Planar algebra basics

Teo Banica

Department of Mathematics, University of Cergy-Pontoise, F-95000 Cergy-Pontoise, France. teo.banica@gmail.com

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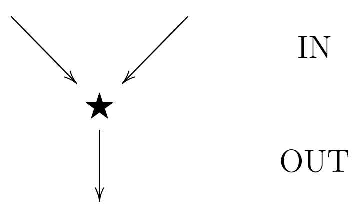
Abstract. This is an introduction to planar algebras, and other algebras formed by diagrams, with emphasis on operator algebra and quantum physics aspects. Following Jones, we introduce the planar algebras as combinatorial devices, with motivation from the Temperley-Lieb algebra, and knot invariants. Then we discuss the relation with subfactor theory, via the heavy axiomatization work there. Aftwerwards, we go into a discussion involving quantum physics, quantum fields, strings and more. Finally, we go back to mathematics, and discuss analytic aspects, in the large index setting.

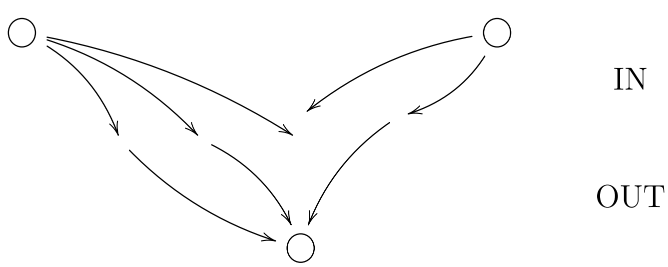
Preface

This is an introduction to the planar algebras, coming from the work of Jones and others in the 80s and 90s, and axiomatized by Jones in his 1999 paper [49]. The idea is very simple. Consider the following device, appearing in mathematics, physics, chemistry, engineering, biology, computer science, economy, industry, and many more:

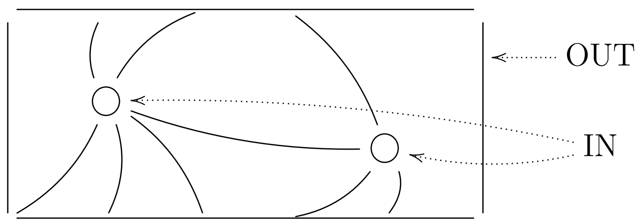
In mathematics and physics, the most such interesting devices are the Feynman diagrams, describing what happens when elementary particles collide:



However, you might have heard about string theory, saying that elementary particles are in fact small vibrating strings, operating on a rather smooth basis:



This happens in 3D, but we can get back to 2D by hugely enlarging the output circle, and projecting everything on it. With the further convention that this output circle is replaced by a box, say for typographical reasons, we are led to a diagram as follows:



And good news, this is what a planar algebra is. In short, what we need is a graded vector space $P = (P_k)$, for the input and output data, and then the above diagrams take care of the computations, producing output from input, k being the number of legs.

All this sounds very good, but the question that you surely have in mind is, what does a planar algebra exactly compute? Well, no one really knows. But the present book is here for introducing you to such beasts, in the hope that, later, you will find.

Generally speaking, planar algebras are best learned from Jones, who spent more than 20 years in formalizing them. His early papers on subfactors and knots [**43**], [**44**], [**45**], [**46**], complemented by [**47**], [**48**], have profoundly reshaped the mathematics and physics of the 80s and 90s, and are a must-read, for any mathematician or physicist. As for the planar algebras themselves, you have here his trilogy [**49**], [**50**], [**51**], totalling about 200 pages, which is a must-read too, for anyone interested in the subject.

The present book stands for what it is worth, with the aim of explaining [**49**], [**50**], [**51**], from a point of view very close to the one of Jones, but not identical to it. Algebrically, we will be more into Lie groups and quantum groups, instead of the traditional topology and knots. Analytically, we will be more into calculus and random matrices, instead of the traditional functional analysis and subfactors. And physically, we will be a bit more into quantum mechanics, than statistical mechanics. Although, regarding this latter point, this is something quite insignificant in the context of the big problems of physics, which seem to require a 50–50 mix of quantum and statistical mechanics, anyway.

