INVARIANTS OF THE HALF-LIBERATED ORTHOGONAL GROUP

TEODOR BANICA AND ROLAND VERGNIOUX

Abstract. The half-liberated orthogonal group O_n^* appears as intermediate quantum group between the orthogonal group O_n, and its free version O_n^+. We discuss here its basic algebraic properties, and we classify its irreducible representations. The classification of representations is done by using a certain twisting-type relation between O_n^* and U_n, a non abelian discrete group playing the role of weight lattice for O_n^*, and a number of methods inspired from the theory of Lie algebras. We use these results for showing that the discrete quantum group dual to O_n^* has polynomial growth.

Introduction

The quantum groups introduced by Drinfeld in [13] have played a prominent role in various areas of mathematics and physics. In addition to Drinfeld's discovery, Woronowicz's axiomatization in [23], [24] of the compact quantum groups has been very influential as well and opened the way for the search of new examples. In particular it allowed the discovery by Wang of the free quantum groups [20], which have been subject of several systematic investigations. Since then other families of examples have been discovered, building up a fast evolving area:

(1) The first two quantum groups are O_n^+, U_n^+, introduced in [20]. These led to a number of general developments, including the study of connections with subfactors, noncommutative geometry and free probability [1], [5], [14] and a number of advances in relation with operator algebras [16], [17], [18].
(2) The third quantum group is S_n^+, introduced in [21]. This led to the quite amazing world of quantum permutation groups, heavily investigated in the last few years. These quantum groups allowed in particular a clarification of the relation with subfactors [3], noncommutative geometry [9] and free probability [15].
(3) The fourth quantum group is H_n^+, recently introduced in [4]. This quantum group gave rise as well to a number of new investigations, which are currently under development. Let us mention here the opening world of quantum reflection groups [8], and the new formalism of easy quantum groups [6].

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In this paper we study the fifth "new" quantum group, namely the half-liberated orthogonal group O_n^*, constructed in [6]. We believe that, as it was the case with its predecessors O_n^+, U_n^+, S_n^+, H_n^+, this quantum group will open up as well a new area: namely, that of the "root systems" for the compact quantum groups.

The quantum group O_n^* is constructed as follows. Consider the basic coordinate functions $u_{ij} : O_n \to \mathbb{C}$. These commute with each other, form an orthogonal matrix, and generate the algebra $C(O_n)$. Wang's algebra $C(O_n^+)$ is obtained by simply removing the "commutativity" assumption. As for obtaining the "half-liberated" algebra $C(O_n^*)$, the commutativity condition $ab = ba$ with $a, b \in \{u_{ij}\}$ should be replaced by the weaker condition $abc = cba$, for a, b, $c \in \{u_{ij}\}$.

Summarizing, the quantum group O_n^* appears via a kind of tricky "‘weakening" of Wang's original relations in [20]. Observe that we have $O_n \subset O_n^* \subset O_n^+$.

One can prove that, under a suitable "easiness" assumption, O_n^* is the only quantum group between O_n and O_n^+. This abstract result, to be proved in this paper, justifies the name "half-liberated", and provides a first motivation for the study of O_n^*. In fact O_n^* was introduced in as one of the 15 easy intermediate quantum groups $S_n \subset G \subset O_n^+$.

In this paper we perform a systematic study of O_n^*, and in particular of its category of representations. After discussing the first features of the definition, we describe in Sections 2 – 4 some Hom-spaces of this category in terms of Brauer diagrams and derive two consequences: a connection with the group U_n via the projective version PO_n^*, and the uniqueness result mentionned above.

Then we undertake the classification of irreducible representations of O_n^*. The main technical novelty is the use of diagonal groups and root systems in a quantum framework. Diagonal groups are meant to be replacements for maximal torii in good situations and are introduced in Section 5. In the case of O_n^* the diagonal group provides a noncommutative weight lattice which is used in Sections 6 and 7 together with a subtle relation to the classical group U_n to classify representations of O_n^*.

Finally we derive in Sections 8, 9 some applications of the classification of irreducible representations to fusion rules, Cayley graph and growth. Let us mention here a quite surprising feature of our results: although O_n^* is an intermediate subgroup between two orthogonal groups O_n, O_n^+ with commutative fusion rules, its fusion rules are noncommutative and its exponent of polynomial growth is the same as for SU_n. This shows also that O_n^* is not monoidally equivalent, in the sense of [10], to any known compact quantum group so far, in particular it is the first original example of a compact quantum group with exponential growth as considered in [7].

1. Half-liberation

Given a compact group G, the algebra of complex continuous functions $C(G)$ is a Hopf algebra, with comultiplication, counit and antipode given by:

$$\begin{aligned}\Delta(\varphi) &= ((g,h) \to \varphi(gh))\\ \varepsilon(\varphi) &= \varphi(1)\\ S(\varphi) &= (g \to \varphi(g^{-1}))\end{aligned}$$

Consider in particular the orthogonal group O_n. This is a real algebraic group, and we denote by $x_{ij} : O_n \to \mathbb{R}$ its basic coordinates, $x_{ij}(g) = g_{ij}$.

The matrix $x = (x_{ij})$ is by definition orthogonal, in the sense that all its entries are self-adjoint, and we have $xx^t = x^t x = 1$. Moreover, it follows from the Stone-Weierstrass theorem that the elements x_{ij} generate $C(O_n)$ as a C^*-algebra.

These observations lead to the following presentation result.

Theorem 1.1. *$C(O_n)$ is the universal unital commutative C^*-algebra generated by the entries of an $n \times n$ orthogonal matrix x. The maps given by*

$$\begin{aligned}\Delta(x_{ij}) &= \sum x_{ik} \otimes x_{kj}\\ \varepsilon(x_{ij}) &= \delta_{ij}\\ S(x_{ij}) &= x_{ji}\end{aligned}$$

are the comultiplication, counit and antipode of $C(O_n)$.

Proof. The first assertion is a direct application of the classical theorems of Stone-Weierstrass and Gelfand. The second assertion follows by transposing the usual rules for the matrix multiplication, unit and inversion. □

The following key definition is due to Wang [20].

Definition 1.2. *$A_o(n)$ is the universal unital C^*-algebra generated by the entries of an $n \times n$ orthogonal matrix u. The maps given by*

$$\begin{aligned}\Delta(u_{ij}) &= \sum u_{ik} \otimes u_{kj}\\ \varepsilon(u_{ij}) &= \delta_{ij}\\ S(u_{ij}) &= u_{ji}\end{aligned}$$

are the comultiplication, counit and antipode of $A_o(n)$.

It is routine to check that $A_o(n)$ satisfies the general axioms of Woronowicz in [23]. This tells us that we have the heuristic formula $A_o(n) = C(O_n^+)$, where O_n^+ is a certain compact quantum group, called free version of O_n. See [20].

It is known that we have $A_o(2) = C(SU_2^{-1})$. More generally, the algebra $A_o(n)$ with $n \geq 2$ arbitrary shares many properties with the algebra $C(SU_2)$. See [1], [5].

The following definition is from the recent paper [6].

Definition 1.3. *The half-liberated orthogonal quantum algebra is*

$$A_o^*(n) = A_o(n) / \left\langle abc = cba \,\middle|\, a, b, c \in \{u_{ij}\} \right\rangle$$

with comultiplication, counit and antipode coming from those of $A_o(n)$.

It is routine to check that the comultiplication, counit and antipode of $A_o(n)$ factorize indeed, and that $A_o^*(n)$ satisfies the general axioms of Woronowicz in [23].

In order to get some insight into the structure of $A_o^*(n)$, we first examine its "cocommutative version". We have the following analogue of Definition 1.3.

Definition 1.4. *We consider the discrete group*

$$L_n = \mathbb{Z}_2^{*n} / \left\langle abc = cba \,\middle|\, a, b, c \in \{g_i\} \right\rangle$$

where $g_1, \ldots, g_n$ with $g_i^2 = 1$ are the standard generators of $\mathbb{Z}_2^{*n}$.

Observe that we have surjective group morphisms $\mathbb{Z}_2^{*n} \to L_n \to \mathbb{Z}_2^n$. As shown in Proposition 1.6 below, these morphisms are not isomorphisms in general.

We recall that any for discrete group Γ, the group algebra $C^*(\Gamma)$ is a Hopf algebra, with comultiplication, counit and antipode given by:

$$\begin{aligned} \Delta(g) &= g \otimes g \\ \varepsilon(g) &= 1 \\ S(g) &= g^{-1} \end{aligned}$$

The interest in the above group L_n comes from the following result.

