [1,\infty]$, the following space, with the convention that functions are identified up to equality almost everywhere,*

$$L^p(X) = \left\{ f : X \to \mathbb{C} \Big| \int_I |f(x)|^p dx < \infty \right\}$$

is a vector space, and the following quantity

$$||f||_p = \left(\int_X |f(x)|^p \right)^{1/p}$$

is a norm on it, making it a Banach space.

PROOF. This follows indeed from Theorem 9.11, with due attention to the null sets, and this because of the first normed space axiom, namely:

$$||x|| = 0 \implies x = 0$$

To be more precise, in order for this axiom to hold, we must identify the functions up to equality almost everywhere, as indicated in the statement. □

Very nice all this. So, we have our examples of Banach spaces, which look definitely interesting, and related to analysis. In the remainder of this chapter we will develop some general Banach space theory, and apply it to the above L^p spaces.

Getting now to work, as a first result about the abstract normed spaces, we would like to talk about the linear maps $T : V \to W$. We first have here:

PROPOSITION 9.13. *For a linear map $T : V \to W$, the following conditions are equivalent, and if they hold, we say that T is bounded:*

1. *T is continuous.*
2. *T is continuous at 0.*
3. *T maps the unit ball of V into something bounded.*
4. *T is bounded, in the sense that $||T|| = \sup_{||x||=1} ||Tx||$ is finite.*

PROOF. Here the equivalences $(1) \iff (2) \iff (3) \iff (4)$ all follow from definitions, by using the linearity of T, and performing various rescalings, and with the number $||T||$ needed in (4) being the bound coming from (3). □

With the above result in hand, we can now formulate:

THEOREM 9.14. *Given two Banach spaces V, W, the bounded linear maps*

$$T : V \to W$$

form a linear space $B(V,W)$, on which the following quantity is a norm,

$$||T|| = \sup_{||x||=1} ||Tx||$$

making $B(V,W)$ a Banach space. When $V = W$, we obtain a Banach algebra.

PROOF. All this is very standard, and in the case $V = W$, for simplifying, which is the one that matters the most, the proof goes as follows:

(1) The fact that we have indeed an algebra, satisfying the product condition in the statement, follows from the following estimates, which are all elementary:

$$||S+T|| \leq ||S|| + ||T||$$
$$||\lambda T|| = |\lambda| \cdot ||T||$$
$$||ST|| \leq ||S|| \cdot ||T||$$

(2) Regarding now the last assertion, if $\{T_n\} \subset B(V)$ is Cauchy then $\{T_n x\}$ is Cauchy for any $x \in V$, so we can define the limit $T = \lim_{n\to\infty} T_n$ by setting:

$$Tx = \lim_{n\to\infty} T_n x$$

Let us first check that the application $x \to Tx$ is linear. We have:

$$\begin{aligned} T(x+y) &= \lim_{n\to\infty} T_n(x+y) \\ &= \lim_{n\to\infty} T_n(x) + T_n(y) \\ &= \lim_{n\to\infty} T_n(x) + \lim_{n\to\infty} T_n(y) \\ &= T(x) + T(y) \end{aligned}$$

Similarly, we have as well the following computation:

$$\begin{aligned} T(\lambda x) &= \lim_{n\to\infty} T_n(\lambda x) \\ &= \lambda \lim_{n\to\infty} T_n(x) \\ &= \lambda T(x) \end{aligned}$$

Thus we have a linear map $T : A \to A$. Now observe that we have:

$$\begin{aligned} & ||T_n - T_m|| \leq \varepsilon\ ,\ \forall n,m \geq N \\ \Longrightarrow\ & ||T_n x - T_m x|| \leq \varepsilon\ ,\ \forall ||x|| = 1\ ,\ \forall n,m \geq N \\ \Longrightarrow\ & ||T_n x - Tx|| \leq \varepsilon\ ,\ \forall ||x|| = 1\ ,\ \forall n \geq N \\ \Longrightarrow\ & ||T_N x - Tx|| \leq \varepsilon\ ,\ \forall ||x|| = 1 \\ \Longrightarrow\ & ||T_N - T|| \leq \varepsilon \end{aligned}$$

As a first consequence, we obtain $T \in B(V)$, because we have:

$$\begin{aligned} ||T|| &= ||T_N + (T - T_N)|| \\ &\leq ||T_N|| + ||T - T_N|| \\ &\leq ||T_N|| + \varepsilon \\ &< \infty \end{aligned}$$

As a second consequence, we obtain $T_N \to T$ in norm, and we are done. □

As a basic example for the above construction, in the case where both our spaces are finite dimensional, $V=\mathbb{C}^N$ and $W=\mathbb{C}^M$, with $N,M<\infty$, we obtain a matrix space:

$$B(\mathbb{C}^N,\mathbb{C}^M)=M_{M\times N}(\mathbb{C})$$

More on this later. On the other hand, of particular interest is as well the case $W=\mathbb{C}$ of the above construction, which leads to the following result:

THEOREM 9.15. *Given a Banach space V, its dual space, constructed as*

$$V^*=\Big\{f:V\to\mathbb{C},\ \text{linear and bounded}\Big\}$$

is a Banach space too, with norm given by:

$$||f||=\sup_{||x||=1}|f(x)|$$

When V is finite dimensional, we have $V\simeq V^*$.

PROOF. This is clear indeed from Theorem 9.14, because we have:

$$V^*=B(V,\mathbb{C})$$

Thus, we are led to the conclusions in the statement. □

In order to better understand the linear forms, we will need:

THEOREM 9.16 (Hahn-Banach). *Given a Banach space V, the following happen:*

(1) *Given $x\in V-\{0\}$, there exists $f\in V^*$ with $f(x)\neq 0$.*
(2) *Given a subspace $W\subset V$, any $f\in W^*$ extends into a $\tilde{f}\in V^*$, of same norm.*

PROOF. This is something quite tricky, the idea being as follows:

(1) As a first observation, (1) is weaker than (2).

(2) As a second observation, (2) can be proved in finite dimensions by using a direct sum decomposition $V=W\oplus U$, and setting $\tilde{f}\in V^*$ to be zero on U.

(3) In general, the proof is quite similar, by using the same ideas. To be more precise, we can first prove (1), and then by using using this, prove (2) as well. □

We can now formulate a key result, as follows:

THEOREM 9.17. *Given a Banach space V, we have an embedding as follows,*

$$V\subset V^{**}$$

which is an isomorphism in finite dimensions, and for the l^p and L^p spaces too.

Proof. There are several things going on here, the idea being as follows:

(1) The fact that we have indeed a vector space embedding $V\subset V^{**}$ is clear from definitions, the formula of this embedding being as follows:

$$i(v)[f]=f(v)$$

(2) However, the fact that this embedding $V\subset V^{**}$ is isometric is something more subtle, which requires the use of the Hahn-Banach result from Theorem 9.16.

(3) Next, the fact that we have $V=V^{**}$ in finite dimensions is clear.

(4) Regarding now the formula $V=V^{**}$ for the various l^p and L^p spaces, this is something quite tricky. Let us start with the simplest case, that of the space $V=l^2$. We know that this space is given by definition by the following formula:

$$l^2=\left\{(x_i)_{i\in\mathbb N}\Big|\sum_i x_i^2<\infty\right\}$$

Now let us look for linear forms $f:l^2\to\mathbb C$. By linearity such a linear form must appear as follows, for certain scalars $a_i\in\mathbb C$, which must be such that f is well-defined:

$$f\left((x_i)_i\right)=\sum_i a_ix_i$$

But, what does the fact that f is well-defined mean? In answer, this means that the values of f must all converge, which in practice means that we must have:

$$\sum_i x_i^2<\infty\implies\left|\sum_i a_ix_i\right|<\infty$$

Moreover, we would like our linear form $f:l^2\to\mathbb C$ to be bounded, and by denoting by $A=||f||<\infty$ the minimal bound, this means that we must have:

$$\left|\sum_i a_ix_i\right|\leq A\sqrt{\sum_i x_i^2}$$

Now recall that the Cauchy-Schwarz inequality tells us that we have:

$$\left|\sum_i a_ix_i\right|\leq\sqrt{\sum_i a_i^2}\cdot\sqrt{\sum_i x_i^2}$$

Thus, the linear form $f:l^2\to\mathbb C$ associated to any $a=(a_i)\in l^2$ will do. Moreover, conversely, by examining the proof of Cauchy-Schwarz, we conclude that this condition $a=(a_i)\in l^2$ is in fact necessary. Thus, we have proved that we have:

$$(l^2)^*=l^2$$

But this gives the $V = V^{**}$ result in the statement for our space $V = l^2$, because by dualizing one more time we obtain, as desired:

$$(l^2)^{**} = (l^2)^* = l^2$$

(5) Getting now to more complicated spaces, let us look, more generally, at $L^2(X)$. We know that this space is given by definition by the following formula:

$$L^2(X) = \left\{ f : X \to \mathbb{C} \Big| \int_X f(x)^2 dx < \infty \right\}$$

As before, when looking for linear forms $\varphi : L^2(X) \to \mathbb{C}$, by linearity, and with some mesure theory helping, our forms must appear via a formula as follows:

$$\varphi(f) = \int_X f(x)\varphi(x)dx$$

Now in order for this integral to converge, as for our map $\varphi : L^2(X) \to \mathbb{C}$ to be well-defined, and with the additional requirement that φ must be actually bounded, we must have an inequality as follows, for a certain positive constant $A < \infty$:

$$\left| \int_X f(x)\varphi(x)dx \right| \leq A \sqrt{\int_X f(x)^2 dx}$$

Now recall that the Cauchy-Schwarz inequality tells us that we have:

$$\left| \int_X f(x)\varphi(x)dx \right| \leq \sqrt{\int_X \varphi(x)^2 dx} \cdot \sqrt{\int_X f(x)^2 dx}$$

Thus, the linear form $\varphi : L^2(X) \to \mathbb{C}$ associated to any $\varphi \in L^2(X)$ will do. Moreover, conversely, by examining the proof of Cauchy-Schwarz, we conclude that this condition $\varphi \in L^2(X)$ is in fact necessary. Thus, we have proved that we have:

$$(L^2)^* = L^2$$

But this gives the $V = V^{**}$ result in the statement for our space $V = L^2$, because by dualizing one more time we obtain, as desired:

$$(L^2)^{**} = (L^2)^* = L^2$$

(6) Before getting further, let us mention that, more generally with respect to our l^2, L^2 computations, we have the following formula, valid for any Hilbert space H:

$$H^* \simeq \bar{H}$$

To be more precise, we can talk about Hilbert spaces, as being those Banach spaces whose norm comes from a scalar product, via $||x|| = \sqrt{< x, x >}$, and we will discuss this in the next chapter. And, the point is that, as we will see in the next chapter, any such Hilbert space has an orthogonal basis, which in practice means that we can write:

$$H = l^2(I)$$

Thus, we are apparently led to $H^* = H$, but this is not exactly true, because the correspondence $a \to f$ that we constructed in (4), and that we would like to rely upon, is antilinear, instead of being linear. Of course, this was not a problem in the context of (4), and nor is this a problem, for the same reasons, for a Hilbert space H given with a basis, and so with an explicit isomorphism $H = l^2(I)$, as above. However, when talking about abstract Hilbert spaces H, coming without a basis, we must correct this, into:

$$H^* \simeq \bar{H}$$

But this gives the $V = V^{**}$ result in the statement for our Hilbert space $V = H$, because by dualizing one more time we obtain, as desired:

$$H^{**} = (\bar{H})^* = \bar{\bar{H}} = H$$

So long for l^2, L^2 spaces, and more general Hilbert spaces H. We will be back to this in the next chapter, when systematically discussing the Hilbert spaces.

(7) Moving ahead now, let us go back to the l^p spaces, as in (4), but now with general exponents $p \in [1, \infty]$, instead of $p = 2$. The space l^p is by definition given by:

$$l^p = \left\{ (x_i)_{i \in \mathbb{N}} \Big| \sum_i |x_i|^p < \infty \right\}$$

Now by arguing as in (4), a linear form $f : l^p \to \mathbb{C}$ must come as follows:

$$f\left((x_i)_i\right) = \sum_i a_i x_i$$

To be more precise, here $a_i \in \mathbb{C}$ are certain scalars, which are subject to an inequality as follows, for a certain constant $A < \infty$, making f well-defined, and bounded:

$$\left| \sum_i a_i x_i \right| \leq A \left(\sum_i |x_i|^p \right)^{1/p}$$

Now recall that the Hölder inequality tells us that we have, with $\frac{1}{p} + \frac{1}{q} = 1$:

$$\left| \sum_i a_i x_i \right| \leq \left(\sum_i |x_i|^p \right)^{1/p} \left(\sum_i |a_i|^p \right)^{1/q}$$

Thus, the linear form $f : l^2 \to \mathbb{C}$ associated to any element $a = (a_i) \in l^q$ will do. Moreover, conversely, by examining the proof of Hölder, we conclude that this condition

$a = (a_i) \in l^q$ is in fact necessary. Thus, we have proved that we have:

$$(l^p)^* = l^q$$

But this gives the $V = V^{**}$ result in the statement for our space $V = l^p$, because by dualizing one more time we obtain, as desired:

$$(l^p)^{**} = (l^q)^* = l^q$$

(8) All this is very nice, and time now to generalize everything that we know, by looking at the general spaces $L^p(X)$, with $p \in [1, \infty]$. These spaces are given by:

$$L^p(X) = \left\{ f : X \to \mathbb{C} \Big| \int_X |f(x)|^p dx < \infty \right\}$$

As before in (5), when looking for linear forms $\varphi : L^p(X) \to \mathbb{C}$, by linearity, and with some mesure theory helping, our forms must appear via a formula as follows:

$$\varphi(f) = \int_X f(x)\varphi(x)dx$$

Now in order for this integral to converge, as for our map $\varphi : L^p(X) \to \mathbb{C}$ to be well-defined, and with the additional requirement that φ must be actually bounded, we must have an inequality as follows, for a certain positive constant $A < \infty$:

$$\left| \int_X f(x)\varphi(x)dx \right| \leq A \left(\int_X |f(x)|^p dx \right)^{1/p}$$

Now recall that the Hölder inequality tells us that we have, with $\frac{1}{p} + \frac{1}{q} = 1$:

$$\left| \int_X f(x)\varphi(x)dx \right| \leq \left(\int_X |f(x)|^p dx \right)^{1/p} \left(\int_X |\varphi(x)|^q dx \right)^{1/q}$$

Thus, the linear form $\varphi : L^p(X) \to \mathbb{C}$ associated to any function $\varphi \in L^q(X)$ will do. Moreover, conversely, by examining the proof of Hölder, we conclude that this condition $\varphi \in L^q(X)$ is in fact necessary. Thus, we have proved that we have:

$$(L^p)^* = L^q$$

But this gives the $V = V^{**}$ result in the statement for our space $V = L^p$, because by dualizing one more time we obtain, as desired:

$$(L^p)^{**} = (L^q)^* = L^p$$

(9) Finally, let us mention that not all Banach spaces satisfy $V = V^{**}$, with a basic counterexample here being the space c_0 of sequences $x_n \in \mathbb{C}$ satisfying $x_n \to 0$, with the sup norm. Indeed, computations show that we have the following formulae:

$$c_0^* = l^1 \quad , \quad (l^1)^* = l^\infty$$

Thus, in this case $V \subset V^{**}$ is the embedding $c_0 \subset l^\infty$, which is not an isomorphism. $\square$

9c. Abstract results

Getting now to more advanced theory, we have many non-trivial things that can be said, about the Banach spaces, with a quick list here being as follows:

(1) The Baire theorem.

(2) The Banach-Steinhaus theorem.

(3) The open mapping theorem.

(4) The closed graph theorem.

9d. Tensor products

There are many interesting questions, regarding the tensor products of Banach spaces. On one hand we have the question of coming with a norm on a tensor product. On the other hand we have a number of concrete questions, related to Fubini.

9e. Exercises

Exercises:

Exercise 9.18.

Exercise 9.19.

Exercise 9.20.

Exercise 9.21.

Exercise 9.22.

Exercise 9.23.

Exercise 9.24.

Exercise 9.25.

Bonus exercise.

CHAPTER 10

Compact operators

10a. Linear algebra

Let us start with some linear algebra. As a first construction, that we would like to generalize to the matrix setting, we have the construction of the modulus, as follows:

$$|z|=\sqrt{z\bar{z}}$$

The point now is that, as we already know from chapter 1, we can indeed generalize this construction, by using the spectral theorem for the normal matrices, as follows:

Theorem 10.1. *Given a matrix $A\in M_N(\mathbb{C})$, we can construct a matrix $|A|$ as follows, by using the fact that A^*A is diagonalizable, with positive eigenvalues:*

$$|A|=\sqrt{A^*A}$$

This matrix $|A|$ is then positive, and its square is $|A|^2=A$. In the case $N=1$, we obtain in this way the usual absolute value of the complex numbers.

Proof. Consider indeed the matrix in the statement A^*A, which is normal. According to the spectral theorem for the normal matrices, that we know well from chapter 1, we can diagonalize this matrix as follows, with $U\in U_N$, and with D diagonal:

$$A=UDU^*$$

From $A^*A\geq 0$ we obtain $D\geq 0$. But this means that the entries of D are real, and positive. Thus we can extract the square root $\sqrt{D}$, and then set:

$$\sqrt{A^*A}=U\sqrt{D}U^*$$

Thus, we are basically done. Indeed, if we call this latter matrix $|A|$, then we are led to the conclusions in the statement. Finally, the last assertion is clear from definitions. □

As a comment here, it is possible to talk as well about $\sqrt{AA^*}$, which is in general different from $\sqrt{A^*A}$. Note that when A is normal, there is no issue, because we have:

$$AA^*=A^*A\implies\sqrt{AA^*}=\sqrt{A^*A}$$

Regarding now the polar decomposition formula, let us start with a weak version of this statement, regarding the invertible matrices, as follows:

THEOREM 10.2. *We have the polar decomposition formula*

$$A = U\sqrt{A^*A}$$

with U being a unitary, for any $A \in M_N(\mathbb{C})$ invertible.

PROOF. According to our definition of the modulus, $|A| = \sqrt{A^*A}$, we have:

$$\begin{aligned} < |A|x, |A|y > &= < x, |A|^2 y > \\ &= < x, A^*Ay > \\ &= < Ax, Ay > \end{aligned}$$

Thus we can define a unitary operator $U \in M_N(\mathbb{C})$ by the following formula:

$$U(|A|x) = Ax$$

But this formula shows that we have $A = U|A|$, as desired. □

Observe that we have uniqueness in the above result, in what regards the choice of the unitary $U \in M_N(\mathbb{C})$, due to the fact that we can write this unitary as follows:

$$U = A(\sqrt{A^*A})^{-1}$$

More generally now, we have the following result:

THEOREM 10.3. *We have the polar decomposition formula*

$$A = U\sqrt{A^*A}$$

with U being a partial isometry, for any $A \in M_N(\mathbb{C})$.