Everything I know about planar algebras, I learned it from Vaughan Jones, his students and collaborators. Nothing can replace attending a talk by him, or a discussion with him, or a beer with him and his group. In the hope however that this book, while written in the rainy northern France, can help communicating a bit of that West Coast spirit.

Many thanks as well to my other colleagues and collaborators. Finally, many thanks go to my cats. No one really understands what happens during their collisions, which seem to defy all laws of mathematics and physics. To be axiomatized, too.

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Planar algebras

How many roads must a man walk down
Before you call him a man
How many seas must a white dove sail
Before she sleeps in the sand

CHAPTER 1

Temperley-Lieb

1a. Combinatorics

Welcome to planar algebras. This is a wide and powerful algebraic theory, having ramifications in most of the modern branches of mathematics and physics, and that you will probably find useful, no matter what mathematics or physics you are doing.

The planar algebras have a long story. They were axiomatixed in the 1999 paper of Jones [49], following more than 20 years of work, by him and others, on quite technical topics from mathematical physics, such as subfactors, and knot invariants. However, their definition is quite simple, barely more complicated than that of rings, algebras, fields and so on. Following [49], or any other kind of introductory text to the subject, we will give their definition, and discuss the relation with subfactors, knots and more, afterwards.

Before starting, a few instructions, depending on the type of reader that you are:

(1) All that follows will be based on standard mathematics as we know it, meaning Rudin [76]. When needing pieces of more complicated mathematics, we will explain them, or give references. The same goes for physics, the basic text here, that we will assume that you are sort of familar with, being the undergraduate course of Feynman [28].

(2) In what regards planar algebras, the original text of Jones [49], having 120 pages, and which was written over a long period of time, and by Jones who was a master in mathematical presentation, is hard to beat. So, have a copy of that handy. Of course, we will be presenting things here from a slightly different perspective, and so on.

(3) Finally, in case you are already familiar with [49] and planar algebras, our table of contents is there, for explaining what we will be doing here. Let me also mention that, at the level of conventions and notations, everything square diagram will go from up to down, I will use boxes instead of disks, and the index will be denoted $N \in [1, \infty]$.

Getting started now, you would probably expect the definition of planar algebras, to start with. That could be an option, but thinking well, at abstract algebraic objects and their definition, would you like for instance to hear the formal definition of a field k, assuming that you only know about the positive integers $\mathbb{N}$, with their addition $+$ and multiplication $\times$, and perhaps a bit about fractions too? Certainly not, the definition of

k comes after some serious study of $\mathbb{Q}, \mathbb{R}, \mathbb{C}$. So, this is the situation, let us be reasonable, and getting back now to planar algebras, their definition can wait a bit, we need examples before anything. And here, coming a bit in advance, what we need to know is:

Fact 1.1. *The simplest example of planar algebra is the Temperley-Lieb algebra, that we must understand first.*

In order to introduce now the Temperley-Lieb algebra, many options are on the table. Group theory, topology, statistical mechanics, as in the original paper of Temperley and Lieb [**81**], operator algebras, quantum mechanics, probability theory, random matrices, and many more, all these can be useful in order to introduce you to this algebra.

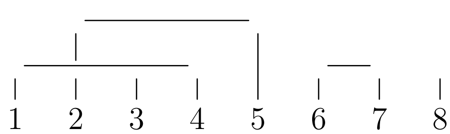
However, for "creation" purposes in mathematics, nothing beats set theory, and the mighty empty set $\emptyset$. So, here is the story that we intend to tell, to start with:

Fact 1.2. *The legend goes that $\emptyset$ produced $\mathbb{N}$ and mathematics, by recursion. In fact, $\emptyset$ produced the Temperley-Lieb algebra too, by recursion and partition.*

Very nice, so eventually, we have a plan. We will talk in this chapter about $\emptyset$ and its various creations, including sets, partitions, and the Temperley-Lieb algebra, that we will study a bit too. Then in chapter 2, based on this knowledge, we will axiomatize the planar algebras, and work out a few other basic examples as well, appearing as variations of the Temperley-Lieb algebra. And then in chapters 3-16 we will study the planar algebras.