Proposition 1.5. *We have quotient maps as follows:*

$$\begin{array}{ccccc} A_o(n) & \longrightarrow & A_o^*(n) & \longrightarrow & C(O_n) \\ \downarrow & & \downarrow & & \downarrow \\ C^*(\mathbb{Z}_2^{*n}) & \longrightarrow & C^*(L_n) & \longrightarrow & C^*(\mathbb{Z}_2^n) \end{array}$$

Proof. The vertical maps can be defined indeed by $u_{ij} \to \delta_{ij} g_i$, by using the universal property of the algebras on top. Observe that these maps are indeed Hopf algebra morphisms, because the formulae of Δ, ε, S in Definition 1.2 reduce to the above cocommutative formulae, after performing the identification $u_{ij} = 0$ for $i \neq j$. □

The group L_n will play an important role in the present paper in relation with the representation theory of O_n^*. In particular we will obtain in section 6 an abstract isomorphism $L_n \simeq \mathbb{Z}^{n-1} \rtimes \mathbb{Z}_2$. Let us start with a simpler statement that we use for Theorem 1.7. Note that for $n = 2$ this already gives $L_2 = D_\infty = \mathbb{Z} \rtimes \mathbb{Z}_2$.

Proposition 1.6. *The groups L_n are as follows:*

(1) *At $n = 2$ we have $\mathbb{Z}_2^{*2} = L_2 \neq \mathbb{Z}_2^2$.*
(2) *At $n \geq 3$, we have $\mathbb{Z}_2^{*n} \neq L_n \neq \mathbb{Z}_2^n$.*

Proof. We denote by g, h the standard generators of $\mathbb{Z}_2^{*2}$.

(1) We know that L_2 appears as quotient of $\mathbb{Z}_2^{*2}$ by the relations $abc = cba$, with a, b, $c \in \{g, h\}$. In the case $a = b$ or $b = c$ this is a trivial relation (of type $k = k$), and in the case $a \neq b$, $b \neq c$ we must have $a = c$, so once again our relation is trivial (of type $klk = klk$). Thus we have $L_2 = \mathbb{Z}_2^{*2}$, which gives the result.

(2) The first assertion is clear, because the equality $g_1g_2g_3 = g_3g_2g_1$ doesn't hold in $\mathbb{Z}_2^{*n}$. Observe now that we have a quotient map $L_3 \to \mathbb{Z}_2^{*2}$, given by $g_1 \to g$, $g_2 \to g$, $g_3 \to h$. This shows that L_3 is not abelian. We deduce that L_n with $n \geq 3$ is not abelian either, so in particular it is not isomorphic to $\mathbb{Z}_2^n$. □

Theorem 1.7. *The algebras $A_o^*(n)$ are as follows:*

1. *At $n = 2$ we have $A_o(2) = A_o^*(2) \neq C(O_2)$.*
2. *At $n \geq 3$, we have $A_o(n) \neq A_o^*(n) \neq C(O_n)$.*

Proof. The three non-equalities in the statement follow from the three non-equalities in Proposition 1.6. In order to prove the remaining statement $A_o^*(2) = A_o(2)$, consider the fundamental corepresentation of $A_o(2)$:

$$u = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$$

The elements x, y, z, t are by definition self-adjoint, and satisfy the relations making u unitary. These unitary relations can be written as follows:

$$\begin{aligned} x^2 &= t^2 \\ y^2 &= z^2 \\ x^2 + y^2 &= 1 \\ xy + zt &= 0 \\ xz + yt &= 0 \end{aligned}$$

With these relations in hand, the verification of the relations of type $abc = cba$ with a, b, $c \in \{x, y, z, t\}$ is routine, and we get $A_o^*(2) = A_o(2)$. □

2. Brauer diagrams

In this section and in the next one we discuss some basic properties of $A_o^*(n)$, by "interpolating" between some well-known results regarding $A_o(n)$ and $C(O_n)$.

For k, l with $k+l$ even we consider the pairings between an upper sequence of k points, and a lower sequence of l points. We make the convention that the $k+l$ points of a pairing p are counted counterclockwise, starting from bottom left. As an example we draw hereafter the diagram corresponding to the pairing $\{\{1, 9\}, \{2, 4\}, \{3, 7\}, \{5, 8\}, \{6, 10\}\}$ for $k = l = 5$. These pairings, also called Brauer diagrams, are taken as usual up to planar isotopy. See e.g. [22].

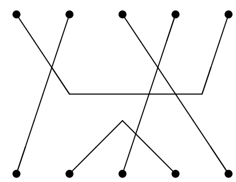


Figure 1. An element of $P(5,5)$ not in $E(5,5)$.

Definition 2.1. *We use the following sets of partitions:*

1. $P(k,l)$*: all pairings.*
2. $E(k,l)$*: all pairings with each string having an even number of crossings.*
3. $N(k,l)$*: all pairings having no crossing at all.*

The partitions in $N(k,l)$ are familiar objects, also called Temperley-Lieb diagrams. Observe that the number of crossings for each string of a pairing is invariant under planar isotopy, so the middle set $E(k,l)$ is indeed well-defined.

We make the convention that for $k+l$ odd the above three sets are defined as well, as being equal to $\emptyset$. Observe that for any k, l we have embeddings as follows:

$$N(k,l) \subset E(k,l) \subset P(k,l)$$

Proposition 2.2. *For $p \in P(k,l)$, the following are equivalent:*

1. *Each string has an even number of crossings (i.e. $p \in E(k,l)$).*
2. *The number of points between the two legs of any string is even.*
3. *When labelling the points ababab.., each string joins an "a" to a "b".*

Proof. This follows from the fact that the above three sets have $((k+l)/2)!$ elements, and the first one includes the second one, which in turn includes the third one. □

The interest in the Brauer diagrams comes from the fact that they encode several key classes of linear maps, according to the following construction.

Definition 2.3. *Associated to any partition $p \in P(k,l)$ and any $n \in \mathbb{N}$ is the linear map $T_p : (\mathbb{C}^n)^{\otimes k} \to (\mathbb{C}^n)^{\otimes l}$ given by*

$$T_p(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta \begin{pmatrix} i_1 & \ldots & i_k \\ & p & \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$, and the δ symbol is defined as follows: $\delta = 1$ if each string of p joins a pair of equal indices, and $\delta = 0$ if not.

Here are a few examples of such linear maps, which are of certain interest for the considerations to follow:

$$\begin{aligned} T_{-}\{\,|\;|\,\}(e_a \otimes e_b) &= e_a \otimes e_b \\ T_{-}\left\{\times\right\}(e_a \otimes e_b) &= e_b \otimes e_a \\ T_{-}\left\{\begin{matrix}\sqcup \\ \sqcap\end{matrix}\right\}(e_a \otimes e_b) &= \delta_{ab}\sum_c e_c \otimes e_c \end{aligned}$$

It is known since Brauer that the linear maps T_p with p ranging over all diagrams in $P(k,l)$ span the tensor category associated to O_n. See [11], [12].

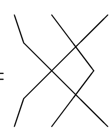
Theorem 2.4. *We have the following results:*

1. *For* $C(O_n)$ *we have* $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}(T_p \mid p \in P(k,l))$.
2. *For* $A_o^*(n)$ *we have* $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}(T_p \mid p \in E(k,l))$.
3. *For* $A_o(n)$ *we have* $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}(T_p \mid p \in N(k,l))$.

Proof. The first assertion is Brauer's theorem, see Section 5.a) of [11]. The third assertion is proved in [1], see Proposition 2 and the following Remarque there. The middle assertion is Theorem 6.9 in [6], the idea being as follows. First, the defining relations $abc = cba$ express the fact that the following operator must intertwine $u^{\otimes 3}$:

$$T(e_i \otimes e_j \otimes e_k) = e_k \otimes e_j \otimes e_i$$

The point is that T comes from the following Brauer diagram:

$p_3 =$ 

A careful examination shows that this diagram "generates" all the diagrams having an even number of crossings, and this proves the result. □

In was pointed out in [6] that $A_o^*(n)$ should appear as some kind of "twist" of $C(U_n)$, due to a certain common occurrence of the symmetrized Rayleigh law, in the asymptotic representation theory of these algebras. In this paper we will present several results in this sense. These results will be all based on the following fact.

Theorem 2.5. *If u, v are the fundamental corepresentations of* $A_o^*(n)$, $C(U_n)$ *then*

$$\mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) = \mathrm{Hom}(v_k, v_l)$$

for any k, l, where $v_k = v \otimes \bar{v} \otimes v \otimes \ldots$ (k terms).