PROOF. As before, in the proof of Theorem 10.2, dealing with the invertible matrix case, we have the following equality, for any two vectors $x, y \in \mathbb{C}^N$:

$$< |A|x, |A|y > = < Ax, Ay >$$

We conclude that the following linear application is well-defined, and isometric:

$$U : Im|A| \to Im(A) \quad , \quad |A|x \to Ax$$

But now we can further extend this linear isometric map U into a partial isometry $U : \mathbb{C}^N \to \mathbb{C}^N$, in a straightforward way, by setting:

$$Ux = 0 \quad , \quad \forall x \in Im|A|^\perp$$

And the point is that, with this convention, the result follows. □

Let us discuss now the singular value theorem, which is a key result in linear algebra, valid for any matrix. This theorem can be formulated, a bit abstractly, as follows:

Theorem 10.4. *We can write the action of any matrix $A \in M_N(\mathbb{C})$ in the following form, with $\{e_n\}$, $\{f_n\}$ being orthonormal families, and with $\lambda_n \searrow 0$:*

$$A(x) = \sum_n \lambda_n < x, e_n > f_n$$

The numbers λ_n, called singular values of A, are the eigenvalues of the modulus $|A|$. In fact, the polar decomposition of A is given by $A = U|A|$, with

$$|A|(x) = \sum_n \lambda_n < x, e_n > e_n$$

and with U being given by $Ue_n = f_n$, and $U = 0$ on the complement of span(e_i).

Proof. This basically comes from what we already have, as follows:

(1) Given two orthonormal families $\{e_n\}$, $\{f_n\}$, and a sequence of real numbers $\lambda_n \searrow 0$, consider the linear map given by the formula in the statement, namely:

$$A(x) = \sum_n \lambda_n < x, e_n > f_n$$

The adjoint of this linear map is the given by the following formula:

$$A^*(x) = \sum_n \lambda_n < x, f_n > e_n$$

Thus, when composing A^* with A, we obtain the following linear map:

$$A^*A(x) = \sum_n \lambda_n^2 < x, e_n > e_n$$

Now by extracting the square root, we obtain the formula in the statement, namely:

$$|A|(x) = \sum_n \lambda_n < x, e_n > e_n$$

(2) Conversely, consider a matrix $A \in M_N(\mathbb{C})$. Then A^*A is self-adjoint, so we have a formula as follows, with $\{e_n\}$ being a certain orthonormal family, and with $\lambda_n \searrow 0$:

$$A^*A(x) = \sum_n \lambda_n^2 < x, e_n > e_n$$

By extracting the square root we obtain the formula of $|A|$ in the statement, namely:

$$|A|(x) = \sum_n \lambda_n < x, e_n > e_n$$

Moreover, with $U(e_n) = f_n$, we obtain a second orthonormal family, $\{f_n\}$, such that:

$$A(x) = U|T| = \sum_n \lambda_n < x, e_n > f_n$$

Thus, our matrix $A \in M_N(\mathbb{C})$ appears indeed as in the statement. □

As a technical remark now, it is possible to slightly improve a part of the above statement. Consider indeed a linear map of the following form, with $\{e_n\}$, $\{f_n\}$ being orthonormal families as before, and with $\lambda_n \to 0$ being now complex numbers:

$$A(x) = \sum_n \lambda_n < x, e_n > f_n$$

The adjoint of this linear map is the given by the following formula:

$$A^*(x) = \sum_n \bar{\lambda}_n < x, f_n > e_n$$

Thus, when composing A^* with A, we obtain the following linear map:

$$A^*A(x) = \sum_n |\lambda_n|^2 < x, e_n > e_n$$

Now by extracting the square root, we conclude that the polar decomposition of A is given by $A = U|A|$, with the modulus $|A|$ being as follows:

$$|A|(x) = \sum_n |\lambda_n| < x, e_n > e_n$$

As for the partial isometry U, this is given by $Ue_n = w_n f_n$, and $U = 0$ on the complement of $span(e_i)$, where $w_n \in \mathbb{T}$ are such that $\lambda_n = |\lambda_n| w_n$.

As already mentioned in the above, there are many possible applications of the singular value theorem. We will be back to this, on several occasions, in what follows.

10b. Finite rank operators

Back now to infinite dimensions, let us start with a basic definition, as follows:

DEFINITION 10.5. *An operator $T \in B(H)$ is said to be of finite rank if its image*

$$Im(T) \subset H$$

is finite dimensional. The set of such operators is denoted $F(H)$.

There are many interesting examples of finite rank operators, the most basic ones being the finite rank projections, on the finite dimensional subspaces $K \subset H$. Observe also that in the case where H is finite dimensional, any operator $T \in B(H)$ is automatically of finite rank. In general, this is of course wrong, but we have the following result:

THEOREM 10.6. *The set of finite rank operators*

$$F(H) \subset B(H)$$

is a two-sided $*$-ideal.

PROOF. We have several assertions to be proved, the idea being as follows:

(1) It is clear from definitions that $F(H)$ is indeed a vector space, with this due to the following formulae, valid for any $S,T\in B(H)$, which are both clear:

$$\dim(Im(S+T))\leq\dim(Im(S))+\dim(Im(T))$$

$$\dim(Im(\lambda T))=\dim(Im(T))$$

(2) Let us prove now that $F(H)$ is stable under $*$. Given $T\in F(H)$, we can regard it as an invertible operator between finite dimensional Hilbert spaces, as follows:

$$T:(\ker T)^\perp\to Im(T)$$

We conclude from this that we have the following dimension equality:

$$\dim((\ker T)^\perp)=\dim(Im(T))$$

Our claim now, in relation with our problem, is that we have equalities as follows:

$$\begin{aligned}\dim(Im(T^*)) &= \dim(\overline{Im(T^*)})\\ &= \dim((\ker T)^\perp)\\ &= \dim(Im(T))\end{aligned}$$

Indeed, the third equality is the one above, and the second equality is something that we know too, from chapter 2. Now by combining these two equalities we deduce that $Im(T^*)$ is finite dimensional, and so the first equality holds as well. Thus, our equalities are proved, and this shows that we have $T^*\in F(H)$, as desired.

(3) Finally, regarding the ideal property, this follows from the following two formulae, valid for any $S,T\in B(H)$, which are once again clear from definitions:

$$\dim(Im(ST))\leq\dim(Im(T))$$

$$\dim(Im(TS))\leq\dim(Im(T))$$

Thus, we are led to the conclusion in the statement. □

10c. Compact operators

Let us discuss now the compact operators, which will be the main topic of discussion, for the present chapter. These are best introduced as follows:

DEFINITION 10.7. *An operator $T\in B(H)$ is said to be compact if the closed set*

$$\overline{T(B_1)}\subset H$$

is compact, where $B_1\subset H$ is the unit ball. The set of such operators is denoted $K(H)$.

Equivalently, an operator $T \in B(H)$ is compact when for any sequence $\{x_n\} \subset B_1$, or more generally for any bounded sequence $\{x_n\} \subset H$, the sequence $\{T(x_n)\}$ has a convergence subsequence. We will see later some further criteria of compactness.

In finite dimensions any operator is compact. In general, as a first observation, any finite rank operator is compact. We have in fact the following result:

PROPOSITION 10.8. *Any finite rank operator is compact,*

$$F(H) \subset K(H)$$

and the finite rank operators are dense inside the compact operators.

PROOF. The first assertion is clear, because if $Im(T)$ is finite dimensional, then the following subset is closed and bounded, and so it is compact:

$$\overline{T(B_1)} \subset Im(T)$$

Regarding the second assertion, let us pick a compact operator $T \in K(H)$, and a number $\varepsilon > 0$. By compactness of T we can find a finite set $S \subset B_1$ such that:

$$T(B_1) \subset \bigcup_{x \in S} B_\varepsilon(Tx)$$

Consider now the orthogonal projection P onto the following finite dimensional space:

$$E = span\left(Tx\Big|x \in S\right)$$

Since the set S is finite, this space E is finite dimensional, and so P is of finite rank, $P \in F(H)$. Now observe that for any norm one $y \in H$ and any $x \in S$ we have:

$$\begin{aligned}
||Ty - Tx||^2 &= ||Ty - PTx||^2 \\
&= ||Ty - PTy + PTy - PTx||^2 \\
&= ||Ty - PTy||^2 + ||PTx - PTy||^2
\end{aligned}$$

Now by picking $x \in S$ such that the ball $B_\varepsilon(Tx)$ covers the point Ty, we conclude from this that we have the following estimate:

$$||Ty - PTy|| \leq ||Ty - Tx|| \leq \varepsilon$$

Thus we have $||T - PT|| \leq \varepsilon$, which gives the density result. □

Quite remarkably, the set of compact operators is closed, and we have:

THEOREM 10.9. *The set of compact operators*

$$K(H) \subset B(H)$$

is a closed two-sided $*$-ideal.

PROOF. We have several assertions here, the idea being as follows:

(1) It is clear from definitions that $K(H)$ is indeed a vector space, with this due to the following formulae, valid for any $S,T\in B(H)$, which are both clear:

$$(S+T)(B_1)\subset S(B_1)+T(B_1)$$

$$(\lambda T)(B_1)=|\lambda|\cdot T(B_1)$$

(2) In order to prove now that $K(H)$ is closed, assume that a sequence $T_n\in K(H)$ converges to $T\in B(H)$. Given $\varepsilon>0$, let us pick $N\in\mathbb N$ such that:

$$||T-T_N||\leq\varepsilon$$

By compactness of T_N we can find a finite set $S\subset B_1$ such that:

$$T_N(B_1)\subset\bigcup_{x\in S}B_\varepsilon(T_Nx)$$

We conclude that for any $y\in B_1$ there exists $x\in S$ such that:

$$\begin{aligned}||Ty-Tx|| &\leq ||Ty-T_Ny||+||T_Ny-T_Nx||+||T_Nx-Tx||\\ &\leq \varepsilon+\varepsilon+\varepsilon\\ &= 3\varepsilon\end{aligned}$$

Thus, we have an inclusion as follows, with $S\subset B_1$ being finite:

$$T(B_1)\subset\bigcup_{x\in S}B_{3\varepsilon}(Tx)$$

But this shows that our limiting operator T is compact, as desired.

(3) Regarding now the fact that $K(H)$ is stable under involution, this follows from Theorem 10.6, Proposition 10.8 and (2). Indeed, by using Proposition 10.7, given $T\in K(H)$ we can write it as a limit of finite rank operators, as follows:

$$T=\lim_{n\to\infty}T_n$$

Now by applying the adjoint, we obtain that we have as well:

$$T^*=\lim_{n\to\infty}T_n^*$$

The point now is that we know, as a consequence of Theorem 10.6, that the operators T_n^* are of finite rank, and so are compact by Proposition 10.8. Thus, by using (2) we obtain that their limit T^* is compact too, as desired.

(4) Finally, regarding the ideal property, this follows from the following two formulae, valid for any $S,T\in B(H)$, which are once again clear from definitions:

$$(ST)(B_1)=S(T(B_1))$$

$$(TS)(B_1)\subset||S||\cdot T(B_1)$$

Thus, we are led to the conclusion in the statement. □

Here is now a second key result regarding the compact operators:

THEOREM 10.10. *A bounded operator $T \in B(H)$ is compact precisely when*

$$Te_n \to 0$$

for any orthonormal system $\{e_n\} \subset H$.

PROOF. We have two implications to be proved, the idea being as follows:

"$\implies$" Assume that T is compact. By contradiction, assume $Te_n \not\to 0$. This means that there exists $\varepsilon > 0$ and a subsequence satisfying $||Te_{n_k}|| > \varepsilon$, and by replacing $\{e_n\}$ with this subsequence, we can assume that the following holds, with $\varepsilon > 0$:

$$||Te_n|| > \varepsilon$$

Since T was assumed to be compact, and the sequence $\{e_n\}$ is bounded, a certain subsequence $\{Te_{n_k}\}$ must converge. Thus, by replacing once again $\{e_n\}$ with a subsequence, we can assume that the following holds, with $x \neq 0$:

$$Te_n \to x$$

But this is a contradiction, because we obtain in this way:

$$\begin{aligned} < x, x > &= \lim_{n \to \infty} < Te_n, x > \\ &= \lim_{n \to \infty} < e_n, T^*x > \\ &= 0 \end{aligned}$$

Thus our assumption $Te_n \not\to 0$ was wrong, and we obtain the result.

"$\impliedby$" Assume $Te_n \to 0$, for any orthonormal system $\{e_n\} \subset H$. In order to prove that T is compact, we use the various results established above, which show that this is the same as proving that T is in the closure of the space of finite rank operators:

$$T \in \overline{F(H)}$$

We do this by contradiction. So, assume that the above is wrong, and so that there exists $\varepsilon > 0$ such that the following holds:

$$S \in F(H) \implies ||T - S|| > \varepsilon$$

As a first observation, by using $S = 0$ we obtain $||T|| > \varepsilon$. Thus, we can find a norm one vector $e_1 \in H$ such that the following holds:

$$||Te_1|| > \varepsilon$$

Our claim, which will bring the desired contradiction, is that we can construct by recurrence vectors $e_1, \ldots, e_n$ such that the following holds, for any i:

$$||Te_i|| > \varepsilon$$

Indeed, assume that we have constructed such vectors $e_1,\ldots,e_n$. Let $E\subset H$ be the linear space spanned by these vectors, and let us set:

$$P=Proj(E)$$

Since the operator TP has finite rank, our assumption above shows that we have:

$$||T-TP||>\varepsilon$$

Thus, we can find a vector $x\in H$ such that the following holds:

$$||(T-TP)x||>\varepsilon$$

We have then $x\notin E$, and so we can consider the following nonzero vector:

$$y=(1-P)x$$

With this nonzero vector y constructed, in this way, now let us set:

$$e_{n+1}=\frac{y}{||y||}$$

This vector e_{n+1} is then orthogonal to E, has norm one, and satisfies:

$$||Te_{n+1}||\geq ||y||^{-1}\varepsilon\geq\varepsilon$$

Thus we are done with our construction by recurrence, and this contradicts our assumption that $Te_n\to 0$, for any orthonormal system $\{e_n\}\subset H$, as desired. □

10d. Singular values

Let us discuss now the spectral theory of the compact operators, in analogy with the known results from linear algebra. We first have the following result:

PROPOSITION 10.11. *Assuming that $T\in B(H)$, with $\dim H=\infty$, is compact and self-adjoint, the following happen:*

(1) *The eigenvalues of T form a sequence $\lambda_n\to 0$.*
(2) *All eigenvalues $\lambda_n\neq 0$ have finite multiplicity.*

PROOF. We prove both the assertions at the same time. For this purpose, we fix a number $\varepsilon>0$, we consider all the eigenvalues satisfying $|\lambda|\geq\varepsilon$, and for each such eigenvalue we consider the corresponding eigenspace $E_\lambda\subset H$. Let us set:

$$E=span\left(E_\lambda\Big||\lambda|\geq\varepsilon\right)$$

Our claim, which will prove both (1) and (2), is that this space E is finite dimensional. In now to prove now this claim, we can proceed as follows:

(1) We know that we have $E\subset Im(T)$. Our claim is that we have:

$$\bar{E}\subset Im(T)$$

Indeed, assume that we have a sequence $g_n \in E$ which converges, $g_n \to g \in \bar{E}$. Let us write $g_n = Tf_n$, with $f_n \in H$. By definition of E, the following condition is satisfied:

$$h \in E \implies ||Th|| \geq \varepsilon ||h||$$

Now since the sequence $\{g_n\}$ is Cauchy we obtain from this that the sequence $\{f_n\}$ is Cauchy as well, and with $f_n \to f$ we have $Tf_n \to Tf$, as desired.

(2) Consider now the projection $P \in B(H)$ onto the closure $\bar{E}$ of the above vector space E. The composition PT is then as follows, surjective on its target:

$$PT : H \to \bar{E}$$

On the other hand since T is compact so must be PT, and if follows from this that the space $\bar{E}$ is finite dimensional. Thus E itself must be finite dimensional too, and as explained in the beginning of the proof, this gives (1) and (2), as desired. □

In order to construct now eigenvalues, we will need:

PROPOSITION 10.12. *If T is compact and self-adjoint, one of the numbers*

$$||T|| \ , \ -||T||$$

must be an eigenvalue of T.

PROOF. We know from the spectral theory of the self-adjoint operators that the spectral radius $||T||$ of our operator T is attained, and so one of the numbers $||T||, -||T||$ must be in the spectrum. In order to prove now that one of these numbers must actually appear as an eigenvalue, we must use the compactness of T, as follows:

(1) First, we can assume $||T|| = 1$. By functional calculus this implies $||T^3|| = 1$ too, and so we can find a sequence of norm one vectors $x_n \in H$ such that:

$$| < T^3 x_n, x_n > | \to 1$$

By using our assumption $T = T^*$, we can rewrite this formula as follows:

$$| < T^2 x_n, T x_n > | \to 1$$

Now since T is compact, and $\{x_n\}$ is bounded, we can assume, up to changing the sequence $\{x_n\}$ to one of its subsequences, that the sequence Tx_n converges:

$$Tx_n \to y$$

Thus, the convergence formula found above reformulates as follows, with $y \neq 0$:

$$| < Ty, y > | = 1$$

(2) Our claim now, which will finish the proof, is that this latter formula implies $Ty = \pm y$. Indeed, by using Cauchy-Schwarz and $||T|| = 1$, we have:

$$| < Ty, y > | \leq ||Ty|| \cdot ||y|| \leq 1$$

We know that this must be an equality, so Ty, y must be proportional. But since T is self-adjoint the proportionality factor must be ± 1, and so we obtain, as claimed:

$$Ty = \pm y$$

Thus, we have constructed an eigenvector for $\lambda = \pm 1$, as desired. □

We can further build on the above results in the following way:

PROPOSITION 10.13. *If T is compact and self-adjoint, there is an orthogonal basis of H made of eigenvectors of T.*

PROOF. We use Proposition 10.12. According to the results there, we can arrange the nonzero eigenvalues of T, taken with multiplicities, into a sequence $\lambda_n \to 0$. Let $y_n \in H$ be the corresponding eigenvectors, and consider the following space:

$$E = \overline{span(y_n)}$$

The result follows then from the following observations:

(1) Since we have $T = T^*$, both E and its orthogonal $E^\perp$ are invariant under T.

(2) On the space E, our operator T is by definition diagonal.

(3) On the space $E^\perp$, our claim is that we have $T = 0$. Indeed, assuming that the restriction $S = T_{E^\perp}$ is nonzero, we can apply Proposition 10.12 to this restriction, and we obtain an eigenvalue for S, and so for T, contradicting the maximality of E. □

With the above results in hand, we can now formulate a first spectral theory result for compact operators, which closes the discussion in the self-adjoint case:

THEOREM 10.14. *Assuming that $T \in B(H)$, with $\dim H = \infty$, is compact and self-adjoint, the following happen:*

1. *The spectrum $\sigma(T) \subset \mathbb{R}$ consists of a sequence $\lambda_n \to 0$.*
2. *All spectral values $\lambda \in \sigma(T) - \{0\}$ are eigenvalues.*
3. *All eigenvalues $\lambda \in \sigma(T) - \{0\}$ have finite multiplicity.*
4. *There is an orthogonal basis of H made of eigenvectors of T.*

PROOF. This follows from the various results established above:

(1) In view of Proposition 10.11 (1), this will follow from (2) below.

(2) Assume that $\lambda \neq 0$ belongs to the spectrum $\sigma(T)$, but is not an eigenvalue. By using Proposition 10.13, let us pick an orthonormal basis $\{e_n\}$ of H consisting of eigenvectors of T, and then consider the following operator:

$$Sx = \sum_n \frac{< x, e_n >}{\lambda_n - \lambda} e_n$$

Then S is an inverse for $T - \lambda$, and so we have $\lambda \notin \sigma(T)$, as desired.

(3) This is something that we know, from Proposition 10.11 (2).