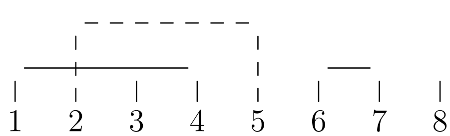
Getting started for good now, as a first definition for this book, we have:

Definition 1.3. *We denote by $P(k)$ the set of partitions of $\{1, \ldots, k\}$, with these partitions $\pi \in P(k)$ being most conveniently being drawn as diagrams,*



with the strings joining the numbers belonging to the same block of π. That is, the above diagram represents the partition $\{1, \ldots, 8\} = \{1, 3, 4\} \cup \{2, 5\} \cup \{6, 7\} \cup \{8\}$.

Observe that there is a bit of care to be taken with this convention, in respect to the crossings. We can either proceed as above, with the $\{2, 5\}$ block being respresented "under" the block $\{1, 3, 4\}$, or use different types of strings, as for instance:



Both conventions are good, and we will be mostly using the first one, that from Definition 1.3. Now, let us study these partitions. And here, surprise, instead of pulling a theorem, as you would expect, and believe me I would have liked as well to have a quick theorem, to start my book, we must formulate something quite modest, as follows:

Proposition 1.4. *The Bell numbers $B_k = |P(k)|$ satisfy the recurrence relation*

$$B_{k+1} = \sum_s \binom{k}{s} B_{k-s}$$

with initial data $B_0 = 1$, $B_1 = 1$, and are numerically as follows:

$$1,\ 1,\ 2,\ 5,\ 15,\ 52,\ 203,\ 877,\ 4140,\ 21147,\ 115975,\ 678570,\ \ldots$$

However, there is no mathematical formula for B_k.

Proof. There are several things going on here, the idea being as follows:

(1) Experiments first, before anything, let us compute a few Bell numbers. Obviously $B_1 = 1$, and then we have $B_2 = 2$, the partitions being as follows:

$$|\,|\quad,\quad\sqcap$$

Next, we have $B_3 = 5$, the partitions at $k = 3$ being as follows:

$$|\,|\,|\quad,\quad\sqcap\,|\quad,\quad\sqcap\!\!\!|\quad,\quad|\,\sqcap\quad,\quad\sqcap\!\sqcap$$

At $k = 4$ now, things become more complex, and it is better to trick. We can count the partitions up to permutations of the corresponding diagrams, and with this convention made, here are the relevant partitions and their multiplicities, leading to $B_4 = 15$:

$$|\,|\,|\,|\times 1\quad,\quad\sqcap\,|\,|\times 6\quad,\quad\sqcap\,\sqcap\times 3\quad,\quad\sqcap\!\sqcap\,|\times 4\quad,\quad\sqcap\!\sqcap\!\sqcap\times 1$$

The same method works at $k = 5$, with the block distributions and multiplicities, which are simpler to draw than partitions, being as follows, leading to $B_5 = 52$:

$$11111 \to 1\ ,\ 2111 \to 10\ ,\ 221 \to 15\ ,\ 311 \to 10\ ,\ 32 \to 10\ ,\ 41 \to 5\ ,\ 5 \to 1$$

As for the case $k = 6$, where $B_6 = 203$, we will leave this as an instructive exercise.

(2) Let us try now to find a recurrence for these Bell numbers. Since a partition of $\{1, \ldots, k+1\}$ appears by choosing s partners for 1, among the k numbers available, and then partitioning the $k - s$ elements left, we have the following formula:

$$B_{k+1} = \sum_s \binom{k}{s} B_{k-s}$$

Observe that this formula forces us to talk about $B_0 = 1$, as done in the statement.