Proof. If α, β are tensor products between v, $\bar{v}$, of length K, L, and we denote by $P(\alpha,\beta) \subset P(K,L)$ the set of pairings such that each string joins a v to a $\bar{v}$, then:

$$\mathrm{Hom}(\alpha,\beta) = \mathrm{span}(T_p \mid p \in P(\alpha,\beta))$$

This is indeed a well-known result, see e.g. Theorem 9.1 of [5] for a recent proof of the version with non-crossing pairings. Now in the particular case $\alpha = v_k$, $\beta = v_l$, we get:

$$\mathrm{Hom}(v_k, v_l) = \mathrm{span}(T_p \mid p \in P(v_k, v_l))$$

On the other hand, Proposition 2.2 shows that we have $P(v_k, v_l) = E(k, l)$. Together with the middle assertion in Theorem 2.4, this gives the result. □

3. The projective version

The projective version of a unitary Hopf algebra (A, u) is the subalgebra $PA \subset A$ generated by the entries of $u \otimes \bar{u}$, with $u \otimes \bar{u}$ as fundamental corepresentation.

In the Proposition below are some basic examples, with (1) justifying the terminology. Recall that $A = C^*(\Gamma)$, with Γ finitely generated, is a Woronowicz algebra with fundamental corepresentation $u = \mathrm{diag}(g_1, \ldots, g_n)$ where the g_i are generators of Γ. On the other hand $A = C(G)$, with $G \subset U_n$ compact subgroup, is a Woronowicz algebra with fundamental corepresentation u given by the embedding into U_n.

Proposition 3.1. *The projective version is as follows:*

(1) *For $G \subset U_n$ we have $PC(G) = C(PG)$.*
(2) *For $\Gamma = \langle g_i \rangle$ we have $PC^*(\Gamma) = C^*(\Lambda)$, with $\Lambda = \langle g_i g_j^{-1} \rangle \subset \Gamma$.*
(3) *For $A = A_o(n)$ and $A_u(n)$ we have $PA = A_{aut}(M_n(\mathbb{C}))$.*

Proof. The first two assertions are well-known, and follow from definitions. The third assertion is known as well: for $A_o(n)$, see Corollary 4.1 in [2] and for $A_u(n)$, use Théorème 1 (iv) in [1]. □

In this section we compute the projective version of $A_o^*(n)$. Our starting point is the following simple observation, coming from definitions.

Proposition 3.2. *$PA_o^*(n)$ is commutative.*

Proof. This follows indeed from the relations $abc = cba$, because $PA_o^*(n)$ is generated by the elements of type ab, and we have $aba'b' = ab'a'b = a'b'ab$. □

Theorem 3.3. *We have $PA_o^*(n) = C(PU_n)$.*

Proof. We will prove that the algebras $A_o^*(n)$ and $C(U_n)$ have the same projective version. In order to "compare" these two algebras, we use Wang's universal algebra $A_u(n)$, having both of them as quotients. Consider indeed the following diagram:

$$\begin{array}{ccc} A_u(n) & \longrightarrow & A_o^*(n) \\ \downarrow & & \downarrow \\ C(U_n) & \longrightarrow & C(O_n) \end{array}$$

We fix $k,\ l \geq 0$ and we consider the words $\alpha = (u \otimes \bar{u})^{\otimes k}$ and $\beta = (u \otimes \bar{u})^{\otimes l}$. According to the above results, the spaces $\mathrm{Hom}(\alpha, \beta)$ of our four algebras appear as span of the operators T_p, with p belonging to the following four sets of diagrams:

$$\begin{array}{ccc} N(2k,2l) & \subset & E(2k,2l) \\ \cap & & \cap \\ E(2k,2l) & \subset & P(2k,2l) \end{array}$$

Summarizing, we have computed the relevant diagrams for the projective versions of our four algebras. So, let us look now at these projective versions:

$$\begin{array}{ccc} PA_u(n) & \longrightarrow & PA_o^*(n) \\ \downarrow & & \downarrow \\ C(PU_n) & \longrightarrow & C(PO_n) \end{array}$$

We can see that the relationship between $PA_o^*(n)$ and $C(PU_n)$ is as follows:

(1) These two algebras appear as quotients of a same algebra.
(2) The relevant diagrams for these two algebras are the same.

It is a well-known application of Woronowicz's results in [24] that these two conditions ensure the fact that our algebras are isomorphic and we are done. □

4. A uniqueness result

In this section we find an abstract characterization of the algebra $A_o^*(n)$: it is in some sense the unique intermediate quotient $A_o(n) \to A \to C(O_n)$. This will justify the terminology "half-liberated" that we use in this paper.

The following result follows from Woronowicz's Tannakian duality in [24]:

Theorem 4.1. *The intermediate Hopf algebras $A_o(n) \to A \to C(O_n)$ are in one-to-one correspondence with the tensor categories C satisfying*

$$\mathrm{span}(T_p \mid p \in N(k,l)) \subset C(k,l) \subset \mathrm{span}(T_p \mid p \in P(k,l))$$

where N denotes as usual the noncrossing pairings, and P, all the pairings.

Here, and in what follows, we use Woronowicz's tensor category formalism in [24]. That is, we call "tensor category" a tensor C^*-category with duals whose monoid of objects is $(\mathbb{N}, +)$, embedded into the tensor C^*-category of Hilbert spaces.

We have the following definition, adapted from [6].

Definition 4.2. *An intermediate Hopf algebra $A_o(n) \to A \to C(O_n)$ is called "easy" when its associated tensor category is of the form*

$$C(k,l) = \mathrm{span}(T_p \mid p \in D(k,l))$$

for a certain collection of subsets $D(k,l) \subset P(k,l)$, with $k,\ l \in \mathbb{N}$.

In other words, we know from Theorem 4.1 that the Hom-spaces $C(k,l)$ associated to A consist of certain linear combinations of partitions. In the case where for any k, l we can exhibit a certain set of partitions $D(k,l) \subset P(k,l)$ such that $C(k,l)$ is spanned by the elements T_p with $p \in D(k,l)$, we call our Hopf algebra "easy".

Observe that the sets to be exhibited can be chosen to be:

$$D(k,l) = \{p \in P(k,l) \mid T_p \in C(k,l)\}$$

Thus, in concrete situations, the check of easiness is in fact straightforward. We refer to [6] for full details regarding this notion, including examples and counterexamples. We use as well the following technical definition.

Definition 4.3. *Let $p \in P(k,l)$ be a partition, with its points counted as usual counterclockwise starting from bottom left. For $i = 1, 2, \ldots, k+l$ we denote by p^i the partition obtained by connecting with a semicircle the i-th and $(i+1)$-th points.*

In this definition we agree of course that the points are counted modulo $k+l$. The partitions p^i will be called "cappings" of p, and will be generically denoted p'.

We denote by P, E, N the collection of sets in Definition 2.1, endowed with the operations of horizontal and vertical concatenation, and upside-down turning. These operations are well-known to correspond via $p \to T_p$ to the tensor product, composition and involution operations in the corresponding Hom-spaces: see e.g. Prop. 1.9 in [6].

Lemma 4.4. *Consider a partition $p \in P - N$, having $s \geq 4$ strings.*

(1) *If $p \in P - E$, there exists a capping $p' \in P - E$.*
(2) *If $p \in E - N$, there exists a capping $p' \in E - N$.*

Proof. First, we can use a rotation — see the proof of Lemma 2.7 in [6] — in order to assume that p has no upper points. In other words, our data is a partition $p \in P(0,2s) - N(0,2s)$, with $s \geq 4$.

(1) The assumption $p \notin E$ means that p has certain strings having an odd number of crossings. We fix such an "odd" string, and we try to cap p, as for this string to remain odd in the resulting partition p'. An examination of all possible pictures shows that this is possible, provided that our partition has $s \geq 3$ strings, and we are done.

(2) The assumption $p \notin N$ means that p has certain crossing strings. We fix such a pair of crossing strings, and we try to cap p, as for these strings to remain crossing in p'. Once again, an examination of all possible pictures shows that this is possible, provided that our partition has $s \geq 4$ strings, and we are done. □

For $p \in P$ we denote by $\langle p \rangle \subset P$ the collection of partitions generated by p and by N, via the above operations of concatenation and upside-down turning. In particular if $q \in \langle p \rangle$ we also have $q' \in \langle p \rangle$ for any capping q' of q. Observe that we have:

$$\langle T_p \rangle = \mathrm{span}(T_q \mid q \in \langle p \rangle)$$

Here the left term is by definition the tensor category generated by T_p. It contains at least all morphisms T_q for $q \in N$ since $T_|$ is the identity morphism and $T_\sqcap$, $T_\sqcup$ are the morphisms describing the duality in the category.