(4) This is something that we know too, from Proposition 10.13. □

Finally, we have the following result, regarding the general case:

THEOREM 10.15. *The compact operators $T\in B(H)$, with $\dim H=\infty$, are the operators of the following form, with $\{e_n\}$, $\{f_n\}$ being orthonormal families, and with $\lambda_n\searrow 0$:*

$$T(x)=\sum_n\lambda_n<x,e_n>f_n$$

The numbers λ_n, called singular values of T, are the eigenvalues of $|T|$. In fact, the polar decomposition of T is given by $T=U|T|$, with

$$|T|(x)=\sum_n\lambda_n<x,e_n>e_n$$

and with U being given by $Ue_n=f_n$, and $U=0$ on the complement of span(e_i).

PROOF. This basically follows from Theorem 10.14, as follows:

(1) Given two orthonormal families $\{e_n\}$, $\{f_n\}$, and a sequence of real numbers $\lambda_n\searrow 0$, consider the linear operator given by the formula in the statement, namely:

$$T(x)=\sum_n\lambda_n<x,e_n>f_n$$

Our first claim is that T is bounded. Indeed, when assuming $|\lambda_n|\leq\varepsilon$ for any n, which is something that we can do if we want to prove that T is bounded, we have:

$$\begin{aligned}
||T(x)||^2&=\left|\sum_n\lambda_n<x,e_n>f_n\right|^2\\
&=\sum_n|\lambda_n|^2|<x,e_n>|^2\\
&\leq\varepsilon^2\sum_n|<x,e_n>|^2\\
&\leq\varepsilon^2||x||^2
\end{aligned}$$

(2) The next observation is that this operator is indeed compact, because it appears as the norm limit, $T_N\to T$, of the following sequence of finite rank operators:

$$T_N=\sum_{n\leq N}\lambda_n<x,e_n>f_n$$

(3) Regarding now the polar decomposition assertion, for the above operator, this follows once again from definitions. Indeed, the adjoint is given by:

$$T^*(x)=\sum_n\lambda_n<x,f_n>e_n$$

Thus, when composing T^* with T, we obtain the following operator:

$$T^*T(x)=\sum_n\lambda_n^2<x,e_n>e_n$$

Now by extracting the square root, we obtain the formula in the statement, namely:

$$|T|(x)=\sum_n\lambda_n<x,e_n>e_n$$

(4) Conversely now, assume that $T\in B(H)$ is compact. Then T^*T, which is self-adjoint, must be compact as well, and so by Theorem 10.14 we have a formula as follows, with $\{e_n\}$ being a certain orthonormal family, and with $\lambda_n\searrow 0$:

$$T^*T(x)=\sum_n\lambda_n^2<x,e_n>e_n$$

By extracting the square root we obtain the formula of $|T|$ in the statement, and then by setting $U(e_n)=f_n$ we obtain a second orthonormal family, $\{f_n\}$, such that:

$$T(x)=U|T|=\sum_n\lambda_n<x,e_n>f_n$$

Thus, our compact operator $T\in B(H)$ appears indeed as in the statement. □

As a technical remark here, it is possible to slightly improve a part of the above statement. Consider indeed an operator of the following form, with $\{e_n\}$, $\{f_n\}$ being orthonormal families as before, and with $\lambda_n\to 0$ being now complex numbers:

$$T(x)=\sum_n\lambda_n<x,e_n>f_n$$

Then the same proof as before shows that T is compact, and that the polar decomposition of T is given by $T=U|T|$, with the modulus $|T|$ being as follows:

$$|T|(x)=\sum_n|\lambda_n|<x,e_n>e_n$$

As for the partial isometry U, this is given by $Ue_n=w_nf_n$, and $U=0$ on the complement of $span(e_i)$, where $w_n\in\mathbb{T}$ are such that $\lambda_n=|\lambda_n|w_n$.

10e. Exercises

Exercises:

Exercise 10.16.

Exercise 10.17.

Exercise 10.18.

Exercise 10.19.

Exercise 10.20.

Exercise 10.21.

Exercise 10.22.

Exercise 10.23.

Bonus exercise.

CHAPTER 11

Trace, determinant

11a. Trace class operators

We have not talked so far about the trace of operators $T\in B(H)$, in analogy with the trace of the usual matrices $M\in M_N(\mathbb C)$. This is because the trace can be finite or infinite, or even not well-defined, and we will discuss this now.

Let us start our discussion here with a standard result, as follows:

PROPOSITION 11.1. *Given a positive operator $T\in B(H)$, the quantity*

$$Tr(T)=\sum_n<Te_n,e_n>\in[0,\infty]$$

is indpendent on the choice of an orthonormal basis $\{e_n\}$.

PROOF. If $\{f_n\}$ is another orthonormal basis, we have:

$$\begin{aligned}\sum_n<Tf_n,f_n> &= \sum_n<\sqrt{T}f_n,\sqrt{T}f_n>\\ &= \sum_n||\sqrt{T}f_n||^2\\ &= \sum_{mn}|<\sqrt{T}f_n,e_m>|^2\\ &= \sum_{mn}|<T^{1/4}f_n,T^{1/4}e_m>|^2\end{aligned}$$

Since this quantity is symmetric in e,f, this gives the result. □

We can now introduce the trace class operators, as follows:

DEFINITION 11.2. *An operator $T\in B(H)$ is said to be of trace class if:*

$$Tr|T|<\infty$$

The set of such operators, also called integrable, is denoted $B_1(H)$.

In finite dimensions, any operator is of course of trace class. In arbitrary dimension, finite or not, we first have the following result, regarding such operators:

PROPOSITION 11.3. *Any finite rank operator is of trace class, and any trace class operator is compact, so that we have embeddings as follows:*

$$F(H) \subset B_1(H) \subset K(H)$$

Moreover, for any compact operator $T \in K(H)$ we have the formula

$$Tr|T| = \sum_n \lambda_n$$

where $\lambda_n \geq 0$ are the singular values, and so $T \in B_1(H)$ precisely when $\sum_n \lambda_n < \infty$.

PROOF. We have several assertions here, the idea being as follows:

(1) If T is of finite rank, it is clearly of trace class.

(2) In order to prove now the second assertion, assume first that $T > 0$ is of trace class. For any orthonormal basis $\{e_n\}$ we have:

$$\begin{aligned}\sum_n ||\sqrt{T}e_n||^2 &= \sum_n < Te_n, e_n > \\ &\leq Tr(T) \\ &< \infty\end{aligned}$$

But this shows that we have a convergence as follows:

$$\sqrt{T}e_n \to 0$$

Thus the operator $\sqrt{T}$ is compact. Now observe that we have:

$$T = \sqrt{T} \cdot \sqrt{T}$$

Since we know from chapter 10 that the compact operators form an ideal, it follows that this operator $T = \sqrt{T} \cdot \sqrt{T}$ is compact as well, as desired.

(3) In order to prove now the second assertion in general, assume that $T \in B(H)$ is of trace class. Then $|T|$ is also of trace class, and so compact by (2), and since we have $T = U|T|$ by polar decomposition, it follows that T is compact too.

(4) Finally, in order to prove the last assertion, assume that T is compact. The singular value decomposition of $|T|$, from chapter 10, is then as follows:

$$|T|(x) = \sum_n \lambda_n < x, e_n > e_n$$

But this gives the formula for $Tr|T|$ in the statement, and proves the last assertion. $\square$

Here is a useful reformulation of the above result, or rather of the above result coupled with the singular value decomposition, without reference to compact operators:

THEOREM 11.4. *The trace class operators are precisely the operators of the form*

$$|T|(x) = \sum_n \lambda_n < x, e_n > f_n$$

with $\{e_n\}$, $\{f_n\}$ *being orthonormal systems, and with* $\lambda \searrow 0$ *being a sequence satisfying:*

$$\sum_n \lambda_n < \infty$$

Moreover, for such an operator we have the following estimate:

$$|Tr(T)| \leq Tr|T| = \sum_n \lambda_n$$

PROOF. This follows indeed from Proposition 11.3, or rather for step (4) in the proof of Proposition 11.3, coupled with the singular value decomposition theorem. □

11b. Ideal property

Next, we have the following result, which comes as a continuation of Proposition 11.3, and is our central result here, regarding the trace class operators:

THEOREM 11.5. *The space of trace class operators, which appears as an intermediate space between the finite rank operators and the compact operators,*

$$F(H) \subset B_1(H) \subset K(H)$$

is a two-sided $*$*-ideal of* $K(H)$*. The following is a Banach space norm on* $B_1(H)$*,*

$$||T||_1 = Tr|T|$$

satisfying $||T|| \leq ||T||_1$*, and for* $T \in B_1(H)$ *and* $S \in B(H)$ *we have:*

$$||ST||_1 \leq ||S|| \cdot ||T||_1$$

Also, the subspace $F(H)$ *is dense inside* $B_1(H)$*, with respect to this norm.*

PROOF. There are several assertions here, the idea being as follows:

(1) In order to prove that $B_1(H)$ is a linear space, and that $||T||_1 = Tr|T|$ is a norm on it, the only non-trivial point is that of proving the following inequality:

$$Tr|S + T| \leq Tr|S| + Tr|T|$$

For this purpose, consider the polar decompositions of these operators:

$$S = U|S| \quad , \quad T = V|T| \quad , \quad S + T = W|S + T|$$

Given an orthonormal basis $\{e_n\}$, we have the following formula:

$$\begin{aligned}
Tr|S+T| &= \sum_n < |S+T|e_n, e_n > \\
&= \sum_n < W^*(S+T)e_n, e_n > \\
&= \sum_n < W^*U|S|e_n, e_n > + \sum_n < W^*V|T|e_n, e_n >
\end{aligned}$$

The point now is that the first sum can be estimated as follows:

$$\begin{aligned}
& \sum_n < W^*U|S|e_n, e_n > \\
= & \sum_n < \sqrt{|S|}e_n, \sqrt{|S|}U^*We_n > \\
\leq & \sum_n \left|\left|\sqrt{|S|}e_n\right|\right| \cdot \left|\left|\sqrt{|S|}U^*We_n\right|\right| \\
\leq & \sqrt{\sum_n \left|\left|\sqrt{|S|}e_n\right|\right|^2} \cdot \sqrt{\sum_n \left|\left|\sqrt{|S|}U^*We_n\right|\right|^2}
\end{aligned}$$

In order to estimate the terms on the right, we can proceed as follows:

$$\begin{aligned}
\sum_n \left|\left|\sqrt{|S|}U^*We_n\right|\right|^2 &= \sum_n < W^*U|S|U^*We_n, e_n > \\
&= Tr(W^*U|S|U^*W) \\
&\leq Tr(U|S|U^*) \\
&\leq Tr(|S|)
\end{aligned}$$

The second sum in the above formula of $Tr|S+T|$ can be estimated in the same way, and in the end we obtain, as desired:

$$Tr|S+T| \leq Tr|S| + Tr|T|$$

(2) The estimate $||T|| \leq ||T||_1$ can be established as follows:

$$\begin{aligned}
||T|| &= \left|\left||T|\right|\right| \\
&= \sup_{||x||=1} < |T|x, x > \\
&\leq Tr|T|
\end{aligned}$$

(3) The fact that $B_1(H)$ is indeed a Banach space follows by constructing a limit for any Cauchy sequence, by using the singular value decomposition.

(4) The fact that $B_1(H)$ is indeed closed under the involution follows from:

$$\begin{aligned} Tr(T^*) &= \sum_n < T^*e_n, e_n > \\ &= \sum_n < e_n, Te_N > \\ &= \overline{Tr(T)} \end{aligned}$$

(5) In order to prove now the ideal property of $B_1(H)$, we use the standard fact, that we know well from chapter 5, that any bounded operator $T \in B(H)$ can be written as a linear combination of 4 unitary operators, as follows:

$$T = \lambda_1 U_1 + \lambda_2 U_2 + \lambda_3 U_3 + \lambda_4 U_4$$

Indeed, by taking the real and imaginary part we can first write T as a linear combination of 2 self-adjoint operators, and then by functional calculus each of these 2 self-adjoint operators can be written as a linear linear combination of 2 unitary operators.

(6) With this trick in hand, we can now prove the ideal property of $B_1(H)$. Indeed, it is enough to prove that we have:

$$T \in B_1(H), U \in U(H) \implies UT, TU \in B_1(H)$$

But this latter result follows by using the polar decomposition theorem.

(7) With a bit more care, we obtain from this the estimate $||ST||_1 \leq ||S|| \cdot ||T||_1$ from the statement. As for the last assertion, this is clear as well. □

This was for the basic theory of the trace class operators. Much more can be said, and we refer here to the literature, such as Lax [**68**]. In what concerns us, we will be back to these operators later in this book, in Part IV, when discussing operator algebras.

11c. Hilbert-Schmidt

As a further topic for this chapter, let us discuss yet another important class of operators, namely the Hilbert-Schmidt ones. These operators, that we will need on several occasions in later on, when talking operator algebras, are introduced as follows:

Definition 11.6. *An operator $T \in B(H)$ is said to be Hilbert-Schmidt if:*

$$Tr(T^*T) < \infty$$

The set of such operators is denoted $B_2(H)$.

As before with other sets of operators, in finite dimensions we obtain in this way all the operators. In general, we have the following result, regarding such operators:

THEOREM 11.7. *The space $B_2(H)$ of Hilbert-Schmidt operators, which appears as an intermediate space between the trace class operators and the compact operators,*

$$F(H) \subset B_1(H) \subset B_2(H) \subset K(H)$$

is a two-sided $*$-ideal of $K(H)$. This ideal has the property

$$S, T \in B_2(H) \implies ST \in B_1(H)$$

and conversely, each $T \in B_1(H)$ appears as product of two operators in $B_2(H)$. In terms of the singular values (λ_n), the Hilbert-Schmidt operators are characterized by:

$$\sum_n \lambda_n^2 < \infty$$

Also, the following formula, whose output is finite by Cauchy-Schwarz,

$$< S, T >= Tr(ST^*)$$

defines a scalar product of $B_2(H)$, making it a Hilbert space.

PROOF. All this is quite standard, from the results that we have already, and more specifically from the singular value decomposition theorem, and its applications. To be more precise, the proof of the various assertions goes as follows:

(1) First of all, the fact that the space of Hilbert-Schmidt operators $B_2(H)$ is stable under taking sums, and so is a vector space, follows from:

$$\begin{aligned} (S+T)^*(S+T) &\leq (S+T)^*(S+T) + (S-T)^*(S-T) \\ &= (S^*+T^*)(S+T) + (S^*-T^*)(S-T) \\ &= 2(S^*S + T^*T) \end{aligned}$$

Regarding now multiplicative properties, we can use here the following inequality:

$$(ST)^*(ST) = T^*S^*ST \leq ||S||^2 T^*T$$

Thus, the space $B_2(H)$ is a two-sided $*$-ideal of $K(H)$, as claimed.

(2) In order to prove now that the product of any two Hilbert-Schmidt operators is a trace class operator, we can use the following formula, which is elementary:

$$S^*T = \sum_{k=1}^4 i^k (S - iT)^*(S - iT)$$

Conversely, given an arbitrary trace class operator $T \in B_1(H)$, we have:

$$T \in B_1(H) \implies |T| \in B_1(H) \implies \sqrt{|T|} \in B_2(H)$$

Thus, by using the polar decomposition $T = U|T|$, we obtain the following decomposition for T, with both components being Hilbert-Schmidt operators:

$$T = U|T| = U\sqrt{|T|} \cdot \sqrt{|T|}$$

(3) The condition for the singular values is clear.

(4) The fact that we have a scalar product is clear as well.

(5) The proof of the completness property is routine as well. □

We have as well the following key result, regarding the Hilbert-Schmidt operators:

THEOREM 11.8. *We have the following formula,*

$$Tr(ST) = Tr(TS)$$

valied for any Hilbert-Schmidt operators $S, T \in B_2(H)$.

PROOF. We can prove this in two steps, as follows:

(1) Assume first that $|S|$ is trace class. Consider the polar decomposition $S = U|S|$, and choose an orthonormal basis $\{x_i\}$ for the image of U, suitably extended to an orthonormal basis of H. We have then the following computation, as desired:

$$\begin{aligned} Tr(ST) &= \sum_i < U|S|Tx_i, x_i > \\ &= \sum_i < |S|TUU^*x_i, U^*x_i > \\ &= Tr(|S|TU) \\ &= Tr(TU|S|) \\ &= Tr(TS) \end{aligned}$$

(2) Assume now that we are in the general case, where S is only assumed to be Hilbert-Schmidt. For any finite rank operator S' we have then:

$$\begin{aligned} |Tr(ST) - Tr(TS)| &= |Tr((S - S')T) - Tr(T(S - S'))| \\ &\leq 2||S - S'||_2 \cdot ||T||_2 \end{aligned}$$

Thus by choosing S' with $||S - S'||_2 \to 0$, we obtain the result. □

This was for the basic theory of bounded operators on a Hilbert space, $T \in B(H)$. In the remainder of this book we will be interested in examples, and in the operator algebras $A \subset B(H)$ that these operators can form. This is of course related to operator theory, because we can, at least in theory, take $A =< T >$, and then study T via the properties of A. Actually, this is something that we already did a few times, when doing spectral theory, and notably when talking about functional calculus for normal operators.

11d. Determinants

Determinants.

11e. Exercises

Exercises:

Exercise 11.9.

Exercise 11.10.

Exercise 11.11.

Exercise 11.12.

Exercise 11.13.

Exercise 11.14.

Exercise 11.15.

Exercise 11.16.

Bonus exercise.

CHAPTER 12

Some geometry

12a.

12b.

12c.

12d.

12e. Exercises

Exercises:

EXERCISE 12.1.

EXERCISE 12.2.

EXERCISE 12.3.

EXERCISE 12.4.

EXERCISE 12.5.

EXERCISE 12.6.

EXERCISE 12.7.

EXERCISE 12.8.

Bonus exercise.

Part IV

Operator algebras

There is no pain, you are receding
A distant ship, smoke on the horizon
You are only coming through in waves
Your lips move, but I can't hear what you're saying

CHAPTER 13

C*-algebras

13a. C*-algebras

We have seen that the study of the bounded operators $T \in B(H)$ often leads to the consideration of the algebras $< T >\subset B(H)$ generated by such operators, the idea being that the study of $A =< T >$ can lead to results about T itself. In the remainder of this book we focus on the study of such algebras $A \subset B(H)$. Let us start our discussion with the following broad definition, obtained by imposing the "minimal" set of axioms:

DEFINITION 13.1. *An operator algebra is an algebra of bounded operators $A \subset B(H)$ which contains the unit, is closed under taking adjoints,*

$$T \in A \implies T^* \in A$$

and is closed as well under the norm.

Here, as in the previous chapters, $B(H)$ is the algebra of linear operators $T : H \to H$ which are bounded, in the sense that the norm $||T|| = \sup_{||x||=1} ||Tx||$ is finite. This algebra has an involution $T \to T^*$, with the adjoint operator $T^* \in B(H)$ being defined by the formula $< Tx, y >=< x, T^*y >$, and in the above definition, the assumption $T \in A \implies T^* \in A$ refers to this involution. Thus, A must be a $*$-algebra.

As a first result now regarding the operator algebras, in relation with the normal operators, where most of the non-trivial results that we have so far are, we have:

THEOREM 13.2. *The operator algebra $< T >\subset B(H)$ generated by a normal operator $T \in B(H)$ appears as an algebra of continuous functions,*

$$< T >= C(\sigma(T))$$

where $\sigma(T) \subset \mathbb{C}$ denotes as usual the spectrum of T.