(3) As for the last assertion, regarding the non-computability of the Bell numbers, take this as a physics fact. Mankind has tried to find a formula for these numbers, had not found anything, and we are reporting here this finding, which is of course rock-solid. $\square$

All the above does not look very good. We seem to be going on some sort of wrong way with our partitions, most likely into one of the numerous fringe branches of mathematics. However, as a ray of light, we have the following theorem, connecting the partitions and Bell numbers to the central objects in discrete probability, the Poisson laws:

THEOREM 1.5. *The moments of the Poisson law are the Bell numbers:*

$$p_1=\frac{1}{e}\sum_{k\in\mathbb N}\frac{\delta_k}{k!}\quad:\quad M_k(p_1)=|P(k)|$$

More generally, the moments of the Poisson law of parameter $t>0$ are as follows,

$$p_t=e^{-t}\sum_{k\in\mathbb N}\frac{t^k}{k!}\,\delta_k\quad:\quad M_k(p_t)=\sum_{\pi\in P(k)}t^{|\pi|}$$

where $|.|$ is the number of blocks.

PROOF. The moments of p_1 are given by the following formula:

$$M_k=\frac{1}{e}\sum_r\frac{r^k}{r!}$$

We therefore have the following recurrence formula for these moments:

$$\begin{aligned}M_{k+1}&=\frac{1}{e}\sum_r\frac{r^k}{r!}\left(1+\frac{1}{r}\right)^k\\&=\frac{1}{e}\sum_r\frac{r^k}{r!}\sum_s\binom{k}{s}r^{-s}\\&=\sum_s\binom{k}{s}M_{k-s}\end{aligned}$$

But the Bell numbers $B_k=|P(k)|$ satisfy the same recurrence, so we have $M_k=B_k$, as claimed. Next, the moments of p_t with $t>0$ are given by:

$$N_k=e^{-t}\sum_r\frac{t^rr^k}{r!}$$

We therefore have the following recurrence formula for these moments:

$$\begin{aligned} N_{k+1} &= e^{-t}\sum_r \frac{t^{r+1}r^k}{r!}\left(1+\frac{1}{r}\right)^k \\ &= e^{-t}\sum_r \frac{t^{r+1}r^k}{r!}\sum_s \binom{k}{s} r^{-s} \\ &= t\sum_s \binom{k}{s} N_{k-s} \end{aligned}$$

But the numbers $S_k=\sum_{\pi\in P(k)} t^{|\pi|}$ are easily seen to satisfy the same recurrence, with the same initial values, namely t and $t+t^2$, so we have $N_k=S_k$, as claimed. □

Summarizing, we have some partition mathematics going on, for sure, but with the Poisson laws being something quite deep, we are not exactly into the simple and conceptual things we were wishing for. Let us record our conclusions as follows:

Conclusion 1.6. *The set partitions $\pi\in P(k)$ are something quite complicated, and better not mess with them, unless doing advanced probability.*

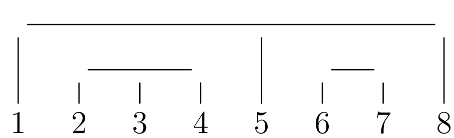
Shall we give up? Certainly not. When looking for a bug in our theory, after some thinking, that bug is in fact in Definition 1.3 and in the comments afterwards, regarding the annoyance caused by the crossings, when drawing our partitions. So, forgetting all the mathematics that we know, back to primary school, and we would prefer our partitions to be noncrossing, as for us to be able to draw them quicker. Obviously.

This might sound of course overly futile, but you know what, sometimes kids are right, and adults are wrong. So, let us record this thought, as follows:

Thought 1.7. *The partitions $\pi\in P(k)$ look like non-topological objects, but when it comes to drawing them, they are topological, with the crossings causing the mess.*

And in what follows, we will trust this thought. Which teaches us something very simple, namely that in order to reach to simpler objects, we must remove the crossings. So, let us update Definition 1.3, in wishing for a better theory, as follows:

Definition 1.8. *We denote by $NC(k)$ the set of noncrossing partitions of $\{1,\ldots,k\}$, that is, of the partitions $\pi\in P(k)$ which can be drawn as noncrossing diagrams,*



with the strings joining as usual the numbers belonging to the same block of π. The above diagram represents the partition $\{1,\ldots,8\}=\{1,5,8\}\cup\{2,3,4\}\cup\{6,7\}$.