Let us quote two examples used for the proof of the next Lemma: for $p = \diagup\!\!\!\!\diagdown$ we clearly have $\langle p \rangle = P$, and if $p = p_3$ is the diagram pictured in the proof of Theorem 2.4, we have $\langle p \rangle = E$ as already stated there.

Lemma 4.5. *Consider a partition $p \in P(k,l) - N(k,l)$.*

(1) *If $p \in P(k,l) - E(k,l)$ then $\langle p \rangle = P$.*
(2) *If $p \in E(k,l) - N(k,l)$ then $\langle p \rangle = E$.*

Proof. This can be proved by recurrence on the number of strings, $s = (k+l)/2$. Indeed, by using Lemma 4.4, for $s \geq 4$ we have a descent procedure $s \to s-1$, and this leads to the situation $s \in \{1,2,3\}$, where the statement is clear from the examples above. □

Theorem 4.6. *$A_o^*(n)$ is the unique easy Hopf algebra between $A_o(n)$ and $C(O_n)$.*

Proof. Let A be such an easy Hopf algebra, and consider the sets $D(k,l) \subset P(k,l)$, as in Definition 4.2. We have three cases:

(1) Assume first that we have $D(k,l) \subset N(k,l)$, for any k, l. We can apply Theorem 4.1 and we get $A = A_o(n)$.

(2) Assume now that there exist k, l and $p \in P(k,l) - E(k,l)$. From Lemma 4.5 (1) we get $\langle p \rangle = P$, and by applying Theorem 4.1 we get $A = C(O_n)$.

(3) Finally, assume that we have $D(k,l) \subset E(k,l)$ for any k, l, and that there exist k', l' and $p \in E(k',l') - N(k',l')$. From Lemma 4.5 (2) we get $\langle p \rangle = E$, and by applying Theorem 4.1 we get $A = A_o^*(n)$. □

5. Diagonal groups

In this section and in the next two ones we present a classification result for the irreducible corepresentations of $A_o^*(n)$, which is reminiscent of the classification by highest weights of the irreducible representations of compact Lie groups.

We begin with some general considerations. In order to simplify the presentation, all the Woronowicz algebras to be considered will be assumed to be full.

Theorem 5.1. *Let (A,u) be a Woronowicz algebra. Put*

$$A' = A \,/\, \langle u_{ij} = 0, \forall i \neq j \rangle$$

and denote by g_i the image of u_{ii} in A'.

(1) *A' is a cocommutative Hopf algebra quotient and the unitaries g_i generate a group L such that $A' \simeq C^*(L)$.*
(2) *If the elements $g_i \in L$ are pairwise distinct, then A' is maximal as a cocommutative Hopf algebra quotient of A.*

Proof. (1) Denote by J the closed, two-sided ideal generated by the u_{ij} with $i \neq j$. Denote by $q : A \to A'$ the quotient map. We first have to prove that $(q \otimes q)\Delta(a) = 0$ for all $a \in J$, and it suffices to consider $a = u_{ij}$ with $i \neq j$. But then for any k at least one of $q(u_{ik})$, $q(u_{kj})$ vanishes so that

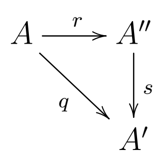
$$(q \otimes q)\Delta(u_{ij}) = \sum q(u_{ik}) \otimes q(u_{kj}) = 0$$

Hence Δ factors to a coproduct $\Delta' : A' \to A' \otimes A'$. Moreover the elements g_i are group-like in A':

$$(q \otimes q)\Delta(u_{ii}) = \sum q(u_{ik}) \otimes q(u_{ki}) = g_i \otimes g_i$$

Since the g_i generate A', this shows that A' is cocommutative.

(2) Assume q factors through another Hopf algebra quotient map $r : A \to A''$ with A'' cocommutative:



Denote by u', u'' the images of u in $M_n(A')$, $M_n(A'')$. We have by definition $u' = \mathrm{diag}(g_i)$, and since A'' is cocommutative u'' can also be decomposed into one-dimensional corepresentations: we write $u'' = P^{-1}\,\mathrm{diag}(h_i)P$ with $h_i \in A''$ and $P \in U_n$.

By commutativity of the diagram above we have $(id \otimes s)(u'') = u'$ hence the elements $s(h_i)$ give the decomposition of u' into irreducible subcorepresentations, so that we can find $\sigma \in S_n$ such that $s(h_i) = g_{\sigma(i)}$. Let $P_\sigma \in U_n$ denote the corresponding permutation matrix.

We have by construction

$$\begin{aligned} u' &= (id \otimes s)(P^{-1}\,\mathrm{diag}(h_i)P) = P^{-1}\,\mathrm{diag}(s(h_i))P \\ &= P^{-1}P_\sigma^{-1}\,\mathrm{diag}(g_i)P_\sigma P = P^{-1}P_\sigma^{-1}u'P_\sigma P \end{aligned}$$

Denoting $Q = P_\sigma P$ this yields $Q_{ij}g_j = g_iQ_{ij}$, and if the g_i are pairwise distinct we obtain $Q_{ij} = 0$ for $i \neq j$. Hence we have $P = P_\sigma$ up to scalar factors, and the identity $u'' = P^{-1}\,\mathrm{diag}(h_i)P$ shows that u'' was already diagonal. As a result $\mathrm{Ker}\, r \supset J$, so we have in fact equality and s is an isomorphism. □

Definition 5.2. *The discrete group L given by*

$$C^*(L) = A \,/\, \langle u_{ij} = 0, \forall i \neq j\rangle$$

is called diagonal group of the Woronowicz algebra A.

Let us now discuss the diagonal groups for standard examples.

Proposition 5.3. *We have the following results:*

(1) *For $A = C^*(\Gamma)$ with Γ finitely generated discrete group we have $L = \Gamma$.*
(2) *For $A = C(G)$ with $G \subset U_n$ compact we have $L = \widehat{T}$, where $T = G \cap \mathbb{T}^n$.*

(3) *For* $A = A_u(n),\ A_o(n),\ A_s(n)$ *we have* $L = F_n,\ \mathbb{Z}_2^{*n},\ \{1\}$ *respectively.*

Proof. This is clear from definitions. □

The interest in the diagonal group comes from Proposition 5.3 (2): for $C(U_n)$, this group is nothing but the dual of the maximal torus of U_n. For a general connected compact subgroup $G \subset U_n$, the diagonal group need not be a dual maximal torus, e.g. $O_n \cap \mathbb{T}^n = \mathbb{Z}_2^n \subset U_n$ is maximal abelian but not a torus. However all maximal torii are clearly duals of diagonal groups, up to conjugation of the fundamental representation u by a matrix $P \in U_n$, and they are known to be maximal abelian. For $G = S_n \subset U_n$ we have $G \cap \mathbb{T}^n = \{1\}$, which is not maximal abelian.

In the quantum case there is no clear notion of what a torus should be, however there are cases where diagonal groups are clearly too small, e.g. for $A = A_s(n)$. We will attack the issue by considering the potential applications of diagonal groups to representation theory: we introduce below a map Φ which should be injective for "good" diagonal subgroups.

We denote by $R^+(A)$ the set of equivalence classes of finite dimensional smooth corepresentations of A, endowed with the operations of sum and tensor product. We use the character map $\chi : R^+(A) \to A$, given by $\chi(r) = Tr(r)$.

Definition 5.4. *Associated to a Woronowicz algebra* (A, u) *is the map*

$$\Phi : R^+(A) \to \mathbb{N}[L]$$

given by $r \to \chi(r)'$, *where* L *is the diagonal group.*

In this definition $x \to x'$ is the canonical map $A \to A' = C^*(L)$, constructed in the previous section. Observe that an alternative definition for Φ could be $\Phi(r) = \chi(r')$, where $r' \in M_n(A')$ is the corepresentation induced by $r \in M_n(A)$.

The target of Φ is indeed $\mathbb{N}[L]$, because characters of corepresentations of $C^*(L)$ are sums of elements of L. For the same reason the elements $\Phi(r) \in \mathbb{N}[L]$ can also be considered as subsets with repetitions of L, which we will denote by $\Sigma(r)$.

Observe finally that Φ is a morphism of semirings, due to the additivity and multiplicativity properties of the character map $w \to \chi(w)$.

Theorem 5.5. *In the following situations,* Φ *is injective and* $C^*(L)$ *is a maximal cocommutative quotient:*

(1) *For* $A = C^*(\Gamma)$, *with* Γ *discrete group of finite type.*
(2) *For* $A = C(G)$, *with* $G \subset U_n$ *connected such that* $G \cap \mathbb{T}$ *is a maximal torus.*
(3) *For the free quantum algebras* $A_o(n),\ A_u(n)$.