PROOF. This is an abstract reformulation of the continuous functional calculus theorem for the normal operators, that we know since chapter 3. Indeed, that theorem tells us that we have a continuous morphism of $*$-algebras, as follows:

$$C(\sigma(T)) \to B(H) \quad , \quad f \to f(T)$$

Moreover, by the general properties of the continuous calculus, also established in chapter 3, this morphism is injective, and its image is the norm closed algebra $< T >$ generated by T, T^*. Thus, we obtain the isomorphism in the statement. $\square$

The above result is very nice, and it is possible to further build on it, by using this time the spectral theorem for families of normal operators, as follows:

THEOREM 13.3. *The operator algebra $< T_i >\subset B(H)$ generated by a family of normal operators $T_i \in B(H)$ appears as an algebra of continuous functions,*

$$< T >= C(X)$$

where $X \subset \mathbb{C}$ is a certain compact space associated to the family $\{T_i\}$. Equivalently, any commutative operator algebra $A \subset B(H)$ is of the form $A = C(X)$.

PROOF. We have two assertions here, the idea being as follows:

(1) Regarding the first assertion, this follows exactly as in the proof of Theorem 13.2, by using this time the spectral theorem for families of normal operators.

(2) As for the second assertion, this is clear from the first one, because any commutative algebra $A \subset B(H)$ is generated by its elements $T \in A$, which are all normal. □

All this is good to know, but Theorem 13.2 and Theorem 13.3 remain something quite heavy, based on the spectral theorem. We would like to present now an alternative proof for these results, which is rather elementary, and has the advantage of reconstructing the compact space X directly from the knowledge of the algebra A. Let us start with:

DEFINITION 13.4. *A C^*-algebra is an complex algebra A, given with:*

1. *A norm $a \to ||a||$, making it into a Banach algebra.*
2. *An involution $a \to a^*$, related to the norm by the formula $||aa^*|| = ||a||^2$.*

Here by Banach algebra we mean a complex algebra with a norm satisfying all the conditions for a vector space norm, along with $||ab|| \leq ||a|| \cdot ||b||$ and $||1|| = 1$, and which is such that our algebra is complete, in the sense that the Cauchy sequences converge. As for the involution, this must be antilinear, antimultiplicative, and satisfying $a^{**} = a$.

As basic examples, we have the operator algebra $B(H)$, for any Hilbert space H, and more generally, the norm closed $*$-subalgebras $A \subset B(H)$. It is possible to prove that any C^*-algebra appears in this way, but this is a non-trivial result, called GNS theorem, and more on this later. Note in passing that this result tells us that there is no need to memorize the above axioms for the C^*-algebras, because these are simply the obvious things that can be said about $B(H)$, and its norm closed $*$-subalgebras $A \subset B(H)$.

As a second class of basic examples, which are of particular interest, we have:

PROPOSITION 13.5. *If X is a compact space, the algebra $C(X)$ of continuous functions $f : X \to \mathbb{C}$ is a C^*-algebra, with the usual norm and involution, namely:*

$$||f|| = \sup_{x\in X} |f(x)| \quad , \quad f^*(x) = \overline{f(x)}$$

This algebra is commutative, in the sense that $fg = gf$, for any $f, g \in C(X)$.

PROOF. All this is clear from definitions. Observe that we have indeed:

$$||ff^*|| = \sup_{x\in X} |f(x)|^2 = ||f||^2$$

Thus, the axioms are satisfied, and finally $fg = gf$ is clear. □

In general, the C^*-algebras can be thought of as being algebras of operators, over some Hilbert space which is not present. By using this philosophy, one can emulate spectral theory in this setting, with extensions of the various results from chapter 3:

THEOREM 13.6. *Given element $a \in A$ of a C^*-algebra, define its spectrum as:*

$$\sigma(a) = \left\{\lambda \in \mathbb{C}\Big| a - \lambda \notin A^{-1}\right\}$$

The following spectral theory results hold, exactly as in the $A = B(H)$ case:

1. *We have $\sigma(ab) \cup \{0\} = \sigma(ba) \cup \{0\}$.*
2. *We have polynomial, rational and holomorphic calculus.*
3. *As a consequence, the spectra are compact and non-empty.*
4. *The spectra of unitaries ($u^* = u^{-1}$) and self-adjoints ($a = a^*$) are on $\mathbb{T}, \mathbb{R}$.*
5. *The spectral radius of normal elements ($aa^* = a^*a$) is given by $\rho(a) = ||a||$.*

In addition, assuming $a \in A \subset B$, the spectra of a with respect to A and to B coincide.

PROOF. This is something that we know well since chapter 3, in the case of the full operator algebra $A = B(H)$, and in general, the proof is similar, as follows:

(1) Regarding the assertions (1-5), which are of course formulated a bit informally, the proofs here are perfectly similar to those for the full operator algebra $A = B(H)$. All this is standard material, and in fact, things in chapters 3 were written in such a way as for their extension now, to the general C^*-algebra setting, to be obvious.

(2) Regarding the last assertion, the inclusion $\sigma_B(a) \subset \sigma_A(a)$ is clear. For the converse, assume $a - \lambda \in B^{-1}$, and consider the following self-adjoint element:

$$b = (a - \lambda)^*(a - \lambda)$$

The difference between the two spectra of $b \in A \subset B$ is then given by:

$$\sigma_A(b) - \sigma_B(b) = \left\{\mu \in \mathbb{C} - \sigma_B(b)\Big|(b - \mu)^{-1} \in B - A\right\}$$

Thus this difference in an open subset of $\mathbb{C}$. On the other hand b being self-adjoint, its two spectra are both real, and so is their difference. Thus the two spectra of b are equal, and in particular b is invertible in A, and so $a - \lambda \in A^{-1}$, as desired. □

We can now get back to the commutative C^*-algebras, and we have the following result, due to Gelfand, which will be of crucial importance for us:

THEOREM 13.7. *The commutative C^*-algebras are exactly the algebras of the form*

$$A = C(X)$$

with the "spectrum" X of such an algebra being the space of characters $\chi : A \to \mathbb{C}$, with topology making continuous the evaluation maps $ev_a : \chi \to \chi(a)$.

PROOF. Given a commutative C^*-algebra A, we can define X as in the statement. Then X is compact, and $a \to ev_a$ is a morphism of algebras, as follows:

$$ev : A \to C(X)$$

(1) We first prove that ev is involutive. We use the following formula, which is similar to the $z = Re(z) + iIm(z)$ formula for the usual complex numbers:

$$a = \frac{a + a^*}{2} + i \cdot \frac{a - a^*}{2i}$$

Thus it is enough to prove the equality $ev_{a^*} = ev_a^*$ for self-adjoint elements a. But this is the same as proving that $a = a^*$ implies that ev_a is a real function, which is in turn true, because $ev_a(\chi) = \chi(a)$ is an element of $\sigma(a)$, contained in $\mathbb{R}$.

(2) Since A is commutative, each element is normal, so ev is isometric:

$$||ev_a|| = \rho(a) = ||a||$$

(3) It remains to prove that ev is surjective. But this follows from the Stone-Weierstrass theorem, because $ev(A)$ is a closed subalgebra of $C(X)$, which separates the points. □

In view of the Gelfand theorem, we can formulate the following key definition:

DEFINITION 13.8. *Given an arbitrary C^*-algebra A, we write*

$$A = C(X)$$

and call X a compact quantum space.

This might look like something informal, but it is not. Indeed, we can define the category of compact quantum spaces to be the category of the C^*-algebras, with the arrows reversed. When A is commutative, the above space X exists indeed, as a Gelfand spectrum, $X = Spec(A)$. In general, X is something rather abstract, and our philosophy here will be that of studying of course A, but formulating our results in terms of X. For instance whenever we have a morphism $\Phi : A \to B$, we will write $A = C(X), B = C(Y)$, and rather speak of the corresponding morphism $\phi : Y \to X$. And so on.

As a first concrete consequence of the Gelfand theorem, we have:

THEOREM 13.9. *Assume that $a \in A$ is normal, and let $f \in C(\sigma(a))$.*

1. *We can define $f(a) \in A$, with $f \to f(a)$ being a morphism of C^*-algebras.*
2. *We have the "continuous functional calculus" formula $\sigma(f(a)) = f(\sigma(a))$.*

PROOF. Since a is normal, the C^*-algebra $< a >$ that is generates is commutative, so if we denote by X the space formed by the characters $\chi :< a >\to \mathbb{C}$, we have:

$$< a >= C(X)$$

Now since the map $X \to \sigma(a)$ given by evaluation at a is bijective, we obtain:

$$< a >= C(\sigma(a))$$

Thus, we are dealing with usual functions, and this gives all the assertions. □

As another consequence of the Gelfand theorem, we have:

THEOREM 13.10. *For a normal element $a \in A$, the following are equivalent:*

(1) *a is positive, in the sense that $\sigma(a) \subset [0, \infty)$.*
(2) *$a = b^2$, for some $b \in A$ satisfying $b = b^*$.*
(3) *$a = cc^*$, for some $c \in A$.*

PROOF. This is very standard, exactly as in $A = B(H)$ case, as follows:

(1) $\implies$ (2) Since $f(z) = \sqrt{z}$ is well-defined on $\sigma(a) \subset [0, \infty)$, we can set $b = \sqrt{a}$.

(2) $\implies$ (3) This is trivial, because we can set $c = b$.

(3) $\implies$ (1) We can proceed here by contradiction. Indeed, by multiplying c by a suitable element of $< cc^* >$, we are led to the existence of an element $d \neq 0$ satisfying $-dd^* \geq 0$. By writing now $d = x + iy$ with $x = x^*, y = y^*$ we have:

$$dd^* + d^*d = 2(x^2 + y^2) \geq 0$$

Thus $d^*d \geq 0$, contradicting the fact that $\sigma(dd^*), \sigma(d^*d)$ must coincide outside $\{0\}$, that we know well to hold for $A = B(H)$, and whose proof in general is similar. □

13b. Basic results

In order to develop some general theory, let us start by investigating the finite dimensional case. Here the ambient algebra is $B(H) = M_N(\mathbb{C})$, any linear subspace $A \subset B(H)$ is automatically closed, for the norm topology, and we have the following result:

THEOREM 13.11. *The $*$-algebras $A \subset M_N(\mathbb{C})$ are exactly the algebras of the form*

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

depending on parameters $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \mathbb{N}$ satisfying

$$n_1 + \ldots + n_k = N$$

embedded into $M_N(\mathbb{C})$ via the obvious block embedding, twisted by a unitary $U \in U_N$.

PROOF. We have two assertions to be proved, the idea being as follows:

(1) Given numbers $n_1,\ldots,n_k\in\mathbb N$ satisfying $n_1+\ldots+n_k=N$, we have indeed an obvious embedding of $*$-algebras, via matrix blocks, as follows:

$$M_{n_1}(\mathbb C)\oplus\ldots\oplus M_{n_k}(\mathbb C)\subset M_N(\mathbb C)$$

In addition, we can twist this embedding by a unitary $U\in U_N$, as follows:

$$M\to UMU^*$$

(2) In the other sense now, consider a $*$-algebra $A\subset M_N(\mathbb C)$. It is elementary to prove that the center $Z(A)=A\cap A'$, as an algebra, is of the following form:

$$Z(A)\simeq\mathbb C^k$$

Consider now the standard basis $e_1,\ldots,e_k\in\mathbb C^k$, and let $p_1,\ldots,p_k\in Z(A)$ be the images of these vectors via the above identification. In other words, these elements $p_1,\ldots,p_k\in A$ are central minimal projections, summing up to 1:

$$p_1+\ldots+p_k=1$$

The idea is then that this partition of the unity will eventually lead to the block decomposition of A, as in the statement. We prove this in 4 steps, as follows:

Step 1. We first construct the matrix blocks, our claim here being that each of the following linear subspaces of A are non-unital $*$-subalgebras of A:

$$A_i=p_iAp_i$$

But this is clear, with the fact that each A_i is closed under the various non-unital $*$-subalgebra operations coming from the projection equations $p_i^2=p_i^*=p_i$.

Step 2. We prove now that the above algebras $A_i\subset A$ are in a direct sum position, in the sense that we have a non-unital $*$-algebra sum decomposition, as follows:

$$A=A_1\oplus\ldots\oplus A_k$$

As with any direct sum question, we have two things to be proved here. First, by using the formula $p_1+\ldots+p_k=1$ and the projection equations $p_i^2=p_i^*=p_i$, we conclude that we have the needed generation property, namely:

$$A_1+\ldots+A_k=A$$

As for the fact that the sum is indeed direct, this follows as well from the formula $p_1+\ldots+p_k=1$, and from the projection equations $p_i^2=p_i^*=p_i$.

Step 3. Our claim now, which will finish the proof, is that each of the $*$-subalgebras $A_i=p_iAp_i$ constructed above is a full matrix algebra. To be more precise here, with $n_i=rank(p_i)$, our claim is that we have isomorphisms, as follows:

$$A_i\simeq M_{n_i}(\mathbb C)$$

In order to prove this claim, recall that the projections $p_i \in A$ were chosen central and minimal. Thus, the center of each of the algebras A_i reduces to the scalars:

$$Z(A_i) = \mathbb{C}$$

But this shows, either via a direct computation, or via the bicommutant theorem, that the each of the algebras A_i is a full matrix algebra, as claimed.

Step 4. We can now obtain the result, by putting together what we have. Indeed, by using the results from Step 2 and Step 3, we obtain an isomorphism as follows:

$$A \simeq M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

Moreover, a more careful look at the isomorphisms established in Step 3 shows that at the global level, that of the algebra A itself, the above isomorphism simply comes by twisting the following standard multimatrix embedding, discussed in the beginning of the proof, (1) above, by a certain unitary matrix $U \in U_N$:

$$M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C}) \subset M_N(\mathbb{C})$$

Now by putting everything together, we obtain the result. □

In terms of our usual C^*-algebra formalism, the above result tells us that we have:

THEOREM 13.12. *The finite dimensional C^*-algebras are exactly the algebras*

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

with norm $||(a_1, \ldots, a_k)|| = \sup_i ||a_i||$, and involution $(a_1, \ldots, a_k)^* = (a_1^*, \ldots, a_k^*)$.

PROOF. This is indeed a reformulation of what we know from Theorem 13.11, in terms of our usual C^*-algebra formalism, from the beginning of this chapter. □

Let us record as well the quantum space formulation of our result:

THEOREM 13.13. *The finite quantum spaces are exactly the disjoint unions of type*

$$X = M_{n_1} \sqcup \ldots \sqcup M_{n_k}$$

where M_n is the finite quantum space given by $C(M_n) = M_n(\mathbb{C})$.

PROOF. This is a reformulation of Theorem 13.12, by using the quantum space philosophy. Indeed, for a compact quantum space X, coming from a C^*-algebra A via the formula $A = C(X)$, being finite can only mean that the following number is finite:

$$|X| = \dim_{\mathbb{C}} A < \infty$$

Thus, by using Theorem 13.12, we are led to the conclusion that we must have:

$$C(X) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

But since direct sums of algebras A correspond to disjoint unions of quantum spaces X, via the correspondence $A = C(X)$, this leads to the conclusion in the statement. □

As a first application now of Theorem 13.12, we have the following result:

THEOREM 13.14. *Consider a $*$-algebra $A \subset M_N(\mathbb{C})$, written as above:*

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

The commutant of this algebra is then, with respect with the block decomposition used,

$$A' = \mathbb{C} \oplus \ldots \oplus \mathbb{C}$$

and by taking one more time the commutant we obtain A itself, $A = A''$.

PROOF. Let us decompose indeed our algebra A as in Theorem 13.12:

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

The center of each matrix algebra being reduced to the scalars, the commutant of this algebra is then as follows, with each copy of $\mathbb{C}$ corresponding to a matrix block:

$$A' = \mathbb{C} \oplus \ldots \oplus \mathbb{C}$$

By taking once again the commutant we obtain A itself, and we are done. □

As another interesting application of Theorem 13.12, clarifying this time the relation with operator theory, in finite dimensions, we have the following result:

THEOREM 13.15. *Given an operator $T \in B(H)$ in finite dimensions, $H = \mathbb{C}^N$, the operator algebra $A =< T >$ that it generates inside $B(H) = M_N(\mathbb{C})$ is*

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

with the sizes of the blocks $n_1, \ldots, n_k \in \mathbb{N}$ coming from the spectral theory of the associated matrix $M \in M_N(\mathbb{C})$. In the normal case $TT^* = T^*T$, this decomposition comes from

$$T = UDU^*$$

with $D \in M_N(\mathbb{C})$ diagonal, and with $U \in U_N$ unitary.

PROOF. This is something which is routine, by using basic linear algebra:

(1) The fact that $A =< T >$ decomposes into a direct sum of matrix algebras is something that we already know, coming from Theorem 13.12.

(2) By using standard linear algebra, we can compute the block sizes $n_1, \ldots, n_k \in \mathbb{N}$, from the knowledge of the spectral theory of the associated matrix $M \in M_N(\mathbb{C})$.

(3) In the normal case, $TT^* = T^*T$, we can simply invoke the spectral theorem, and by suitably changing the basis, we are led to the conclusion in the statement. □

Let us prove now a key result, called GNS representation theorem, stating that any C^*-algebra appears as an operator algebra. As a first result, we have:

PROPOSITION 13.16. *Let A be a commutative C^*-algebra, write $A = C(X)$, with X being a compact space, and let μ be a positive measure on X. We have then*

$$A \subset B(H)$$

where $H = L^2(X)$, with $f \in A$ corresponding to the operator $g \to fg$.

PROOF. Given a continuous function $f \in C(X)$, consider the operator $T_f(g) = fg$, on $H = L^2(X)$. Observe that T_f is indeed well-defined, and bounded as well, because:

$$||fg||_2 = \sqrt{\int_X |f(x)|^2|g(x)|^2 d\mu(x)} \leq ||f||_\infty ||g||_2$$

The application $f \to T_f$ being linear, involutive, continuous, and injective as well, we obtain in this way a C^*-algebra embedding $A \subset B(H)$, as claimed. □

In order to prove the GNS representation theorem, we must extend the above construction, to the case where A is not necessarily commutative. Let us start with:

DEFINITION 13.17. *Consider a C^*-algebra A.*

(1) *$\varphi : A \to \mathbb{C}$ is called positive when $a \geq 0 \implies \varphi(a) \geq 0$.*
(2) *$\varphi : A \to \mathbb{C}$ is called faithful and positive when $a \geq 0, a \neq 0 \implies \varphi(a) > 0$.*

In the commutative case, $A = C(X)$, the positive elements are the positive functions, $f : X \to [0, \infty)$. As for the positive linear forms $\varphi : A \to \mathbb{C}$, these appear as follows, with μ being positive, and strictly positive if we want φ to be faithful and positive:

$$\varphi(f) = \int_X f(x) d\mu(x)$$

In general, the positive linear forms can be thought of as being integration functionals with respect to some underlying "positive measures". We can use them as follows:

PROPOSITION 13.18. *Let $\varphi : A \to \mathbb{C}$ be a positive linear form.*

(1) *$< a, b >= \varphi(ab^*)$ defines a generalized scalar product on A.*
(2) *By separating and completing we obtain a Hilbert space H.*
(3) *$\pi(a) : b \to ab$ defines a representation $\pi : A \to B(H)$.*
(4) *If φ is faithful in the above sense, then π is faithful.*

PROOF. Almost everything here is straightforward, as follows:

(1) This is clear from definitions, and from the basic properties of the positive elements $a \geq 0$, which can be established exactly as in the $A = B(H)$ case.