Proof. We use the explicit computations of L given at Proposition 5.3.

(1) Here $L = \Gamma$, and the quotient map as well as Φ are actually isomorphisms.

(2) Here L is the weight lattice of G with respect to the maximal torus $T = G \cap \mathbb{T}$, and Φ is the character map, which is known to classify representations of G. The maximality result holds because maximal torii are maximal abelian.
(3) Here $L = \mathbb{Z}_2^{*n}$, F_n, and the injectivity is easily proved using the fusion rules of $A_o(n)$, $A_u(n)$. The maximality results from Theorem 5.1 (2).

□

6. Representation theory

We have seen in the previous section that, at least for certain Woronowicz algebras and up to conjugation of the fundamental corepresentation, the diagonal group is a reasonable candidate for a "dual maximal torus". More precisely, we can say that we have a dual maximal torus when the quotient is maximal cocommutative and Φ is injective.

In what follows we will prove that these requirements are fulfilled in the case of the algebra $A_o^*(n)$. This result, besides of being of independent theoretical interest, can be regarded as a concrete classification of the corepresentations of $A_o^*(n)$, in terms of "combinatorial data".

We begin with a study of the diagonal group. The next Proposition shows in particular that Theorem 5.1 (2) applies in the case of $A_o^*(n)$, hence the diagonal quotient $C^*(L_n)$ of $A_o^*(n)$ is maximal cocommutative. Recall that the diagonal group L_n of $A_o^*(n)$ was already introduced at Definition 1.4.

Proposition 6.1. *Write $\mathbb{Z}_2 = \{1, \tau\}$ and let τ act on $\mathbb{Z}^n$ by $\tau \cdot \lambda = -\lambda$. Consider the subsets $L_n^\circ = \{(\lambda_i) \cdot 1 \mid \sum \lambda_i = 0\}$ and $L_n^\tau = \{(\lambda_i) \cdot \tau \mid \sum \lambda_i = 1\}$ of $\mathbb{Z}^n \rtimes \mathbb{Z}_2$.*

(1) *The group L_n embeds into $\mathbb{Z}^n \rtimes \mathbb{Z}_2$, via $g_i = e_i \cdot \tau$.*
(2) *Its image is $L_n^\circ \cup L_n^\tau$.*
(3) *We have $L_n \simeq \mathbb{Z}^{n-1} \rtimes \mathbb{Z}$.*

Proof. It follows from the definition of the semidirect product that the elements $\gamma_i = e_i \cdot \tau$ multiply according to the following formula:

$$\gamma_{i_1} \ldots \gamma_{i_k} = \left(e_{i_1} - e_{i_2} + \ldots + (-1)^{k+1} e_{i_k}\right) \cdot \tau^k$$

In particular with $k = 2, 3$ we get: $\gamma_a \gamma_b = (e_a - e_b) \cdot 1$, $\gamma_a \gamma_b \gamma_c = (e_a - e_b + e_c) \cdot \tau$. Thus we can define a morphism $\varphi : L_n \to \mathbb{Z}^n \rtimes \mathbb{Z}_2$ by $\varphi(g_i) = \gamma_i$. Moreover, the above formula shows that the image of φ is the subgroup in the statement.

If w is a word on $g_1, \ldots, g_n$, we denote by w^{odd}, w^{even} the subwords formed by letters at odd and even positions respectively, and by w_i the number of occurrences of g_i in w. Then the map φ is given by the following formula, where x is the unique element of $\mathbb{Z}_2$ making $\varphi(w)$ an element of $\varphi(L_n)$:

$$\varphi(w) = (w_i^{\text{odd}} - w_i^{\text{even}})_i \cdot x$$

Indeed, the above formula holds for $w = g_i$, and an easy computation shows that the expression on the right is multiplicative in w.

Now we can prove that $\mathrm{Ker}(\varphi)$ is trivial. If a word w lies in $\mathrm{Ker}(\varphi)$, the above formula shows that each g_i appears an equal number of times at odd and even positions of w. But by definition of L_n the letters of w^{odd} can be permuted without changing the group element, hence we can bring pairs of g_i's in w^{even} and w^{odd} side-by-side and simplify them according to the relation $g_i^2 = 1$, and we get $w = 1$.

Finally it is clear that $\varphi(L_n) = \{xy \mid x \in L_n^\circ,\ y \in \{1, e_1 \cdot \tau\}\}$. Since $L_n^\circ \simeq \mathbb{Z}^{n-1}$ and $\{1, e_1 \cdot \tau\} \simeq \mathbb{Z}_2$ we have $\varphi(L_n) \simeq \mathbb{Z}^{n-1} \rtimes \mathbb{Z}_2$ and an easy check shows that the action of $\mathbb{Z}_2$ on $\mathbb{Z}^{n-1}$ is indeed given by $\tau \cdot \lambda = -\lambda$. □

In this section and the next ones we will make frequent use of the following map, which connects, in a sense to be precised, the corepresentation theory of $A_o^*(n)$ to the representation theory of U_n:

$$\psi : L_n \to \mathbb{Z}^n,\ (\lambda_i) \cdot x \mapsto (\lambda_i)$$

Note that ψ is injective, and that it is not a group morphism.

Theorem 6.2. *Φ is injective for the algebra $A_o^*(n)$.*

Proof. Let v be the fundamental corepresentation of $C(U_n)$, and consider the k-fold tensor product $v_k = v \otimes \bar{v} \otimes v \otimes \ldots$ According to Theorem 2.5, we have:

$$\mathrm{End}(u^{\otimes k}) = \mathrm{End}(v_k)$$

Now recall that the subcorepresentations of a corepresentation w are of the form $(p \otimes 1)w$, with $p \in \mathrm{End}(w)$ projection. This shows that we have a one-to-one additive correspondence J between subobjects of $u^{\otimes k}$ and subobjects of v_k, by setting:

$$J((p \otimes 1)u^{\otimes k}) = J(p \otimes 1)v_k$$

We claim that when k varies, these J maps are compatible with each other. Indeed, let p, q be projections yielding irreducible subrepresentations of v_k, v_l. The same diagrammatic identifications as before show that:

$$q\,\mathrm{Hom}(u^{\otimes l}, u^{\otimes k})p = q\,\mathrm{Hom}(v_l, v_k)p$$

This shows that $(p \otimes 1)u^{\otimes k} = (q \otimes 1)u^{\otimes k}$ is equivalent to $(p \otimes 1)v_k = (q \otimes 1)v_k$, so the J maps are indeed compatible with each other. Summarizing, we have constructed an embedding of additive semirings:

$$J : R^+(A_o^*(n)) \to R^+(C(U_n))$$

Now consider the map $\psi : L_n \to \mathbb{Z}^n$ introduced above, and extend it by linearity to $\mathbb{N}[L_n]$ and $\mathbb{C}[L_n]$. We claim that the following diagram is commutative, so that the injectivity of Φ for $A_o^*(n)$ follows from the one for $C(U_n)$, which is known from the classical theory:

$$\begin{array}{ccc} R^+(A_o^*(n)) & \xrightarrow{J} & R^+(C(U_n)) \\ \downarrow \Phi & & \downarrow \Phi \\ \mathbb{N}[L_n] & \xrightarrow{\psi} & \mathbb{N}[\mathbb{Z}^n] \end{array}$$

Indeed, let us consider the linear extension $\psi : \mathbb{C}[L_n] \to \mathbb{C}[\mathbb{Z}_n]$. We have the following computation, with $\tilde{v} = v$ or $\bar{v}$ depending on the parity of k:

$$\begin{aligned} \psi((u_{i_1j_1} \dots u_{i_kj_k})') &= \delta_{i_1j_1} \dots \delta_{i_kj_k} \psi(g_{i_1} \dots g_{i_k}) \\ &= \delta_{i_1j_1} \dots \delta_{i_kj_k} (e_{i_1} - e_{i_2} + \dots + (-1)^{k+1} e_{i_k}) \\ &= (v_{i_1j_1} \bar{v}_{i_2j_2} \dots \tilde{v}_{i_kj_k})' \end{aligned}$$

This shows that for any rank one projection p we have:

$$\psi(((p \otimes 1)u^{\otimes k})') = ((p \otimes 1)v_k)'$$

By linearity this formula must hold for any $p \in \mathrm{End}(u^{\otimes k})$, so we get $\psi(\Phi(r)) = \Phi(J(r))$ for any corepresentation $r \subset u^{\otimes k}$, and the diagram commutes. □

7. Highest weights

We know from the previous section that the corepresentations of $A_o^*(n)$ can be indexed by certain elements of $\mathbb{N}[L_n]$. In this section we furter refine this result, by indexing the irreducible corepresentations of $A_o^*(n)$ by their "highest weights".

We first recall the general theory for U_n. With the choice of the basis $(e_i - e_{i+1})_i$ for the root system associated to $T = \mathbb{T}^n$, the objects of interest are as follows.

Definition 7.1. *Associated to U_n are the following objects.*

(1) *Dual maximal torus: $X = \mathbb{Z}^n$.*
(2) *Root system: $X_* = \{e_i - e_j \mid i \neq j\}$.*
(3) *Root lattice: $X^\circ = \{(\lambda_i) \in X \mid \sum \lambda_i = 0\}$.*
(4) *Positive weights: $X_+ = \{(\lambda_i) \in X^\circ \mid \lambda_1 \geq 0, \lambda_1 + \lambda_2 \geq 0, \dots, \sum_1^{n-1} \lambda_i \geq 0\}$.*
(5) *Dominant weights: $X_{++} = \{(\lambda_i)_i \in X \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n\}$.*

Recall that $X_* \cup \{0\}$ is the set of weights $\Sigma(v \otimes \bar{v})$ of the adjoint representation of U_n, with the notation $\Sigma(\cdot)$ introduced after Definition 5.6. The set X_* generates the root lattice $X^\circ = \{(\lambda_i) \mid \sum \lambda_i = 0\} \subset X$ and $X = \bigsqcup_{k \in \mathbb{Z}} (X^\circ + k e_1)$.

Observe that X_+ is the subset of X° consisting of elements which have positive coefficients with respect to $(e_i - e_{i+1})_i$. We endow X with the partial order $x \geq y$ if $x - y \in X_+$.

Theorem 7.2. *For any irreducible representation w of U_n, the set with repetitions $\Sigma(w)$ has a greatest element $\lambda_w \in X$, called highest weight of w. For w, w' irreducible we have $w \simeq w'$ iff $\lambda_w = \lambda_{w'}$. Finally, the set of highest weights is X_{++}.*

Proof. This is a well-known result concerning the classification of irreducible representations of U_n. □

We make the following definition.

Definition 7.3. *Associated to $A_o^*(n)$ are the following objects.*

1. *Dual maximal torus: $L = L_n \subset \mathbb{Z}^n \rtimes \mathbb{Z}_2$.*
2. *Root system: $L_* = \{(e_i - e_j) \cdot 1 \mid i \neq j\}$.*
3. *Root lattice: $L^\circ = \{(\lambda_i)_i \cdot 1 \in L \mid \sum \lambda_i = 0\}$.*
4. *Positive weights: $L_+ = \{(\lambda_i)_i \cdot 1 \in L^\circ \mid \lambda_1 \geq 0, \lambda_1 + \lambda_2 \geq 0, \ldots, \sum_1^{n-1} \lambda_i \geq 0\}$.*
5. *Dominant weights: $L_{++} = \{(\lambda_i)_i \cdot x \in L \mid \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\}$.*

Observe that we have as in the classical case $L_* \cup \{0\} = \Sigma(u \otimes \bar{u})$, where $u = \bar{u}$ is the fundamental corepresentation of $A_o^*(n)$. The set L_* generates the subgroup $L^\circ = L_n^\circ$ and we have $L = L^\circ \sqcup L^\tau$ with $L^\tau = L_n^\tau = \{(\lambda_i) \cdot \tau \mid \sum \lambda_i = 1\}$.

Note also that L_+ in contained in L° and consists of elements having positive coefficients with respect to the basis $(e_i - e_{i+1})_i \subset L_*$ of L°. We endow L with the partial order $x \geq y$ if $x - y \in L_+$.

Theorem 7.4. *For any irreducible corepresentation w of $A_o^*(n)$, the set with repetitions $\Sigma(w)$ has a greatest element $\lambda_w \in L$, called highest weight of w. For w, w' irreducible we have $w \simeq w'$ iff $\lambda_w = \lambda_{w'}$. Finally, the set of highest weights is L_{++}.*

Proof. Recall that we use the map $\psi : L \to X$, $(\lambda_i) \cdot x \mapsto (\lambda_i)$ and observe that we have $\psi(L_+) = X_+$ and

$$\psi((\lambda \cdot 1)(\mu \cdot x)) = \psi(\lambda \cdot 1) + \psi(\mu \cdot x)$$

Thus ψ respects the orders. Now since $\Sigma(w)$ has a greatest element for any irreducible representation w of U_n, and this highest weight characterizes w up to isomorphism, the first two assertions follow as in Theorem 6.2.

Moreover, we see that the images under ψ of the highest weights of $A_o^*(n)$ are precisely the highest weights of the irreducible subobjects of the representations v_k of U_n.

Restricting representations of U_n to $Z(U_n) = \mathbb{T}$ corresponds to projecting weights in the quotient $\mathbb{Z} = \mathbb{Z}^n / \langle e_i = e_{i+1} \rangle$, and hence we see that the weights (λ_i) of $v^{\otimes k} \otimes \bar{v}^{\otimes l}$ satisfy $\sum \lambda_i = k - l$. Since all the irreducible representations appear as subobjects of some $v^{\otimes k} \otimes \bar{v}^{\otimes l}$, we conclude that the highest weights of the subobjects of v_{2k} (resp. v_{2k+1}) are exactly the dominant weights $(\lambda_i) \in X_{++}$ such that $\sum \lambda_i = 0$ (resp. 1).

Thus the highest weights we are looking for are the dominant weights of U_n which lie in the image of ψ, and the result is proved. □

As a first application, observe that the highest weights in L°_{++} correspond to the irreducible subobjects of even tensor powers $u^{\otimes 2k}$, and they are mapped by ψ to the highest weights of subobjects of tensor powers of the adjoint representation $v \otimes \bar{v}$ of U_n. Hence we recover the identification $PA_o^*(n) = C(PU_n)$ from Theorem 3.3, at the level of irreducible corepresentations.

8. Fusion rules

In this section we explain how the fusion rules of $A_o^*(n)$ are related to those of U_n, in terms of highest weights. From this we will be able to draw the Cayley graph associated with $R^+(A_o^*(n))$. This is the point where the group structure of the "weight lattice" L, and in particular its non-commutativity, come into play.

According to Theorem 7.4, decomposing into irreducibles and taking highest weights yields an additive bijection from $R^+(A_o^*(n))$ to the free abelian semigroup $\mathbb{N}[L_{++}]$ generated by L_{++}. We endow $\mathbb{N}[L_{++}]$ with the associative, biadditive product $\otimes$ given by the tensor product in $R^+(A_o^*(n))$, and with the additive involution $\lambda \mapsto \bar{\lambda}$ given by the conjugation in $R^+(A_o^*(n))$. We proceed similarly for U_n in $\mathbb{N}[X_{++}]$.

Theorem 8.1. *Denote by $\psi : \mathbb{N}[L_{++}] \to \mathbb{N}[X_{++}]$ the natural additive injective map given by $\psi((\lambda_i)_i \cdot x) = (\lambda_i)_i$ on L_{++}. For $\nu \in X$, $\lambda \in L$ put $\nu^\lambda = \bar{\nu}$ if $\lambda \in L^\tau$ and $\nu^\lambda = \nu$ if $\lambda \in L^\circ$. Then we have for all λ, $\mu \in L_{++}$ the following equalities in $\mathbb{N}[X_{++}]$*

$$\begin{aligned} \psi(\lambda \otimes \mu) &= \psi(\lambda) \otimes \psi(\mu)^\lambda \\ \psi(\bar{\lambda}) &= \overline{\psi(\lambda)}^\lambda \end{aligned}$$

Proof. Denote by v_λ (resp. $w_{\psi(\lambda)}$) an irreducible corepresentation of $A_o^*(n)$ (resp. representation of U_n) with highest weight $\lambda \in L_{++}$ (resp. $\psi(\lambda) \in X_{++}$).

It is clear from the definition of the maps Φ that we have

$$\begin{aligned} \Phi(v_\lambda \otimes v_\mu) &= \Phi(v_\lambda)\Phi(v_\mu) \in \mathbb{N}[L] \\ \Phi(w_{\psi(\lambda)} \otimes w_{\psi(\mu)}) &= \Phi(w_{\psi(\lambda)})\Phi(w_{\psi(\mu)}) \in \mathbb{N}[X] \\ \Phi(w^\lambda_{\psi(\mu)}) &= \Phi(w_{\psi(\mu)})^\lambda \end{aligned}$$

where $w^\lambda = \bar{w}$ or w and $\nu^\lambda = \pm\nu \in X$ are the natural actions of L on $R^+(U_n)$ and X via $\mathbb{Z}_2$. Moreover we know from the proof of Theorem 6.2 that $\psi(\Phi(v_\lambda)) = \Phi(w_{\psi(\lambda)})$ for all $\lambda \in X_{++}$.