(2) This is a standard procedure, which works for any scalar product, the idea being that of dividing by the vectors satisfying $< x, x >= 0$, then completing.

(3) All the verifications here are standard algebraic computations, in analogy with what we have seen many times, for the multiplication operators.

(4) Assuming that we have $a\neq 0$, we have then $\pi(aa^*)\neq 0$, which in turn implies by faithfulness that we have $\pi(a)\neq 0$, which gives the result. □

In order to establish the embedding theorem, it remains to prove that any C^*-algebra has a faithful positive linear form $\varphi:A\to\mathbb C$. This is something more technical:

PROPOSITION 13.19. *Let A be a C^*-algebra.*

1. *Any positive linear form $\varphi:A\to\mathbb C$ is continuous.*
2. *A linear form φ is positive iff there is a norm one $h\in A_+$ such that $||\varphi||=\varphi(h)$.*
3. *For any $a\in A$ there exists a positive norm one form φ such that $\varphi(aa^*)=||a||^2$.*
4. *If A is separable there is a faithful positive form $\varphi:A\to\mathbb C$.*

PROOF. The proof here is quite technical, inspired from the existence proof of the probability measures on abstract compact spaces, the idea being as follows:

(1) This follows from Proposition 13.18, via the following estimate:

$$|\varphi(a)|\leq||\pi(a)||\varphi(1)\leq||a||\varphi(1)$$

(2) In one sense we can take $h=1$. Conversely, let $a\in A_+$, $||a||\leq 1$. We have:

$$|\varphi(h)-\varphi(a)|\leq||\varphi||\cdot||h-a||\leq\varphi(h)$$

Thus we have $Re(\varphi(a))\geq 0$, and with $a=1-h$ we obtain:

$$Re(\varphi(1-h))\geq 0$$

Thus $Re(\varphi(1))\geq||\varphi||$, and so $\varphi(1)=||\varphi||$, so we can assume $h=1$. Now observe that for any self-adjoint element a, and any $t\in\mathbb R$ we have, with $\varphi(a)=x+iy$:

$$\begin{aligned}\varphi(1)^2(1+t^2||a||^2)&\geq\varphi(1)^2||1+t^2a^2||\\&=||\varphi||^2\cdot||1+ita||^2\\&\geq|\varphi(1+ita)|^2\\&=|\varphi(1)-ty+itx|\\&\geq(\varphi(1)-ty)^2\end{aligned}$$

Thus we have $y=0$, and this finishes the proof of our remaining claim.

(3) We can set $\varphi(\lambda aa^*)=\lambda||a||^2$ on the linear space spanned by aa^*, then extend this functional by Hahn-Banach, to the whole A. The positivity follows from (2).

(4) This is standard, by starting with a dense sequence (a_n), and taking the Cesàro limit of the functionals constructed in (3). We have $\varphi(aa^*)>0$, and we are done. □

With these ingredients in hand, we can now state and prove:

THEOREM 13.20. *Any C^*-algebra appears as a norm closed $*$-algebra of operators*

$$A\subset B(H)$$

over a certain Hilbert space H. When A is separable, H can be taken to be separable.

PROOF. This result, called called GNS representation theorem after Gelfand, Naimark and Segal, follows indeed by combining Proposition 13.18 with Proposition 13.19. □

Many other things can be said, along these lines. We will be back to this later, when talking von Neumann algebras, and their relation with the C^*-algebras.

13c. Group algebras

Let us discuss now some basic examples of C^*-algebras. We first have:

THEOREM 13.21. *Let Γ be a discrete group, and consider the complex group algebra $\mathbb{C}[\Gamma]$, with involution given by the fact that all group elements are unitaries, $g^*=g^{-1}$.*

1. *The maximal C^*-seminorm on $\mathbb{C}[\Gamma]$ is a C^*-norm, and the closure of $\mathbb{C}[\Gamma]$ with respect to this norm is a C^*-algebra, denoted $C^*(\Gamma)$.*
2. *When Γ is abelian, we have an isomorphism $C^*(\Gamma)\simeq C(G)$, where $G=\widehat{\Gamma}$ is its Pontrjagin dual, formed by the characters $\chi:\Gamma\to\mathbb{T}$.*

PROOF. All this is very standard, the idea being as follows:

(1) In order to prove the result, we must find a $*$-algebra embedding $\mathbb{C}[\Gamma]\subset B(H)$, with H being a Hilbert space. For this purpose, consider the space $H=l^2(\Gamma)$, having $\{h\}_{h\in\Gamma}$ as orthonormal basis. Our claim is that we have an embedding, as follows:

$$\pi:\mathbb{C}[\Gamma]\subset B(H)\quad,\quad\pi(g)(h)=gh$$

Indeed, since $\pi(g)$ maps the basis $\{h\}_{h\in\Gamma}$ into itself, this operator is well-defined, bounded, and is an isometry. It is also clear from the formula $\pi(g)(h)=gh$ that $g\to\pi(g)$ is a morphism of algebras, and since this morphism maps the unitaries $g\in\Gamma$ into isometries, this is a morphism of $*$-algebras. Finally, the faithfulness of π is clear.

(2) Since Γ is abelian, the corresponding group algebra $A=C^*(\Gamma)$ is commutative. Thus, we can apply the Gelfand theorem, and we obtain $A=C(X)$, with:

$$X=Spec(A)$$

But the spectrum $X=Spec(A)$, consisting of the characters $\chi:C^*(\Gamma)\to\mathbb{C}$, can be identified with the Pontrjagin dual $G=\widehat{\Gamma}$, and this gives the result. □

The above result suggests the following definition:

DEFINITION 13.22. *Given a discrete group Γ, the compact quantum space G given by*

$$C(G)=C^*(\Gamma)$$

is called abstract dual of Γ, and is denoted $G=\widehat{\Gamma}$.

With this, we can now talk about quantum tori, as follows:

Theorem 13.23. *The basic tori are all group duals, as follows,*

$$\begin{array}{ccc} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{array} \quad = \quad \begin{array}{ccc} \widehat{L_N} & \longrightarrow & \widehat{F_N} \\ \uparrow & & \uparrow \\ \mathbb{Z}_2^N & \longrightarrow & \mathbb{T}^N \end{array}$$

where $F_N = \mathbb{Z}^{*N}$ is the free group on N generators, and $L_N = \mathbb{Z}_2^{*N}$ is its real version.

Proof. The basic tori appear indeed as group duals, and together with the Fourier transform identifications from Theorem 13.21 (2), this gives the result. □

Moving ahead, now that we have our formalism, we can start developing free geometry. As a first objective, we would like to better understand the relation between the classical and free tori. In order to discuss this, let us introduce the following notion:

Definition 13.24. *Given a compact quantum space X, its classical version is the usual compact space $X_{class} \subset X$ obtained by dividing $C(X)$ by its commutator ideal:*

$$C(X_{class}) = C(X)/I \quad , \quad I = < [a,b] >$$

In this situation, we also say that X appears as a "liberation" of X.

In other words, the space X_{class} appears as the Gelfand spectrum of the commutative C^*-algebra $C(X)/I$. Observe in particular that X_{class} is indeed a classical space.

In relation now with our tori, we have the following result:

Theorem 13.25. *We have inclusions between the various tori, as follows,*

$$\begin{array}{ccc} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{array}$$

and the free tori on top appear as liberations of the tori on the bottom.

Proof. This is indeed clear from definitions, because commutativity of a group algebra means precisely that the group in question is abelian. □

In order to extend now the free geometries that we have, real and complex, let us begin with the spheres. We have the following notions:

Definition 13.26. *We have free real and complex spheres, defined via*

$$C(S^{N-1}_{\mathbb{R},+})=C^*\left(x_1,\ldots,x_N\Big|x_i=x_i^*,\sum_ix_i^2=1\right)$$

$$C(S^{N-1}_{\mathbb{C},+})=C^*\left(x_1,\ldots,x_N\Big|\sum_ix_ix_i^*=\sum_ix_i^*x_i=1\right)$$

where the symbol C^* stands for universal enveloping C^*-algebra.

Here the fact that these algebras are indeed well-defined comes from the following estimate, which shows that the biggest C^*-norms on these $*$-algebras are bounded:

$$||x_i||^2=||x_ix_i^*||\leq\left|\left|\sum_ix_ix_i^*\right|\right|=1$$

As a first result now, regarding the above free spheres, we have:

Theorem 13.27. *We have embeddings of compact quantum spaces, as follows,*

$$\begin{array}{ccc}
S^{N-1}_{\mathbb{R},+}&\longrightarrow&S^{N-1}_{\mathbb{C},+}\\
\uparrow&&\uparrow\\
S^{N-1}_{\mathbb{R}}&\longrightarrow&S^{N-1}_{\mathbb{C}}
\end{array}$$

and the spaces on top appear as liberations of the spaces on the bottom.

Proof. The first assertion, regarding the inclusions, comes from the fact that at the level of the associated C^*-algebras, we have surjective maps, as follows:

$$\begin{array}{ccc}
C(S^{N-1}_{\mathbb{R},+})&\longleftarrow&C(S^{N-1}_{\mathbb{C},+})\\
\downarrow&&\downarrow\\
C(S^{N-1}_{\mathbb{R}})&\longleftarrow&C(S^{N-1}_{\mathbb{C}})
\end{array}$$

For the second assertion, we must establish the following isomorphisms, where the symbol C^*_{comm} stands for "universal commutative C^*-algebra generated by":

$$C(S^{N-1}_{\mathbb{R}})=C^*_{comm}\left(x_1,\ldots,x_N\Big|x_i=x_i^*,\sum_ix_i^2=1\right)$$

$$C(S^{N-1}_{\mathbb{C}})=C^*_{comm}\left(x_1,\ldots,x_N\Big|\sum_ix_ix_i^*=\sum_ix_i^*x_i=1\right)$$

It is enough to establish the second isomorphism. So, consider the second universal commutative C^*-algebra A constructed above. Since the standard coordinates on $S^{N-1}_\mathbb{C}$ satisfy the defining relations for A, we have a quotient map of as follows:

$$A \to C(S^{N-1}_\mathbb{C})$$

Conversely, let us write $A = C(S)$, by using the Gelfand theorem. The variables $x_1, \ldots, x_N$ become in this way true coordinates, providing us with an embedding $S \subset \mathbb{C}^N$. Also, the quadratic relations become $\sum_i |x_i|^2 = 1$, so we have $S \subset S^{N-1}_\mathbb{C}$. Thus, we have a quotient map $C(S^{N-1}_\mathbb{C}) \to A$, as desired, and this gives all the results. □

By using the free spheres constructed above, we can now formulate:

Definition 13.28. *A real algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$ is a closed quantum subspace defined, at the level of the corresponding C^*-algebra, by a formula of type*

$$C(X) = C(S^{N-1}_{\mathbb{C},+}) \Big/ \Big\langle f_i(x_1, \ldots, x_N) = 0 \Big\rangle$$

for certain family of noncommutative polynomials, as follows:

$$f_i \in \mathbb{C} < x_1, \ldots, x_N >$$

We denote by $\mathcal{C}(X)$ the $*$-subalgebra of $C(X)$ generated by the coordinates $x_1, \ldots, x_N$.

As a basic example here, we have the free real sphere $S^{N-1}_{\mathbb{R},+}$. The classical spheres $S^{N-1}_\mathbb{C}, S^{N-1}_\mathbb{R}$, and their real submanifolds, are covered as well by this formalism. At the level of the general theory, we have the following version of the Gelfand theorem:

Theorem 13.29. *If $X \subset S^{N-1}_{\mathbb{C},+}$ is an algebraic manifold, as above, we have*

$$X_{class} = \left\{ x \in S^{N-1}_\mathbb{C} \Big| f_i(x_1, \ldots, x_N) = 0 \right\}$$

and X appears as a liberation of X_{class}.

Proof. This is something that we already met, in the context of the free spheres. In general, the proof is similar, by using the Gelfand theorem. Indeed, if we denote by X'_{class} the manifold constructed in the statement, then we have a quotient map of C^*-algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X_{class}) \to C(X'_{class})$$

Conversely now, from $X \subset S^{N-1}_{\mathbb{C},+}$ we obtain $X_{class} \subset S^{N-1}_\mathbb{C}$. Now since the relations defining X'_{class} are satisfied by X_{class}, we obtain an inclusion $X_{class} \subset X'_{class}$. Thus, at the level of algebras of continuous functions, we have a quotient map of C^*-algebras as follows, mapping standard coordinates to standard coordinates:

$$C(X'_{class}) \to C(X_{class})$$

Thus, we have constructed a pair of inverse morphisms, and we are done. □

Finally, once again at the level of the general theory, we have:

Definition 13.30. *We agree to identify two real algebraic submanifolds $X, Y \subset S^{N-1}_{\mathbb{C},+}$ when we have a $*$-algebra isomorphism between $*$-algebras of coordinates*

$$f : \mathcal{C}(Y) \to \mathcal{C}(X)$$

mapping standard coordinates to standard coordinates.

We will see later the reasons for making this convention, coming from amenability. Now back to the tori, as constructed before, we can see that these are examples of algebraic manifolds, in the sense of Definition 13.28. In fact, we have the following result:

Theorem 13.31. *The four main quantum spheres produce the main quantum tori*

$$\begin{array}{ccc} S^{N-1}_{\mathbb{R},+} & \longrightarrow & S^{N-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{C}} \end{array} \quad \to \quad \begin{array}{ccc} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{array}$$

via the formula $T = S \cap \mathbb{T}_N^+$, with the intersection being taken inside $S^{N-1}_{\mathbb{C},+}$.

Proof. This comes from the above results, the situation being as follows:

(1) Free complex case. Here the formula in the statement reads $\mathbb{T}_N^+ = S^{N-1}_{\mathbb{C},+} \cap \mathbb{T}_N^+$. But this is something trivial, because we have $\mathbb{T}_N^+ \subset S^{N-1}_{\mathbb{C},+}$.

(2) Free real case. Here the formula in the statement reads $T_N^+ = S^{N-1}_{\mathbb{R},+} \cap \mathbb{T}_N^+$. But this is clear as well, the real version of $\mathbb{T}_N^+$ being T_N^+.

(3) Classical complex case. Here the formula in the statement reads $\mathbb{T}_N = S^{N-1}_{\mathbb{C}} \cap \mathbb{T}_N^+$. But this is clear as well, the classical version of $\mathbb{T}_N^+$ being $\mathbb{T}_N$.

(4) Classical real case. Here the formula in the statement reads $T_N = S^{N-1}_{\mathbb{R}} \cap \mathbb{T}_N^+$. But this follows by intersecting the formulae from the proof of (2) and (3). $\square$

We will be back to free geometry, later in this book.

13d. Cuntz algebras

We would like to end this chapter with an interesting class of C^*-algebras, discovered by Cuntz in [28], and heavily used since then, for various technical purposes:

Definition 13.32. *The Cuntz algebra O_n is the C^*-algebra generated by isometries $S_1, \ldots, S_n$ satisfying the following condition:*

$$S_1 S_1^* + \ldots + S_n S_n^* = 1$$

That is, $O_n \subset B(H)$ is generated by n isometries whose ranges sum up to H.

Observe that H must be infinite dimensional, in order to have isometries as above. In what follows we will prove that O_n is independent on the choice of such isometries, and also that this algebra is simple. We will restrict the attention to the case $n=2$, the proof in general being similar. Let us start with some simple computations, as follows:

PROPOSITION 13.33. *Given a word $i=i_1\ldots i_k$ with $i_l\in\{1,2\}$, we associate to it the element $S_i=S_{i_1}\ldots S_{i_k}$ of the algebra O_2. Then S_i are isometries, and we have*

$$S_i^*S_j=\delta_{ij}1$$

for any two words i,j having the same lenght.

PROOF. We use the relations defining the algebra O_2, namely:

$$S_1^*S_1=S_2^*S_2=1\quad,\quad S_1S_1^*+S_2S_2^*=1$$

The fact that S_i are isometries is clear, here being the check for $i=12$:

$$\begin{aligned}S_{12}^*S_{12}&=(S_1S_2)^*(S_1S_2)\\&=S_2^*S_1^*S_1S_2\\&=S_2^*S_2\\&=1\end{aligned}$$

Regarding the last assertion, by recurrence we just have to establish the formula there for the words of length 1. That is, we want to prove the following formulae:

$$S_1^*S_2=S_2^*S_1=0$$

But these two formulae follow from the fact that the projections $P_i=S_iS_i^*$ satisfy by definition $P_1+P_2=1$. Indeed, we have the following computation:

$$\begin{aligned}P_1+P_2=1&\implies P_1P_2=0\\&\implies S_1S_1^*S_2S_2^*=0\\&\implies S_1^*S_2=S_1^*S_1S_1^*S_2S_2^*S_2=0\end{aligned}$$

Thus, we have the first formula, and the proof of the second one is similar. □

We can use the formulae in Proposition 13.33 as follows:

PROPOSITION 13.34. *Consider words in O_2, meaning products of S_1,S_1^*,S_2,S_2^*.*

(1) *Each word in O_2 is of form 0 or $S_iS_j^*$ for some words i,j.*
(2) *Words of type $S_iS_j^*$ with $l(i)=l(j)=k$ form a system of $2^k\times 2^k$ matrix units.*
(3) *The algebra A_k generated by matrix units in (2) is a subalgebra of A_{k+1}.*

PROOF. Here the first two assertions follow from the formulae in Proposition 13.33, and for the last assertion, we can use the following formula:

$$S_iS_j^*=S_i1S_j^*=S_i(S_1S_1^*+S_2S_2^*)S_j^*$$

Thus, we obtain an embedding of algebras A_k, as in the statement. □

Observe now that the embedding constructed in (3) above is compatible with the matrix unit systems in (2). Consider indeed the following diagram:

$$\begin{array}{ccc} A_{k+1} & \simeq & M_{2^{k+1}}(\mathbb{C}) \\ \cup & & \cup \\ A_k & \simeq & M_{2^k}(\mathbb{C}) \end{array}$$

With the notation $e_{ix,yj} = e_{ij} \otimes e_{xy}$, the inclusion on the right is given by:

$$\begin{aligned} e_{ij} &\to e_{i1,1h} + e_{i2,2j} \\ &= e_{ij} \otimes e_{11} + e_{ij} \otimes e_{22} \\ &= e_{ij} \otimes 1 \end{aligned}$$

Thus, with standard tensor product notations, the inclusion on the right is the canonical inclusion $m \to m \otimes 1$, and so the above diagram becomes:

$$\begin{array}{ccc} A_{k+1} & \simeq & M_2(\mathbb{C})^{\otimes k+1} \\ \cup & & \cup \\ A_k & \simeq & M_2(\mathbb{C})^{\otimes k} \end{array}$$

The passage from the algebra $A = \cup_k A_k \simeq M_2(\mathbb{C})^{\otimes\infty}$ coming from this observation to the full the algebra O_2 that we are interested in can be done by using:

PROPOSITION 13.35. *Each element $X \in < S_1, S_2 > \subset O_2$ decomposes as a finite sum*

$$X = \sum_{i>0} S_1^{*i} X_{-i} + X_0 + \sum_{i>0} X_i S_1^i$$

where each X_i is in the union A of algebras A_k.