Hence by injectivity of Φ it suffices to show that

$$\psi(\Phi(v_\lambda)\Phi(v_\mu)) = \psi(\Phi(v_\lambda))\psi(\Phi(v_\mu))^\lambda$$

We observe that $\Phi(v_\lambda) \subset \mathbb{N}[L^x]$ if $\lambda \in L^x_{++}$, for $x = 1$, τ: this is e.g. a consequence of the fact that the elements of $\Sigma(v_\lambda)$ are smaller than λ. Now the result follows from the identity $\psi(\nu\nu') = \psi(\nu)\psi(\nu')^\nu$ which is clear from the definition of the product in $L \subset \mathbb{Z}^n \rtimes \mathbb{Z}_2$.

Denote by $(x \to x^\sharp)$ the linear extension of $(\lambda \mapsto \lambda^{-1})$ to $\mathbb{N}[L]$, and of $(\lambda \to -\lambda)$ to $\mathbb{N}[X]$. We have then

$$\begin{aligned}\Phi(\bar{w}_\lambda) &= \Phi(w_\lambda)^\sharp \\ \Phi(\bar{w}_{\psi(\lambda)}) &= \Phi(w_{\psi(\lambda)})^\sharp\end{aligned}$$

Hence the second identity of the statement follows as the first one from the identity $\psi(\nu^\sharp) = (\psi(\nu)^\sharp)^\nu$. □

We see in particular that all corepresentations of $A_o^*(n)$ with weights in L^τ are selfadjoint. As an other application let us prove that the fusion rules of $A_o^*(n)$ are not commutative. This might seem quite surprising, because $A_o^*(n)$ appears as intermediate subalgebra between $A_o(n)$ and $C(O_n)$, both having commutative fusion rules.

Proposition 8.2. *For $n \geq 3$ the fusion rules of $A_o^*(n)$ are not commutative. In the case $n = 3$, we have $u \otimes w \not\simeq w \otimes u$, where u is the fundamental corepresentation, and w is the irreducible subobject of $u^{\otimes 4}$ with highest weight $(1, 1, -2) \cdot 1$.*

Proof. First recall that

$$\begin{aligned}\psi(L^\circ_{++}) &= \{(\lambda_i)_i \in X_{++} \mid \textstyle\sum \lambda_i = 0\} \\ \psi(L^\tau_{++}) &= \{(\lambda_i)_i \in X_{++} \mid \textstyle\sum \lambda_i = 1\}\end{aligned}$$

Thus by Theorem 8.1 it is enough to produce irreducible representations v, w of U_n with highest weights in $\psi(L^\tau_{++})$, $\psi(L^\circ_{++})$ respectively such that $v \otimes w \not\simeq v \otimes \bar{w}$.

In fact for any irreducible representations v, w, w' of U_n we have $v \otimes w \simeq v \otimes w'$ iff $w \simeq w'$. Indeed if λ, μ are the highest weights of v, w, the (non-irreducible) representation $v \otimes w$ admits $\lambda + \mu$ as highest weight.

In particular if u is the fundamental corepresentation of $A_o^*(n)$, and w is an irreducible corepresentation of $C(PU_n) \simeq PA_o^*(n) \subset A_o^*(n)$ such that $\bar{w} \not\simeq w$, we have $u \otimes w \not\simeq w \otimes u$ in $R^+(A_o^*(n))$. Such representations w of PU_n always exist when $n \geq 3$.

In the case $n = 3$, the tensor square $u^{\otimes 2}$ is the sum of the trivial representation and a selfadjoint representation, but the power $u^{\otimes 4}$ has nonselfadjoint irreducible subrepresentations. Recall indeed that $\lambda_{\bar{w}} = -w_0(\lambda_w)$, where $w_0(\lambda_1, \ldots, \lambda_n) = (\lambda_n, \ldots, \lambda_1)$. Hence w_1, w_2 with highest weights $(1, 1, -2)$, $(2, -1, -1)$ are subrepresentations of $u^{\otimes 4}$ which are nonselfadjoint. □

Now let us turn to Cayley graphs. Recall the following definition:

Definition 8.3. *Let (A, u) be a Woronowicz algebra and fix a self-adjoint corepresentation u_1 of A not containing the trivial corepresentation. The Cayley graph associated to this data is defined as follows:*

1. *the vertices are classes of irreducible corepresentations,*
2. *if $w \subset v \otimes u_1$, we draw $\dim \mathrm{Hom}(w, v \otimes u_1)$ edges from v to w.*

In the case of $C(U_n)$ we take $u_1 = v \oplus \bar{v}$, where v is the fundamental representation of U_n on $\mathbb{C}^n$. In the case of $C(PU_n)$ we take for u_1 the unique nontrivial irreducible subrepresentation of $v \otimes \bar{v}$. In the case of $A_o^*(n)$ we take $u_1 = u = \bar{u}$, where u is the fundamental corepresentation. Identifying irreducible corepresentations with their highest weights we can draw the corresponding Cayley graphs in L_{++}, X_{++}. We moreover identify L_{++} with a subset of X_{++} via the map ψ as previously.

We refer to the pictures of Section 9 for an illustration of the Propositions below in the case $n = 3$.

Proposition 8.4. *The Cayley graph of $A_o^*(n)$ corresponds to the full subgraph of the Cayley graph of $C(U_n)$ whose vertices are elements $(\lambda_i)_i$ lying in $\psi(L_{++})$, i.e. such that $\sum \lambda_i = 0$ or 1. In particular there is an edge of the Cayley graph of $A_o^*(n)$ between two such elements λ, μ of X_{++} iff $\lambda - \mu = \pm e_i$ for some i.*

Proof. We already know that the vertices of the Cayley graph of $A_o^*(n)$ correspond to the image of $\psi : L_{++} \to X_{++}$, i.e. to the elements $(\lambda_i)_i \in X_{++}$ such that $\sum \lambda_i \in \{0, 1\}$.

Let (λ, μ) be an edge between two such vertices in the Cayley graph of $C(U_n)$. Let w_λ, w_μ be irreducible representations of U_n with highest weights λ, μ. We have $w_\mu \subset w_\lambda \otimes v$ or $w_\mu \subset w_\lambda \otimes \bar{v}$, hence $\mu \in \Sigma(w_\lambda) + \{e_i\}$ or $\mu \in \Sigma(w_\lambda) + \{-e_i\}$ respectively, which yields $\sum \mu_i = \sum \lambda_i \pm 1$ because $\sum \nu_i$ is constant for $\nu \in \Sigma(w_\lambda)$.

So we have $w_\mu \subset w_\lambda \otimes v$ if $\lambda \in \psi(L_{++}^\circ)$ and $w_\mu \subset w_\lambda \otimes \bar{v}$ otherwise. According to Theorem 8.1 this reads $w'_\mu \subset w'_\lambda \otimes u$ in both cases, where w'_λ, w'_μ are irreducible corepresentations of $A_o^*(n)$ with highest weights λ, μ. Hence (λ, μ) is also an edge in the Cayley graph of $A_o^*(n)$.

The last assertion holds for $A_o^*(n)$ because it holds for U_n. As a matter of fact if w is a representation of U_n with highest weight $\lambda \in X_{++}$, then the highest weights of irreducible subobjects of $w \otimes (v \oplus \bar{v})$ are the elements of $\{\lambda \pm e_i\} \cap X_{++}$. □

Proposition 8.5. *Consider in the Cayley graph of $A_o^*(n)$ the vertices with highest weight $\lambda \in L_{++}^\circ$, and the paths of length 2 between such vertices. Remove one loop at each vertex. Then the graph obtained coincides with the Cayley graph of $C(PU_n)$.*

Proof. This results clearly from the identification between $PA_o^*(n)$ and PU_n. Note that paths of length 2 from w to w' correspond to inclusions $w' \subset w \otimes v \otimes \bar{v}$, and we have in PU_n the decomposition $w \otimes v \otimes \bar{v} = w \oplus (w \otimes u_1)$, thus we obtain one more loop at each vertex with paths of length 2 than in the Cayley graph of $C(PU_n)$ generated by u_1. □

9. Polynomial growth

Recall the following notion of growth introduced in [19] and [7]. We fix a Woronowicz algebra (A, u) and a self-adjoint corepresentation u_1 not containing the trivial corepresentation. For any irreducible corepresentation w of (A, u) we denote by $l(w)$ the length of w, which is the distance from 1 to w in the Cayley graph associated with A and u_1. In other words we have

$$l(w) = \min\{k \in \mathbb{N} \mid w \subset u_1^{\otimes k}\}$$

Then the volumes of balls in the discrete quantum group associated with (A, u) are defined as follows:

$$b_k = \sum_{l(w) \leq k} \dim(w)^2$$

Definition 9.1. *We say that (A, u) has polynomial growth if the sequence (b_k) has polynomial growth. If there exist constants d, α, $\beta > 0$ such that $\alpha k^d \leq b_k \leq \beta k^d$ for all k, we say that (A, u) has polynomial growth with exponent d. These notions are independent of u_1.*

To prove the next Theorem we will work with representations and highest weights of SU_n. Observe that restricting representations of U_n to SU_n amounts at the level of weights to quotienting $X = \mathbb{Z}^n$ by the line generated by $e_1 + \cdots + e_n$. We denote by $\bar{X}$ the quotient lattice and use it as the lattice of weights of SU_n.