PROOF. By linearity and by using Proposition 13.34 we may assume that X is a nonzero word, say $X = S_i S_j^*$. In the case $l(i) = l(j)$ we can set $X_0 = X$ and we are done. Otherwise, we just have to add at left or at right terms of the form $1 = S_1^* S_1$. For instance $X = S_2$ is equal to $S_2 S_1^* S_1$, and we can take $X_1 = S_2 S_1^* \in A_1$. □

We must show now that the decomposition $X \to (X_i)$ found above is unique, and then prove that each application $X \to X_i$ has good continuity properties. The following formulae show that in both problems we may restrict attention to the case $i = 0$:

$$X_{i+1} = (X S_1^*)_i \qquad\qquad X_{-i-1} = (S_1 X)_i$$

In order to solve these questions, we use the following fact:

PROPOSITION 13.36. *If P is a nonzero projection in $\mathcal{O}_2 =< S_1, S_2 >\subset O_2$, its k-th average, given by the formula*

$$Q = \sum_{l(i)=k} S_i P S_i^*$$

is a nonzero projection in $\mathcal{O}_2$ having the property that the linear subspace QA_kQ is isomorphic to a matrix algebra, and $Y \to QYQ$ is an isomorphism of A_k onto it.

PROOF. We know that the words of form $S_iS_j^*$ with $l(i) = l(j) = k$ are a system of matrix units in A_k. We apply to them the map $Y \to QYQ$, and we obtain:

$$\begin{aligned} QS_iS_j^*Q &= \sum_{pq} S_pPS_p^*S_iS_j^*S_qPS_q^* \\ &= \sum_{pq} \delta_{ip}\delta_{jq}S_pP^2S_q^* \\ &= S_iPS_j^* \end{aligned}$$

The output being a system of matrix units, $Y \to QYQ$ is an isomorphism from the algebra of matrices A_k to another algebra of matrices QA_kQ, and this gives the result. □

Thus any map $Y \to QYQ$ behaves well on the $i = 0$ part of the decomposition on X. It remains to find P such that $Y \to QYQ$ destroys all $i \neq 0$ terms, and we have here:

PROPOSITION 13.37. *Assuming $X_0 \in A_k$, there is a nonzero projection $P \in A$ such that $QXQ = QX_0Q$, where Q is the k-th average of P.*

PROOF. We want $Y \to QYQ$ to map to zero all terms in the decomposition of X, except for X_0. Let us call $M_1, \ldots, M_t \in \mathcal{O}_2 - A$ the terms to be destroyed. We want the following equalities to hold, with the sum over all pairs of length k indices:

$$\sum_{ij} S_iPS_i^*M_qS_jPS_j^* = 0$$

The simplest way is to look for P such that all terms of all sums are 0:

$$S_iPS_i^*M_qS_jPS_j^* = 0$$

By multiplying to the left by S_i^* and to the right by S_j, we want to have:

$$PS_i^*M_qS_jP = 0$$

With $N_z = S_i^*M_qS_j$, where z belongs to some new index set, we want to have:

$$PN_zP = 0$$

Since $N_z \in \mathcal{O}_2 - A$, we can write $N_z = S_{m_z}S_{n_z}^*$ with $l(m_z) \neq l(n_z)$, and we want:

$$PS_{m_z}S_{n_z}^*P = 0$$

In order to do this, we can the projections of form $P = S_rS_r^*$. We want:

$$S_rS_r^*S_{m_z}S_{n_z}^*S_rS_r^* = 0$$

Let K be the biggest length of all m_z, n_z. Assume that we have fixed r, of length bigger than K. If the above product is nonzero then both $S_r^*S_{m_z}$ and $S_{n_z}^*S_r$ must be nonzero, which gives the following equalities of words:

$$r_1 \ldots r_{l(m_z)} = m_z \quad , \quad r_1 \ldots r_{l(n_z)} = n_z$$

Assuming that these equalities hold indeed, the above product reduces as follows:

$$S_rS_{r_{l(r)}}^* \ldots S_{r_{l(m_z)+1}}^* S_{r_{l(n_z)+1}} \ldots S_{r_{l(r)}} S_r^*$$

Now if this product is nonzero, the middle term must be nonzero:

$$S_{r_{l(r)}}^* \ldots S_{r_{l(m_z)+1}}^* S_{r_{l(n_z)+1}} \ldots S_{r_{l(r)}} \neq 0$$

In order for this for hold, the indices starting from the middle to the right must be equal to the indices starting from the middle to the left. Thus r must be periodic, of period $|l(m_z) - l(n_z)| > 0$. But this is certainly possible, because we can take any aperiodic infinite word, and let r be the sequence of first M letters, with M big enough. □

We can now start solving our problems. We first have:

PROPOSITION 13.38. *The decomposition of X is unique, and we have*

$$||X_i|| \leq ||X||$$

for any i.

PROOF. It is enough to do this for $i = 0$. But this follows from the previous result, via the following sequence of equalities and inequalities:

$$\begin{aligned} ||X_0|| &= ||QX_0Q|| \\ &= ||QXQ|| \\ &\leq ||X|| \end{aligned}$$

Thus we got the inequality in the statement. As for the uniqueness part, this follows from the fact that $X_0 \to QX_0Q = QXQ$ is an isomorphism. □

Remember now we want to prove that the Cuntz algebra O_2 does not depend on the choice of the isometries S_1, S_2. In order to do so, let $\overline{\mathcal{O}}_2$ be the completion of the $*$-algebra $\mathcal{O}_2 =< S_1, S_2 >\subset O_2$ with respect to the biggest C^*-norm. We have:

PROPOSITION 13.39. *We have the equivalence*

$$X = 0 \iff X_i = 0, \forall i$$

valid for any element $X \in \overline{\mathcal{O}}_2$.

PROOF. Assume $X_i = 0$ for any i, and choose a sequence $X^k \to X$ with $X^k \in \mathcal{O}_2$. For $\lambda \in \mathbb{T}$ we define a representation ρ_λ in the following way:

$$\rho_\lambda : S_i \to \lambda S_i$$

We have then $\rho_\lambda(Y) = Y$ for any element $Y \in A$. We fix norm one vectors ξ, η and we consider the following continuous functions $f : \mathbb{T} \to \mathbb{C}$:

$$f^k(\lambda) = < \rho_\lambda(X^k)\xi, \eta >$$

From $X^k \to X$ we get, with respect to the usual sup norm of $C(\mathbb{T})$:

$$f^k \to f$$

Each $X^k \in \mathcal{O}_2$ can be decomposed, and f^k is given by the following formula:

$$f^k(\lambda) = \sum_{i>0} \lambda^{-i} < S_1^{*i} X_{-i}^k \xi, \eta > + < X_0 \xi, \eta > + \sum_{i>0} \lambda^i < X_i^k S_1^i \xi, \eta >$$

This is a Fourier type expansion of f^k, that can we write in the following way:

$$f^k(\lambda) = \sum_{j=-\infty}^{\infty} a_j^k \lambda^j$$

By using Proposition 13.38 we obtain that with $k \to \infty$, we have:

$$|a_j^k| \leq ||X_j^k|| \to ||X_j^\infty|| = 0$$

On the other hand we have $a_j^k \to a_j$ with $k \to \infty$. Thus all Fourier coefficients a_j of f are zero, so $f = 0$. With $\lambda = 1$ this gives the following equality:

$$< X\xi, \eta >= 0$$

This is true for arbitrary norm one vectors ξ, η, so $X = 0$ and we are done. □

We can now formulate the Cuntz theorem, from [28], as follows:

THEOREM 13.40 (Cuntz). *Let S_1, S_2 be isometries satisfying $S_1 S_1^* + S_2 S_2^* = 1$.*

1. *The C^*-algebra O_2 generated by S_1, S_2 does not depend on the choice of S_1, S_2.*
2. *For any nonzero $X \in O_2$ there are $A, B \in O_2$ with $AXB = 1$.*
3. *In particular O_2 is simple.*

PROOF. This basically follows from the various results established above:

(1) Consider the canonical projection map $\pi : \overline{O}_2 \to O_2$. We know that π is surjective, and we will prove now that π is injective. Indeed, if $\pi(X) = 0$ then $\pi(X)_i = 0$ for any i. But $\pi(X)_i$ is in the dense $*$-algebra A, so it can be regarded as an element of $\overline{O}_2$, and with this identification, we have $\pi(X)_i = X_i$ in $\overline{O}_2$. Thus $X_i = 0$ for any i, so $X = 0$. Thus π is an isomorphism. On the other hand $\overline{O}_2$ depends only on $\mathcal{O}_2$, and the above formulae in $\mathcal{O}_2$, for algebraic calculus and for decomposition of an arbitrary $X \in \mathcal{O}_2$, show that $\mathcal{O}_2$ does not depend on the choice of S_1, S_2. Thus, we obtain the result.

(2) Choose a sequence $X^k \to X$ with $X^k \in \mathcal{O}_2$. We have the following formula:

$$(X^*X)_0 = \lim_{k\to\infty}\left(\sum_{i>0} X_{-i}^{k*}X_{-i}^k + X_0^{k*}X_0^k + \sum_{i>0} S_1^{*i}X_i^{k*}X_i^kS_1^i\right)$$

Thus $X \neq 0$ implies $(X^*X)_0 \neq 0$. By linearity we can assume that we have:

$$||(X^*X)_0|| = 1$$

Now choose a positive element $Y \in \mathcal{O}_2$ which is close enough to X^*X:

$$||X^*X - Y|| < \varepsilon$$

Since $Z \to Z_0$ is norm decreasing, we have the following estimate:

$$||Y_0|| > 1 - \varepsilon$$

We apply Proposition 13.37 to our positive element $Y \in \mathcal{O}_2$. We obtain in this way a certain projection Q such that $QY_0Q = QYQ$ belongs to a certain matrix algebra. We have $QYQ > 0$, so we can diagonalize this latter element, as follows:

$$QYQ = \sum \lambda_i R_i$$

Here λ_i are positive numbers and R_i are minimal projections in the matrix algebra. Now since $||QYQ|| = ||Y_0||$, there must be an eigenvalue greater that $1-\varepsilon$:

$$\lambda_0 > 1 - \varepsilon$$

By linear algebra, we can pass from a minimal projection to another:

$$U^*U = R_i \quad , \quad UU^* = S_1^kS_1^{*k}$$

The element $B = QU^*S_1^k$ has norm ≤ 1, and we get the following inequality:

$$\begin{aligned} ||1 - B^*X^*XB|| &\leq ||1 - B^*YB|| + ||B^*YB - B^*X^*XB|| \\ &< ||1 - B^*YB|| + \varepsilon \end{aligned}$$

The last term can be computed by using the diagonalization of QYQ, as follows:

$$\begin{aligned} B^*YB &= S_1^{*k}UQYQU^*S_1^k \\ &= S_1^{*k}\left(\sum \lambda_i UR_iU^*\right)S_1^k \\ &= \lambda_0 S_1^{*k}S_1^kS_1^{*k}S_1^k \\ &= \lambda_0 \end{aligned}$$

From $\lambda_0 > 1 - \varepsilon$ we get $||1 - B^*YB|| < \varepsilon$, and we obtain the following estimate:

$$||1 - B^*X^*XB|| < 2\varepsilon$$

Thus B^*X^*XB is invertible, say with inverse C, and we have $(B^*X^*)X(BC) = 1$.

(3) This is clear from the formula $AXB = 1$ established in (2). □

13e. Exercises

Exercises:

Exercise 13.41.

Exercise 13.42.

Exercise 13.43.

Exercise 13.44.

Exercise 13.45.

Exercise 13.46.

Exercise 13.47.

Exercise 13.48.

Bonus exercise.

CHAPTER 14

Von Neumann algebras

14a. Von Neumann algebras

Instead of further building on the above results, which are already quite non-trivial, let us return to our modest status of apprentice operator algebraists, and declare ourselves unsatisfied with the formalism from chapter 13, on the following intuitive grounds:

Thought 14.1. *Our assumption that $A\subset B(H)$ is norm closed is not satisfying, because we would like A to be stable under polar decomposition, under taking spectral projections, and more generally, under measurable functional calculus.*

So, let us get now into this, topologies on $B(H)$, and fine-tunings of our operator algebra formalism, based on them. The result that we will need is as follows:

Proposition 14.2. *For a subalgebra $A\subset B(H)$, the following are equivalent:*

1. *A is closed under the weak operator topology, making each of the linear maps $T\to<Tx,y>$ continuous.*
2. *A is closed under the strong operator topology, making each of the linear maps $T\to Tx$ continuous.*

In the case where these conditions are satisfied, A is closed under the norm topology.

Proof. There are several statements here, the proof being as follows:

(1) It is clear that the norm topology is stronger than the strong operator topology, which is in turn stronger than the weak operator topology. At the level of the subsets $S\subset B(H)$ which are closed things get reversed, in the sense that weakly closed implies strongly closed, which in turn implies norm closed. Thus, we are left with proving that for any algebra $A\subset B(H)$, strongly closed implies weakly closed.

(2) Consider the Hilbert space obtained by summing n times H with itself:

$$K=H\oplus\ldots\oplus H$$

The operators over K can be regarded as being square matrices with entries in $B(H)$, and in particular, we have a representation $\pi:B(H)\to B(K)$, as follows:

$$\pi(T)=\begin{pmatrix}T&&\\&\ddots&\\&&T\end{pmatrix}$$

Assume now that we are given an operator $T \in \bar{A}$, with the bar denoting the weak closure. We have then, by using the Hahn-Banach theorem, for any $x \in K$:

$$\begin{aligned} T \in \bar{A} &\implies \pi(T) \in \overline{\pi(A)} \\ &\implies \pi(T)x \in \overline{\pi(A)x} \\ &\implies \pi(T)x \in \overline{\pi(A)x}^{||.||} \end{aligned}$$

Now observe that the last formula tells us that for any $x = (x_1, \ldots, x_n)$, and any $\varepsilon > 0$, we can find $S \in A$ such that the following holds, for any i:

$$||Sx_i - Tx_i|| < \varepsilon$$

Thus T belongs to the strong operator closure of A, as desired. □

Observe that in the above the terminology is a bit confusing, because the norm topology is stronger than the strong operator topology. As a solution, we agree to call the norm topology "strong", and the weak and strong operator topologies "weak", whenever these two topologies coincide. With this convention made, the algebras $A \subset B(H)$ in Proposition 14.2 are those which are weakly closed. Thus, we can now formulate:

Definition 14.3. *A von Neumann algebra is an operator algebra*

$$A \subset B(H)$$

which is closed under the weak topology.

These algebras will be our main objects of study, in what follows. As basic examples, we have the algebra $B(H)$ itself, then the singly generated algebras, $A =< T >$ with $T \in B(H)$, and then the multiply generated algebras, $A =< T_i >$ with $T_i \in B(H)$. But for the moment, let us keep things simple, and build directly on Definition 14.3, by using basic functional analysis methods. We will need the following key result:

Theorem 14.4. *For an operator algebra $A \subset B(H)$, we have*

$$A'' = \bar{A}$$

with A'' being the bicommutant inside $B(H)$, and $\bar{A}$ being the weak closure.

Proof. We can prove this by double inclusion, as follows:

"$\supset$" Since any operator commutes with the operators that it commutes with, we have a trivial inclusion $S \subset S''$, valid for any set $S \subset B(H)$. In particular, we have:

$$A \subset A''$$

Our claim now is that the algebra A'' is closed, with respect to the strong operator topology. Indeed, assuming that we have $T_i \to T$ in this topology, we have:

$$\begin{aligned} T_i \in A'' &\implies ST_i = T_iS,\ \forall S \in A' \\ &\implies ST = TS,\ \forall S \in A' \\ &\implies T \in A \end{aligned}$$

Thus our claim is proved, and together with Proposition 14.2, which allows us to pass from the strong to the weak operator topology, this gives $\bar{A} \subset A''$, as desired.

"$\subset$" Here we must prove that we have the following implication, valid for any $T \in B(H)$, with the bar denoting as usual the weak operator closure:

$$T \in A'' \implies T \in \bar{A}$$

For this purpose, we use the same amplification trick as in the proof of Proposition 14.2. Consider the Hilbert space obtained by summing n times H with itself:

$$K = H \oplus \ldots \oplus H$$

The operators over K can be regarded as being square matrices with entries in $B(H)$, and in particular, we have a representation $\pi : B(H) \to B(K)$, as follows:

$$\pi(T) = \begin{pmatrix} T && \\ & \ddots & \\ && T \end{pmatrix}$$

The idea will be that of doing the computations in this representation. First, in this representation, the image of our algebra $A \subset B(H)$ is given by:

$$\pi(A) = \left\{ \begin{pmatrix} T && \\ & \ddots & \\ && T \end{pmatrix} \Big| T \in A \right\}$$

We can compute the commutant of this image, exactly as in the usual scalar matrix case, and we obtain the following formula:

$$\pi(A)' = \left\{ \begin{pmatrix} S_{11} & \ldots & S_{1n} \\ \vdots && \vdots \\ S_{n1} & \ldots & S_{nn} \end{pmatrix} \Big| S_{ij} \in A' \right\}$$

We conclude from this that, given an operator $T \in A''$ as above, we have:

$$\begin{pmatrix} T && \\ & \ddots & \\ && T \end{pmatrix} \in \pi(A)''$$

In other words, the conclusion of all this is that we have:

$$T \in A'' \implies \pi(T) \in \pi(A)''$$

Now given a vector $x \in K$, consider the orthogonal projection $P \in B(K)$ on the norm closure of the vector space $\pi(A)x \subset K$. Since the subspace $\pi(A)x \subset K$ is invariant under the action of $\pi(A)$, so is its norm closure inside K, and we obtain from this:

$$P \in \pi(A)'$$

By combining this with what we found above, we conclude that we have:

$$T \in A'' \implies \pi(T)P = P\pi(T)$$

Since this holds for any $x \in K$, we conclude that any operator $T \in A''$ belongs to the strong operator closure of A. By using now Proposition 14.2, which allows us to pass from the strong to the weak operator closure, we conclude that we have:

$$A'' \subset \bar{A}$$

Thus, we have the desired reverse inclusion, and this finishes the proof. □

Now by getting back to the von Neumann algebras, from Definition 14.3, we have the following result, which is a reformulation of Theorem 14.4, by using this notion:

Theorem 14.5. *For an operator algebra $A \subset B(H)$, the following are equivalent:*

(1) *A is weakly closed, so it is a von Neumann algebra.*
(2) *A equals its algebraic bicommutant A'', taken inside $B(H)$.*

Proof. This follows from the formula $A'' = \bar{A}$ from Theorem 14.4, along with the trivial fact that the commutants are automatically weakly closed. □

The above statement, called bicommutant theorem, and due to von Neumann [91], is quite interesting, philosophically speaking. Among others, it shows that the von Neumann algebras are exactly the commutants of the self-adjoint sets of operators:

Proposition 14.6. *Given a subset $S \subset B(H)$ which is closed under $*$, the commutant*

$$A = S'$$

is a von Neumann algebra. Any von Neumann algebra appears in this way.

Proof. We have two assertions here, the idea being as follows:

(1) Given $S \subset B(H)$ satisfying $S = S^*$, the commutant $A = S'$ satisfies $A = A^*$, and is also weakly closed. Thus, A is a von Neumann algebra. Note that this follows as well from the following "tricommutant formula", which follows from Theorem 14.5:

$$S''' = S'$$

(2) Given a von Neumann algebra $A \subset B(H)$, we can take $S = A'$. Then S is closed under the involution, and we have $S' = A$, as desired. □

Observe that Proposition 14.6 can be regarded as yet another alternative definition for the von Neumann algebras, and with this definition being probably the best one when talking about quantum mechanics, where the self-adjoint operators $T : H \to H$ can be though of as being "observables" of the system, and with the commutants $A = S'$ of the sets of such observables $S = \{T_i\}$ being the algebras $A \subset B(H)$ that we are interested in. And with all this actually needing some discussion about self-adjointness, and about boundedness too, but let us not get into this here, and stay mathematical, as before.

As another interesting consequence of Theorem 14.5, we have:

PROPOSITION 14.7. *Given a von Neumann algebra $A \subset B(H)$, its center*

$$Z(A) = A \cap A'$$

regarded as an algebra $Z(A) \subset B(H)$, is a von Neumann algebra too.

PROOF. This follows from the fact that the commutants are weakly closed, that we know from the above, which shows that $A' \subset B(H)$ is a von Neumann algebra. Thus, the intersection $Z(A) = A \cap A'$ must be a von Neumann algebra too, as claimed. $\square$

In order to develop some general theory, let us start by investigating the finite dimensional case. Here the ambient algebra is $B(H) = M_N(\mathbb{C})$, any linear subspace $A \subset B(H)$ is automatically closed, for all 3 topologies in Proposition 14.2, and we have:

THEOREM 14.8. *The $*$-algebras $A \subset M_N(\mathbb{C})$ are exactly the algebras of the form*

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

depending on parameters $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \mathbb{N}$ satisfying

$$n_1 + \ldots + n_k = N$$

embedded into $M_N(\mathbb{C})$ via the obvious block embedding, twisted by a unitary $U \in U_N$.

PROOF. This is something algebraic, that we know from chapter 13, and which, retrospectively thinking, is based on the "center philosophy" from Proposition 14.7. $\square$

In relation with the bicommutant theorem, we have the following result, which fully clarifies the situation, with a very explicit proof, in finite dimensions:

PROPOSITION 14.9. *Consider a $*$-algebra $A \subset M_N(\mathbb{C})$, written as above:*

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

The commutant of this algebra is then, with respect with the block decomposition used,

$$A' = \mathbb{C} \oplus \ldots \oplus \mathbb{C}$$

and by taking one more time the commutant we obtain A itself, $A = A''$.

PROOF. Let us decompose indeed our algebra A as in Theorem 14.8:

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

The center of each matrix algebra being reduced to the scalars, the commutant of this algebra is then as follows, with each copy of $\mathbb{C}$ corresponding to a matrix block:

$$A' = \mathbb{C} \oplus \ldots \oplus \mathbb{C}$$

By taking once again the commutant we obtain A itself, and we are done. ☐

As another interesting application of Theorem 14.8, clarifying this time the relation with operator theory, in finite dimensions, we have the following result:

THEOREM 14.10. *Given an operator $T \in B(H)$ in finite dimensions, $H = \mathbb{C}^N$, the von Neumann algebra $A = < T >$ that it generates inside $B(H) = M_N(\mathbb{C})$ is*

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

with the sizes of the blocks $n_1, \ldots, n_k \in \mathbb{N}$ coming from the spectral theory of the associated matrix $M \in M_N(\mathbb{C})$. In the normal case $TT^* = T^*T$, this decomposition comes from

$$T = UDU^*$$

with $D \in M_N(\mathbb{C})$ diagonal, and with $U \in U_N$ unitary.

PROOF. This is something which is routine, by using the linear algebra and spectral theory developed in chapter 1, for the matrices $M \in M_N(\mathbb{C})$. To be more precise:

(1) The fact that $A = < T >$ decomposes into a direct sum of matrix algebras is something that we already know, coming from Theorem 14.8.

(2) By using standard linear algebra, we can compute the block sizes $n_1, \ldots, n_k \in \mathbb{N}$, from the knowledge of the spectral theory of the associated matrix $M \in M_N(\mathbb{C})$.

(3) In the normal case, $TT^* = T^*T$, we can simply invoke the spectral theorem, and by suitably changing the basis, we are led to the conclusion in the statement. ☐

Let us get now to infinite dimensions, with Theorem 14.10 as our main source of inspiration. The same argument applies, provided that we are in the normal case, and we have the following result, summarizing our basic knowledge here:

THEOREM 14.11. *Given a bounded operator $T \in B(H)$ which is normal, $TT^* = T^*T$, the von Neumann algebra $A = < T >$ that it generates inside $B(H)$ is*

$$< T > = L^\infty(\sigma(T))$$

with $\sigma(T) \subset \mathbb{C}$ being as usual its spectrum.

PROOF. The measurable functional calculus theorem for the normal operators tells us that we have a weakly continuous morphism of $*$-algebras, as follows:

$$L^\infty(\sigma(T)) \to B(H) \quad , \quad f \to f(T)$$

Moreover, by the general properties of the measurable calculus, also established in chapter 5, this morphism is injective, and its image is the weakly closed algebra $< T >$ generated by T, T^*. Thus, we obtain the isomorphism in the statement. $\square$

More generally now, along the same lines, we have the following result:

THEOREM 14.12. *Given operators $T_i \in B(H)$ which are normal, and which commute, the von Neumann algebra $A =< T_i >$ that these operators generates inside $B(H)$ is*

$$< T_i >= L^\infty(X)$$

with X being a certain measured space, associated to the family $\{T_i\}$.

PROOF. This is once again routine, by using the spectral theory for the families of commuting normal operators $T_i \in B(H)$ developed in chapter 5. $\square$

As a fundamental consequence now of the above results, we have:

THEOREM 14.13. *The commutative von Neumann algebras are the algebras*

$$A = L^\infty(X)$$

with X being a measured space.

PROOF. We have two assertions to be proved, the idea being as follows:

(1) In one sense, we must prove that given a measured space X, we can realize the $A = L^\infty(X)$ as a von Neumann algebra, on a certain Hilbert space H. But this is something that we know since chapter 1, the representation being as follows:

$$L^\infty(X) \subset B(L^2(X)) \quad , \quad f \to (g \to fg)$$

(2) In the other sense, given a commutative von Neumann algebra $A \subset B(H)$, we must construct a certain measured space X, and an identification $A = L^\infty(X)$. But this follows from Theorem 14.12, because we can write our algebra as follows:

$$A =< T_i >$$

To be more precise, A being commutative, any element $T \in A$ is normal, so we can pick a basis $\{T_i\} \subset A$, and then we have $A =< T_i >$ as above, with $T_i \in B(H)$ being commuting normal operators. Thus Theorem 14.12 applies, and gives the result.

(3) Alternatively, and more explicitly, we can deduce this from Theorem 14.11, applied with $T = T^*$. Indeed, by using $T = Re(T) + iIm(T)$, we conclude that any von Neumann algebra $A \subset B(H)$ is generated by its self-adjoint elements $T \in A$. Moreover, by using

measurable functional calculus, we conclude that A is linearly generated by its projections. But then, assuming $A = \overline{span}\{p_i\}$, with p_i being projections, we can set:

$$T = \sum_{i=0}^{\infty} \frac{p_i}{3^i}$$

Then $T = T^*$, and by functional calculus we have $p_0 \in < T >$, then $p_1 \in < T >$, and so on. Thus $A = < T >$, and $A = L^\infty(X)$ comes now via Theorem 14.11, as claimed. $\square$

The above result is the foundation for all the advanced von Neumann algebra theory, that we will discuss in the remainder of this book, and there are many things that can be said about it. To start with, in relation with the general theory of the normed closed algebras, that we developed in the beginning of this chapter, we have:

WARNING 14.14. *Although the von Neumann algebras are norm closed, the theory of norm closed algebras does not always apply well to them. For instance for $A = L^\infty(X)$ Gelfand gives $A = C(\widehat{X})$, with $\widehat{X}$ being a certain technical compactification of X.*

In short, this would be my advice, do not mess up the two theories that we will be developing in this book, try finding different rooms for them, in your brain. At least at this stage of things, because later, do not worry, we will be playing with both.

Now forgetting about Gelfand, and taking Theorem 14.13 as such, tentative foundation for the theory that we want to develop, as a first consequence of this, we have:

THEOREM 14.15. *Given a von Neumann algebra $A \subset B(H)$, we have*

$$Z(A) = L^\infty(X)$$

with X being a certain measured space.

PROOF. We know from Proposition 14.7 that the center $Z(A) \subset B(H)$ is a von Neumann algebra. Thus Theorem 14.13 applies, and gives the result. $\square$

It is possible to further build on this, with a powerful decomposition result as follows, over the measured space X constructed in Theorem 14.15:

$$A = \int_X A_x \, dx$$

But more on this later, after developing the appropriate tools for this program, which is something non-trivial. Among others, before getting into such things, we will have to study the von Neumann algebras A having trivial center, $Z(A) = \mathbb{C}$, called factors, which include the fibers A_x in the above decomposition result. More on this later.

14b. Kaplansky density

Time now for some more advanced von Neumann algebra theory, and hang on, all this will be quite technical. Let us begin our study with some generalities. We first have:

PROPOSITION 14.16. *The weak operator topology on $B(H)$ is the topology having the following equivalent properties:*

1. *It makes $T \to < Tx, y >$ continuous, for any $x, y \in H$.*
2. *It makes $T_n \to T$ when $< T_n x, y > \to < Tx, y >$, for any $x, y \in H$.*
3. *Has as subbase the sets $U_T(x, y, \varepsilon) = \{S : | < (S-T)x, y > | < \varepsilon\}$.*
4. *Has as base $U_T(x_1, \ldots, x_n, y_1, \ldots, y_n, \varepsilon) = \{S : | < (S-T)x_i, y_i > | < \varepsilon, \forall i\}$.*

PROOF. The equivalences $(1) \iff (2) \iff (3) \iff (4)$ all follow from definitions, with of course (1,2) referring to the coarsest topology making that things happen. □

Similarly, in what regards the strong operator topology, we have:

PROPOSITION 14.17. *The strong operator topology on $B(H)$ is the topology having the following equivalent properties:*

1. *It makes $T \to Tx$ continuous, for any $x \in H$.*
2. *It makes $T_n \to T$ when $T_n x \to Tx$, for any $x \in H$.*
3. *Has as subbase the sets $V_T(x, \varepsilon) = \{S : ||(S-T)x|| < \varepsilon\}$.*
4. *Has as base the sets $V_T(x_1, \ldots, x_n, \varepsilon) = \{S : ||(S-T)x_i|| < \varepsilon, \forall i\}$.*

PROOF. Again, the equivalences $(1) \iff (2) \iff (3) \iff (4)$ are all clear, and with (1,2) referring to the coarsest topology making that things happen. □

We know from before that an operator algebra $A \subset B(H)$ is weakly closed if and only if it is strongly closed. Here is a useful generalization of this fact:

THEOREM 14.18. *Given a convex set of bounded operators*

$$C \subset B(H)$$

its weak operator closure and strong operator closure coincide.

PROOF. Since the weak operator topology on $B(H)$ is weaker by definition than the strong operator topology on $B(H)$, we have, for any subset $C \subset B(H)$:

$$\overline{C}^{strong} \subset \overline{C}^{weak}$$

Now by assuming that $C \subset B(H)$ is convex, we must prove that:

$$T \in \overline{C}^{weak} \implies T \in \overline{C}^{strong}$$

In order to do so, let us pick vectors $x_1, \ldots, x_n \in H$ and $\varepsilon > 0$. We let $K = H^{\oplus n}$, and we consider the standard embedding $i : B(H) \subset B(K)$, given by:

$$iT(y_1, \ldots, y_n) = (Ty_1, \ldots, Ty_n)$$

We have then the following implications, which are all trivial:

$$T \in \overline{C}^{weak} \implies iT \in \overline{iC}^{weak} \implies iT(x) \in \overline{iC(x)}^{weak}$$

Now since the set $C \subset B(H)$ was assumed to be convex, the set $iC(x) \subset K$ is convex too, and by the Hahn-Banach theorem, for compact sets, it follows that we have:

$$iT(x) \in \overline{iC(x)}^{||.||}$$

Thus, there exists an operator $S \in C$ such that we have, for any i:

$$||Sx_i - Tx_i|| < \varepsilon$$

But this shows that we have $S \in V_T(x_1, \ldots, x_n, \varepsilon)$, and since $x_1, \ldots, x_n \in H$ and $\varepsilon > 0$ were arbitrary, by Proposition 14.17 it follows that we have $T \in \overline{C}^{strong}$, as desired. □

We will need as well the following standard result:

PROPOSITION 14.19. *Given a vector space $E \subset B(H)$, and a linear form $f : E \to \mathbb{C}$, the following conditions are equivalent:*

(1) *f is weakly continuous.*
(2) *f is strongly continuous.*
(3) *$f(T) = \sum_{i=1}^n < Tx_i, y_i >$, for certain vectors $x_i, y_i \in H$.*

PROOF. This is something standard, using the same tools at those already used in chapter 5, namely basic functional analysis, and amplification tricks:

(1) $\implies$ (2) Since the weak operator topology on $B(H)$ is weaker than the strong operator topology on $B(H)$, weakly continuous implies strongly continuous. To be more precise, assume $T_n \to T$ strongly. Then $T_n \to T$ weakly, and since f was assumed to be weakly continuous, we have $f(T_n) \to f(T)$. Thus f is strongly continuous, as desired.

(2) $\implies$ (3) Assume indeed that our linear form $f : E \to \mathbb{C}$ is strongly continuous. In particular f is strongly continuous at 0, and Proposition 14.17 provides us with vectors $x_1, \ldots, x_n \in H$ and a number $\varepsilon > 0$ such that, with the notations there:

$$f(V_0(x_1, \ldots, x_n, \varepsilon)) \subset D_0(1)$$

That is, we can find vectors $x_1, \ldots, x_n \in H$ and a number $\varepsilon > 0$ such that:

$$||Tx_i|| < \varepsilon, \forall i \implies |f(T)| < 1$$

But this shows that we have the following estimate:

$$\sum_{i=1}^n ||Tx_i||^2 < \varepsilon^2 \implies |f(T)| < 1$$

By linearity, it follows from this that we have the following estimate:

$$|f(T)| < \frac{1}{\varepsilon}\sqrt{\sum_{i=1}^n ||Tx_i||^2}$$

Consider now the direct sum $H^{\oplus n}$, and inside it, the following vector:

$$x = (x_1, \ldots, x_n) \in H^{\oplus n}$$

Consider also the following linear space, written in tensor product notation:

$$K = \overline{(E \otimes 1)x} \subset H^{\oplus n}$$

We can define a linear form $f' : K \to \mathbb{C}$ by the following formula, and continuity:

$$f'(Tx_1, \ldots, Tx_n) = f(T)$$

We conclude that there exists a vector $y \in K$ such that the following happens:

$$f'\big((T \otimes 1)y\big) =< (T \otimes 1)x, y >$$

But in terms of the original linear form $f : E \to \mathbb{C}$, this means that we have:

$$f(T) = \sum_{i=1}^n < Tx_i, y_i >$$

(3) $\implies$ (1) This is clear, because we have, with respect to the weak topology:

$$\begin{aligned} T_n \to T \quad &\implies \quad < T_n x_i, y_i > \to < Tx_i, y_i >, \forall i \\ &\implies \quad \sum_{i=1}^n < T_n x_i, y_i > \to \sum_{i=1}^n < Tx_i, y_i > \\ &\implies \quad f(T_n) \to f(T) \end{aligned}$$

Thus, our linear form f is weakly continuous, as desired. □

Here is one more well-known result, that we will need as well:

Theorem 14.20. *The unit ball of $B(H)$ is weakly compact.*

Proof. If we denote by $B_1 \subset B(H)$ the unit ball, and by $D_1 \subset \mathbb{C}$ the unit disk, we have a morphism as follows, which is continuous with respect to the weak topology on B_1, and with respect to the product topology on the set on the right:

$$B_1 \subset \prod_{||x||,||y|| \leq 1} D_1 \quad , \quad T \to (< Tx, y >)_{x,y}$$

Since the set on the right is compact, by Tychonoff, it is enough to show that the image of B_1 is closed. So, let $(c_{xy}) \in \bar{B}_1$. We can then find $T_i \in B_1$ such that:

$$< T_i x, y > \to c_{xy} \quad , \quad \forall x, y$$

But this shows that the following map is a bounded sesquilinear form:

$$H \times H \to \mathbb{C} \quad , \quad (x,y) \to c_{xy}$$

Thus, we can find an operator $T \in B(H)$, and so $T \in B_1$, such that $< Tx, y >= c_{xy}$ for any $x, y \in H$, and this shows that we have $(c_{xy}) \in B_1$, as desired. $\square$

Getting back to operator algebras, we have the following result, due to Kaplansky, which is something very useful, and of independent interest as well:

THEOREM 14.21. *Given an operator algebra $A \subset B(H)$, the following happen:*

1. *The unit ball of A is strongly dense in the unit ball of A''.*
2. *The same happens for the self-adjoint parts of the above unit balls.*

PROOF. This is something quite tricky, the idea being as follows:

(1) Consider the self-adjoint part $A_{sa} \subset A$. By taking real parts of operators, and using the fact that $T \to T^*$ is weakly continuous, we have then:

$$\overline{A_{sa}}^w \subset \left(\overline{A}^w\right)_{sa}$$

Now since the set A_{sa} is convex, and by Theorem 14.18 all weak operator topologies coincide on the convex sets, we conclude that we have in fact equality:

$$\overline{A_{sa}}^w = \left(\overline{A}^w\right)_{sa}$$

(2) With this result in hand, let us prove now the second assertion of the theorem. For this purpose, consider an element $T \in \overline{A}^w$, satisfying $T = T^*$ and $||T|| \leq 1$. Consider as well the following function, going from the interval $[-1, 1]$ to itself:

$$f(t) = \frac{2t}{1+t^2}$$

By functional calculus we can find an element $S \in \left(\overline{A}^w\right)_{sa}$ such that:

$$f(S) = T$$

In other words, we can find an element $S \in \left(\overline{A}^w\right)_{sa}$ such that:

$$T = \frac{2S}{1+S^2}$$

Now given arbitrary vectors $x_1, \ldots, x_n \in H$ and an arbitrary number $\varepsilon > 0$, let us pick an element $R \in A_{sa}$, subject to the following two inequalities:

$$||RTx_i - STx_i|| \leq \varepsilon \quad , \quad \left|\left|\frac{R}{1+S^2}x_i - \frac{S}{1+S^2}x_i\right|\right| \leq \varepsilon$$

Finally, consider the following element, which has norm ≤ 1:

$$L = \frac{2R}{1+R^2}$$

We have then the following computation, using the above formulae:

$$\begin{aligned} L-T &= \frac{2R}{1+R^2}-\frac{2S}{1+S^2} \\ &= 2\left(\frac{1}{1+R^2}\big(R(1+S^2)-(1+S^2)R\big)\frac{1}{1+S^2}\right) \\ &= 2\left(\frac{1}{1+R^2}(R-S)\frac{1}{1+S^2}+\frac{R}{1+R^2}(S-R)\frac{S}{1+S^2}\right) \\ &= \frac{2}{1+R^2}(R-S)\frac{1}{1+S^2}+\frac{L}{2}(S-R)T \end{aligned}$$

Thus, we have the following estimate, for any $i\in\{1,\ldots,n\}$:

$$||(L-T)x_i||\leq\varepsilon$$

But this gives the second assertion of the theorem, as desired.