Since all irreducible representations of SU_n are restrictions of irreducible representations of U_n, the set of dominant weights $\bar{X}_{++} \subset \bar{X}$ is the image of X_{++} in $\bar{X}$. Moreover the quotient map is injective on the subset of dominant weights $(\lambda_i)_i$ such that $0 \leq \sum \lambda_i \leq n-1$, and in particular it is injective on the image of $\psi : L_{++} \to X_{++}$. Notice that the embedding

$$L^\circ_{++} \subset L_{++} \longrightarrow X_{++} \longrightarrow \bar{X}_{++}$$

is induced by the canonical embedding $R^+(C(PU_n)) \to R^+(C(SU_n))$ modulo the identification $PA_o^*(n) \simeq C(PU_n)$.

Moreover the irreducible corepresentations of $C(SU_n)$, $C(U_n)$, $C(PU_n)$ and $A_o^*(n)$ that correspond to each other in this picture have the same dimensions: this is clear for the classical groups since restricting or factoring a representation does not change its dimension, and for $A_o^*(n)$ this results from the proof of Theorem 6.2, where the correspondence between two irreducible corepresentations w, w' of $C(U_n)$, $A_o^*(n)$ is defined by the common subspace $p(\mathbb{C}^{\otimes nk})$ on which they act.

Theorem 9.2. *$A_o^*(n)$ has exponential growth with exponent $d = n^2 - 1$.*

Proof. We proceed by comparison with SU_n, whose growth exponent $d = n^2 - 1$ is known by Theorem 2.1 in [7].

Denote by B_k^{PU}, B_k^{SU}, B_k^A the balls of radius k in the Cayley graphs of $C(PU_n)$, $C(SU_n)$, $A_o^*(n)$, and by b_k^{PU}, b_k^{SU}, b_k^A the growth sequences as above. We have $B_k^{PU} \subset B_{2k}^A \subset B_{2k}^{SU}$ by Propositions 8.4 and 8.5, hence $b_k^{PU} \leq b_{2k}^A \leq b_{2k}^{SU}$.

Now it remains to control (b_k^{SU}) by (b_k^{PU}), and this is a problem in the classical representation theory of SU_n. We present here an ad-hoc argument.

Let first w be an irreducible representation of PU_n, hence also of SU_n, such that $w \in B_k^{SU}$. Since the fundamental representation v of SU_n generates its category of representations, we can find a constant a depending only on n and $l \leq ak$ such that $w \subset v^{\otimes l}$. Since w is a representation of PU_n we must have $l = pn$ for some $p \in \mathbb{N}$, and we notice that

$$v^{\otimes n} = v^{\otimes n-1} \otimes v \subset v^{\otimes n-1} \otimes \bar{v}^{\otimes n-1} = (v \otimes \bar{v})^{\otimes n-1}$$

and hence $w \subset (v \otimes \bar{v})^{\otimes p(n-1)} \subset (v \otimes \bar{v})^{\otimes l}$, so $w \in B_{ak}^{PU}$.

Now take an irreducible representation $w \in B_k^{SU}$ of SU_n with highest weight $(\lambda_i)_i$ such that $\sum \lambda_i = -l \in \{-n+1, \ldots, 0\}$. Then the subobjects $w' \subset w \otimes v^{\otimes l}$ have highest weights $(\lambda'_i)_i$ such that $\sum \lambda'_i = 0$, hence they factor to representations of PU_n. Since we clearly have $w' \in B_{k+l}^{SU} \subset B_{k+n}^{SU}$, the preceeding discussion shows that $w' \in B_{a(k+n)}^{PU}$. But for any such w' we also have $w \subset w' \otimes \bar{v}^{\otimes l}$. We have thus proved:

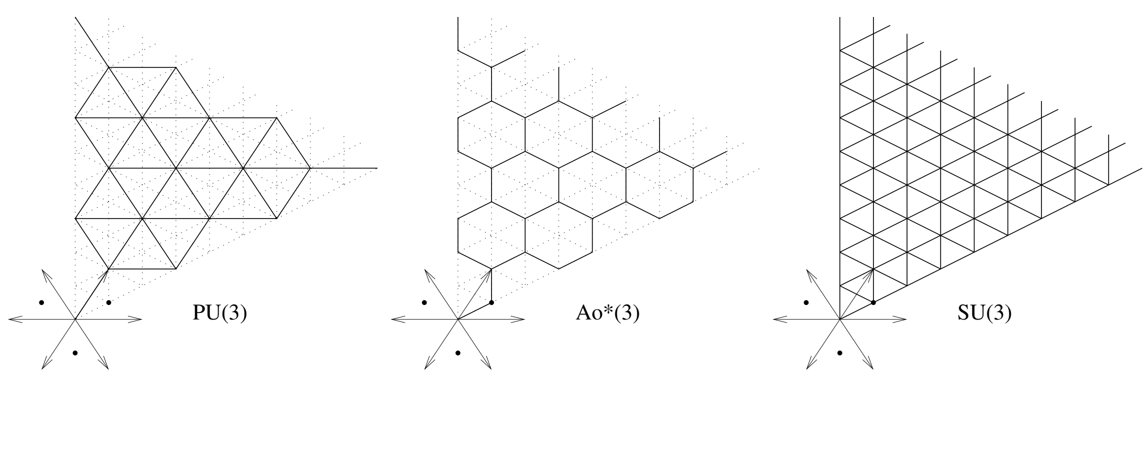
$$B_k^{SU} \subset B_{a(k+n)}^{PU} \otimes U \qquad \text{where} \qquad U = \bigoplus_{l=0}^{n-1} \bar{v}^{\otimes l}$$

From this inclusion it clearly follows $b_k^{SU} \leq (\dim U)^2 b_{a(k+n)}^{PU}$. Hence we have alltogether $b_k^{PU} \leq b_{2k}^{A} \leq b_{2k}^{SU} \leq (\dim U)^2 b_{a(2k+n)}^{PU}$. Since the (b_k) sequences are growing and (b_k^{SU}) has growth exponent $d = n^2 - 1$, this proves that (b_k^{PU}), (b_k^{A}) are polynomially growing with the same exponent. □

Corollary 9.3. *The discrete quantum group associated with $A_o^*(n)$ is amenable and has the Property of rapid decay.*

Proof. See [7], Proposition 2.1 and [19], Proposition 4.4. □

To illustrate the proof of Theorem 9.2 and the Cayley graph computations of Section 8, let us draw the Cayley graphs of $C(PU_3)$, $A_o^*(3)$ and $C(SU_3)$ in $\bar{X}_{++}$. These graphs have no multiplicity and no loops, except in the case of PU_3 where there are 2 loops, that we do not represent, at each vertex different from the origin. The arrows denote the root system of $SU(3)$, and the dots are the images of e_1, e_2, e_3 in $\bar{X}_{++}$.



As a conclusion, let us remark that the results of this paper and the pictures above seem to hint at the existence of some geometrical data behind compact quantum groups, in the spirit of the classical constructions for semisimple Lie algebras. It is tempting to ask whether the theoretical framework of [25] for differential calculus on compact quantum groups can give more insight on the nature of the geometrical objects involved. However this seems to be a very challenging question, and other intermediate examples between the classical world and the free world of compact quantum groups might be needed first.

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T.B.: Department of Mathematics, Toulouse 3 University, 118 route de Narbonne, 31062 Toulouse, France. banica@math.ups-tlse.fr

R.V.: Department of Mathematics, Caen University, BP 5186, 14032 Caen Cedex, France. vergnioux@math.unicaen.fr