(3) Let us prove now the first assertion of the theorem. Given an arbitrary element $T\in\overline{A}^w$, satisfying $||T||\leq 1$, let us look at the following element:

$$T'=\begin{pmatrix}0&T\\T^*&0\end{pmatrix}\in M_2(\overline{A}^w)$$

This element is then self-adjoint, and we can use what we proved in the above, and we are led in this way to the first assertion in the statement, as desired. □

We can go back now to our original question, from the beginning of the present chapter, namely that of abstractly characterizing the von Neumann algebras, and we have:

Theorem 14.22. *A norm closed operator $*$-algebra*

$$A\subset B(H)$$

is a von Neumann algebra precisely when its unit ball is weakly compact.

Proof. This is something which is now clear, coming from the Kaplansky density results established in Theorem 14.21. To be more precise:

(1) In one sense, assuming that $A\subset B(H)$ is a von Neumann algebra, this algebra is weakly closed. But since the unit ball of $B(H)$ is weakly compact, we are led to the conclusion that the unit ball of A is weakly compact too.

(2) Conversely, assume that an operator algebra $A\subset B(H)$ is such that its unit ball is weakly compact. In particular, the unit ball of A is weakly closed. Now if T satisfying $||T||\leq 1$ belongs to the weak closure of A, by Kaplansky density we conclude that we have $T\in A$. Thus our algebra A must be a von Neumann algebra, as claimed. □

14c. Projections, order

In order to further investigate the von Neumann algebras, the key idea, coming from our previous analysis of the finite dimensional algebras, will be that of looking at the projections. Let us start with some generalities. In analogy with what happens in finite dimensions, we have the following notions, over an arbitrary Hilbert space H:

DEFINITION 14.23. *Associated to any two projections $P,Q\in B(H)$ are:*
(1) *The projection $P\wedge Q$, projecting on the common range.*
(2) *The projection $P\vee Q$, projecting on the span of the ranges.*

Abstractly speaking, these two operations can be thought of as being inf and sup type operations, and all the known algebraic formulae for inf and sup hold in this setting. For the moment we will not need all this, and we will be back to it later. Let us record however the following basic formula, which is something very useful:

PROPOSITION 14.24. *We have the following formula,*

$$P+Q=P\wedge Q+P\vee Q$$

valid for any two projections $P,Q\in B(H)$.

PROOF. This is clear from definitions, because when computing $P+Q$ we obtain the projection $P\vee Q$ on the span on the ranges, modulo the fact that the vectors in the common range are obtained twice, which amounts in saying that we must add $P\wedge Q$. □

With the above notions in hand, we have the following result:

THEOREM 14.25. *Consider two projections $P,Q\in B(H)$.*
(1) *In finite dimensions, over $H=\mathbb{C}^N$, we have, in norm:*

$$(PQ)^n\to P\wedge Q$$

(2) *In infinite dimensions, we have the following convergence, for any $x\in H$,*

$$(PQ)^nx\to(P\wedge Q)x$$

but the operators $(PQ)^n$ do not necessarily converge in norm.

PROOF. We have several assertions here, the proof being as follows:

(1) Assume that we are in the case $P,Q\in M_N(\mathbb{C})$. By substracting $P\wedge Q$ from both P,Q, we can assume $P\wedge Q=0$, and we must prove that we have:

$$P\wedge Q=0\implies(PQ)^n\to 0$$

Our claim is that we have $||PQ||<1$. Indeed, we know that we have:

$$||PQ||\leq||P||\cdot||Q||=1$$

Assuming now by contradiction that we have $||PQ|| = 1$, since we are in finite dimensions, we must have, for a certain norm one vector, $||x|| = 1$:

$$||PQx|| = 1$$

Thus, we must have equalities in the following estimate:

$$||PQx|| \leq ||Qx|| \leq ||x||$$

But the second equality tells us that we must have $x \in Im(Q)$, and with this in hand, the first equality tells us that we must have $x \in Im(P)$. But this contradicts $P \wedge Q = 0$, so we have proved our claim, and the convergence $(PQ)^n \to 0$ follows.

(2) In infinite dimensions now, as before by substracting $P \wedge Q$ from both P, Q, we can assume $P \wedge Q = 0$, and we must prove that we have, for any $x \in H$:

$$P \wedge Q = 0 \implies (PQ)^n x \to 0$$

For this purpose, we use a trick. Consider the following operator:

$$R = PQP$$

This operator is positive, because we have $R = (PQ)(PQ)^*$, and we have:

$$||R|| \leq ||P|| \cdot ||Q|| \cdot ||P|| = 1$$

Our claim, which will finish the proof, is that for any $x \in H$ we have:

$$R^n x \to 0$$

In order to prove this claim, let us diagonalize R, by using the spectral theorem for self-adjoint operators, from chapter 1. If all the eigenvalues are < 1 then we are done. If not, this means that we can find a nonzero vector $x \in H$ such that:

$$||Rx|| = ||x||$$

But this condition means that we must have equalities in the following estimate:

$$||PQPx|| \leq ||QPx|| \leq ||Px|| \leq ||x||$$

The point now is that this is impossible, due to our assumption $P \wedge Q = 0$. Indeed, the last equality tells us that we must have $x \in Im(P)$, and with this in hand, the middle equality tells us that we must have $x \in Im(Q)$. But this contradicts $P \wedge Q = 0$, so we have proved our claim, and the convergence $(PQ)^n x \to 0$ follows.

(3) Finally, for a counterexample to $(PQ)^n \to 0$, in infinite dimensions, we can take $H = l^2(\mathbb{N})$, and then find projections P, Q such that $(PQ)^n e_k \to 0$ for any k, but with the convergence arbitrarily slowing down with $k \to \infty$. Thus, $(PQ)^n \not\to 0$. □

As a consequence, in connection with the von Neumann algebras, we have:

THEOREM 14.26. *Given two projections $P, Q \in B(H)$, the projections*

$$P \wedge Q \quad , \quad P \vee Q$$

both belong to the von Neumann algebra generated by P, Q.

PROOF. This comes from the above. Indeed, in what regards $P \wedge Q$, this is something that follows from Theorem 14.25. As for $P \vee Q$, here the result follows from the result for $P \wedge Q$, and from the formula $P + Q = P \wedge Q + P \vee Q$, from Proposition 14.24. □

The idea now will be that of studying the von Neumann algebras $A \subset B(H)$ by using their projections, $p \in A$. Let us start with the following result:

THEOREM 14.27. *Any von Neumann algebra is generated by its projections.*

PROOF. This is something that we know from before, coming from the measurable functional calculus, which can cut any normal operator into projections. □

There are many other things that can be said about projections, in the general setting. In what follows we will just discuss the most important and useful such results. A first such result, providing us with some geometric intuition on projections, is as follows:

THEOREM 14.28. *Given a von Neumann algebra $A \subset B(H)$, and a projection $p \in A$, we have the following equalities, between von Neumann algebras on pH:*

(1) *$pAp = (A'p)'$.*
(2) *$(pAp)' = A'p$.*

PROOF. This is not exactly obvious, but can be proved as follows:

(1) As a first observation, the von Neumann algebras pAp and $A'p$ commute on pH. Thus, we must prove that we have the following implication:

$$x \in (A'p)' \implies x \in pAp$$

For this purpose, consider the element $y = xp$. Then for any $z \in A'$ we have:

$$\begin{aligned} zy &= zxp \\ &= zpxp \\ &= xpzp \\ &= xpz \\ &= yz \end{aligned}$$

But this shows that we have $y \in A$, and so we obtain, as desired:

$$x = pyp \in pAp$$

(2) As before, one of the inclusions being clear, we must prove that we have:

$$x \in (pAp)' \implies x \in A'p$$

By using the standard fact that any bounded operator appears as a linear combination of 4 unitaries, that we know from the end of chapter 3, it is enough to prove this for a unitary element, $x = u$. So, assume that we have a unitary as follows:

$$u \in (pAp)'$$

In order to prove our claim, consider the following vector space:

$$K = \overline{ApH}$$

This space being invariant under both the algebras A, A', we conclude that the projection $q = Proj(K)$ onto it belongs to the center of our von Neumann algebra:

$$q \in Z(A)$$

Our claim now, which will quickly lead to the result that we want to prove, is that we can extend the above unitary $u \in (pAp)'$ to the space $K = \overline{ApH}$ via the following formula, valid for any elements $x_i \in A$, and any vectors $\xi_i \in pH$:

$$v\left(\sum_i x_i\xi_i\right) = \sum_i x_iu\xi_i$$

In order to prove this latter claim, we can use the following computation:

$$\begin{aligned}
\left|\left|v\left(\sum_i x_i\xi_i\right)\right|\right|^2 &= \sum_{ij} < x_iu\xi_i, x_ju\xi_j > \\
&= \sum_{ij} < x_j^*x_iu\xi_i, u\xi_j > \\
&= \sum_{ij} < px_j^*x_ipu\xi_i, u\xi_j > \\
&= \sum_{ij} < upx_j^*x_ip\xi_i, u\xi_j > \\
&= \sum_{ij} < px_j^*x_ip\xi_i, \xi_j > \\
&= \sum_{ij} < x_j^*x_i\xi_i, \xi_j > \\
&= \sum_{ij} < x_i\xi_i, x_j\xi_j > \\
&= \left|\left|\sum_i x_i\xi_i\right|\right|^2
\end{aligned}$$

Thus v is well-defined by the above formula, and is an isometry of K. Now observe that this element v commutes with the algebra A on the space ApH, and so on K. Thus $vq \in A'$, and so $u = vqp$, which proves that we have $u \in A'p$, as desired. $\square$

As a second result now, once again in the general setting, we have:

PROPOSITION 14.29. *Given a von Neumann algebra $A \subset B(H)$, the formula*

$$p \simeq q \iff \exists u, \begin{cases} uu^* = p \\ u^*u = q \end{cases}$$

defines an equivalence relation for the projections $p \in A$.

PROOF. This is something elementary, which follows from definitions, with the transitivity coming by composing the corresponding partial isometries. $\square$

As a third result, once again in the general setting, which once again provides us with some intuition, but this time of somewhat abstract type, we have:

THEOREM 14.30. *Given a von Neumann algebra $A \subset B(H)$, we have a partial order on the projections $p \in A$, constructed as follows, with u being a partial isometry,*

$$p \preceq q \iff \exists u, \begin{cases} uu^* = p \\ u^*u \leq q \end{cases}$$

which is related to the equivalence relation $\simeq$ constructed above by:

$$p \simeq q \iff p \preceq q, \ q \preceq p$$

Thus, $\preceq$ is a partial order on the equivalence classes of projections $p \in A$.

PROOF. We have several assertions here, the idea being as follows:

(1) The fact that we have indeed a partial order is clear, with the transitivity coming, as before, by composing the corresponding partial isometries.

(2) Regarding now the relation with $\simeq$, via the equivalence in the statement, the implication $\implies$ is clear. Thus, we are left with proving $\Longleftarrow$, which reads:

$$p \preceq q, \ q \preceq p \implies p \simeq q$$

Our assumption is that we have partial isometries u, v such that:

$$uu^* = p \quad , \quad u^*u \leq q$$

$$v^*v \leq p \quad , \quad vv^* = q$$

We can construct then two sequences of decreasing projections, as follows:

$$p_0 = p \quad , \quad p_{n+1} = v^*q_nv$$

$$q_0 = q \quad , \quad q_{n+1} = u^*p_nu$$

Consider now the limits of these two sequences of projections, namely:

$$p_\infty = \bigwedge_i p_i \quad , \quad q_\infty = \bigwedge_i q_i$$

In terms of all these projections that we constructed, we have the following decomposition formulae for the original projections p, q:

$$p = (p - p_1) + (p_1 - p_2) + \ldots + p_\infty$$

$$q = (q - q_1) + (q_1 - q_2) + \ldots + q_\infty$$

Now observe that the summands are equivalent, with this being clear from the definition of p_n, q_n at the finite indices $n < \infty$, and with $p_\infty \simeq q_\infty$ coming from:

$$v^* q_\infty v = p_\infty \quad , \quad q_\infty v v^* q_\infty = q_\infty$$

Thus we obtain that we have $p \simeq q$, as desired, by summing.

(3) Finally, the fact that the order relation $\preceq$ factorizes indeed to the equivalence classes under $\simeq$ follows from the equivalence established in (2). $\square$

Summarizing, in view of Theorem 14.27, and of Theorem 14.30, we can formulate:

Conclusion 14.31. *We can think of a von Neumann algebra $A \subset B(H)$ as being a kind of object belonging to "mathematical logic", consisting of equivalence classes of projections $p \in A$, ordered via the relation $\preceq$, and producing A itself via transport by partial isometries, and then linear combinations, and weak limits.*

Which is something quite remarkable, who on Earth could have guessed, when we were struggling with the basics, that we will end up with something that luminous.

Well, that person on Earth who found this was von Neumann himself, back in the 1930s. And his Conclusion 14.31, called "von Neumann vision" of the operator algebras, has been extremely useful ever since, and is still largely used nowadays.

14d. Reduction, factors

In order to further advance, the general idea, which is something quite natural, is that among the von Neumann algebras $A \subset B(H)$, of particular interest are the "free" ones, having trivial center, $Z(A) = \mathbb{C}$. These algebras are called factors:

Definition 14.32. *A factor is a von Neumann algebra $A \subset B(H)$ whose center*

$$Z(A) = A \cap A'$$

which is a commutative von Neumann algebra, reduces to the scalars, $Z(A) = \mathbb{C}$.

This notion is in fact something that we already met in the above, in the context of various comments or exercises, and time now to clarify all this. The idea is that there are two main motivations for the study of factors, with each of them being more than enough, as to serve as a strong motivation. First, at the intuitive level, we have:

PRINCIPLE 14.33 (Freeness). *The following happen:*

(1) *The condition $Z(A)=\mathbb{C}$ defining the factors is, obviously, opposite to the condition $Z(A)=A$ defining the commutative von Neumann algebras.*
(2) *Therefore, the factors are the von Neumann algebras which are "free", meaning as far as possible from the commutative ones.*
(3) *Equivalently, with $A=L^\infty(X)$, the quantum spaces X coming from factors are those which are "free", meaning as far as possible from the classical spaces.*

So, this was for our first principle, which is something reasonable, intuitive, and self-explanatory, and which can surely serve as a strong motivation for the study of factors. In fact, all that has being said above comes straight from the structure theorem for the commutative von Neumann algebras, $A=L^\infty(X)$, with X being a measured space, that we know from before, and the above principle is just a corollary of that theorem.

At a more advanced level, another motivation for the study of factors, which among others justifies the name "factors" for them, comes from the reduction theory of von Neumann [**92**], which is something non-trivial, that can be summarized as follows:

PRINCIPLE 14.34 (Reduction theory). *Given a von Neumann algebra $A\subset B(H)$, if we write its center $Z(A)\subset A$, which is a commutative von Neumann algebra, as*

$$Z(A)=L^\infty(X)$$

with X being a measured space, then the whole algebra decomposes as

$$A=\int_X A_x\,dx$$

with the fibers A_x being factors, that is, satisfying $Z(A_x)=\mathbb{C}$.

As a first comment, we have already seen an instance of such decomposition results in the above, when talking about finite dimensional algebras. Indeed, such algebras decompose, in agreement with the above, as direct sums of matrix algebras, as follows:

$$A=\bigoplus_x M_{n_x}(\mathbb{C})$$

In general, however, things are more complicated than this, and technically speaking, and as opposed to Principle 14.33, which was more of a triviality, Principle 14.34 is a tough theorem, due to von Neumann [**92**]. More on this, later in this book.

Getting to work now, there are many things that can be said about factors. In order to get started, let us first study their projections. We will need the following result:

PROPOSITION 14.35. *Given two projections $p,q\neq 0$ in a factor A, we have*

$$puq\neq 0$$

for a certain unitary $u\in A$.

PROOF. Assume by contradiction $puq=0$, for any unitary $u\in A$. This gives:

$$u^*puq=0$$

By using this for all the unitaries $u\in A$, we obtain the following formula:

$$\left(\bigvee_{u\in U_A}u^*pu\right)q=0$$

On the other hand, from $p\neq 0$ we obtain, by factoriality of A:

$$\bigvee_{u\in U_A}u^*pu=1$$

Thus, our previous formula is in contradiction with $q\neq 0$, as desired. □

Getteing back now to the order on projections from before, and to the whole von Neumann projection philosophy, in the case of factors things simplify, as follows:

THEOREM 14.36. *Given two projections $p,q\in A$ in a factor, we have*

$$p\preceq q\quad\text{or}\quad q\preceq p$$

and so $\preceq$ is a total order on the equivalence classes of projections $p\in A$.

PROOF. This basically follows from Proposition 14.35, and from the Zorn lemma, by using some standard functional analysis arguments. To be more precise:

(1) Consider indeed the following set of partial isometries:

$$S=\left\{u\Big|uu^*\leq p,u^*u\leq q\right\}$$

We can then order this set S by saying that we have $u\leq v$ when $u^*u\leq v^*v$, and when $u=v$ holds on the initial domain u^*uH of u. With this convention made, the Zorn lemma applies, and provides us with a maximal element $u\in S$.

(2) In the case where this maximal element $u\in S$ satisfies $uu^*=p$ or $u^*u=q$, we are led to one of the conditions $p\preceq q$ or $q\preceq p$ in the statement, and we are done.

(3) So, assume that we are in the case left, $uu^*\neq p$ and $u^*u\neq q$. By Proposition 14.35 we obtain a unitary $v\neq 0$ satisfying the following conditions:

$$vv^*\leq p-uu^*$$

$$v^*v\leq q-u^*u$$

But these conditions show that the element $u+v\in S$ is strictly bigger than $u\in S$, which is a contradiction, and we are done. □

Moving ahead now, as explained time and again throughout this book, for a variety of reasons, which can be elementary or advanced, and also mathematical or physical, we are mainly interested in the case where our algebras have traces:

$$tr : A \to \mathbb{C}$$

And in relation with the factors, by leaving aside the rather trivial case of the matrix algebras $A = M_N(\mathbb{C})$, we are led in this way to the following key notion:

DEFINITION 14.37. *A* II_1 *factor is a von Neumann algebra* $A \subset B(H)$ *which:*

(1) *Is infinite dimensional,* $\dim A = \infty$.
(2) *Has trivial center,* $Z(A) = \mathbb{C}$.
(3) *Has a trace* $tr : A \to \mathbb{C}$.

This definition is motivated by some heavy classification work of Murray, von Neumann and Connes, whose conclusion is more or less that everything in von Neumann algebras reduces, via some quite complicated procedures, to the study of the II_1 factors:

FACT 14.38. *The* II_1 *factors are the building blocks of the whole von Neumann algebra theory.*

To be more precise, this statement, that we will get to understand later, is something widely agreed upon, at least among operator algebra experts who are familiar with von Neumann algebras, and with this agreement being something great.

14e. Exercises

Exercises:

EXERCISE 14.39.

EXERCISE 14.40.

EXERCISE 14.41.

EXERCISE 14.42.

EXERCISE 14.43.

EXERCISE 14.44.

EXERCISE 14.45.

EXERCISE 14.46.

Bonus exercise.

CHAPTER 15

Integration theory

15a.

15b.

15c.

15d.

15e. Exercises

Exercises:

EXERCISE 15.1.

EXERCISE 15.2.

EXERCISE 15.3.

EXERCISE 15.4.

EXERCISE 15.5.

EXERCISE 15.6.

EXERCISE 15.7.

EXERCISE 15.8.

Bonus exercise.

CHAPTER 16

Advanced aspects

16a.

16b.

16c.

16d.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter.

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