And surprise here, with this definition in hand, everything illuminates. To start with, the numbers $C_k = |NC(k)|$, called Catalan numbers, are computable, and very interesting. There are many things to be said here, and as a first result on the subject, we have:

THEOREM 1.9. *The Catalan numbers $C_k = |NC(k)|$ satisfy the recurrence relation*

$$C_{k+1} = \sum_{a+b=k} C_a C_b$$

with initial data $C_0 = 1$, $C_1 = 1$, and are numerically as follows:

$$1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430,\ 4862,\ 16796,\ 58786,\ \ldots$$

Moreover, these numbers are given by the formula

$$C_k = \frac{1}{k+1}\binom{2k}{k}$$

coming from the fact that $f(z) = \sum_k C_k z^k$ satisfies $zf^2 - f + 1 = 0$.

PROOF. As before with the Bell numbers, there is no hurry in proving this, and we will take our time, with experiments first, comments, and then proofs:

(1) To start with, let us compute a few Catalan numbers. At $k = 1, 2, 3$ all the partitions are obviously noncrossing, so we have $C_k = B_k$ here, that is:

$$C_1 = 1 \quad , \quad C_2 = 2 \quad , \quad C_3 = 5$$

At $k = 4$ now, we have exactly 1 crossing partition, namely , and we obtain:

$$C_4 = B_4 - 1 = 14$$

At $k = 5$, we can recycle the count for the Bell numbers, from the proof of Proposition 1.4. Taking into account the crossings, this goes as follows, yielding $C_5 = 42$:

$$11111 \to 1\ ,\ 2111 \to 10\ ,\ 221 \to \not{15}\,10\ ,\ 311 \to 10\ ,\ 32 \to \not{10}\,5\ ,\ 41 \to 5\ ,\ 5 \to 1$$

As for the case $k = 6$, where $C_6 = 132$, we will leave this as an instructive exercise.

(2) Before getting into abstract mathematics, let us record a numeric comparison between the Bell and the Catalan numbers. The table here is as follows:

k	1	2	3	4	5	6	7	8	9	10
B_k	1	2	5	15	52	203	877	4140	21147	115975
C_k	1	2	5	14	42	132	429	1430	4862	16796
$B_k - C_k$	0	0	0	1	10	71	448	2710	16285	99179

This table is quite interesting, definitely showing that we are dealing with different beasts here, the point being that, with $k \to \infty$, most of the partitions appear crossing.

(3) Getting now to general theory, let us try to find a recurrence for the Catalan numbers. In order to construct a noncrossing partition of $\{1, \ldots, k+1\}$, we must choose

a number of partners for 1, and by looking at the partner which appears the most at right, we are led to the following recurrence formula for the Catalan numbers:

$$C_{k+1}=\sum_{a+b=k}C_aC_b$$

Observe that this formula forces us to talk about $C_0=1$, as done in the statement.

(4) In order to solve our recurrence, consider the generating series of the Catalan numbers, $f(z)=\sum_{k\geq 0}C_kz^k$. In terms of this generating series, our recurrence gives:

$$\begin{aligned}
zf^2 &= \sum_{a,b\geq 0}C_aC_bz^{a+b+1}\\
&= \sum_{k\geq 1}\sum_{a+b=k-1}C_aC_bz^k\\
&= \sum_{k\geq 1}C_kz^k\\
&= f-1
\end{aligned}$$

(5) By solving the equation $zf^2-f+1=0$ found above, and choosing the solution which is bounded at $z=0$, we obtain the following formula for our series:

$$f(z)=\frac{1-\sqrt{1-4z}}{2z}$$

(6) In order to compute now this function, we use the generalized binomial formula, which is as follows, with $p\in\mathbb{R}$ being an arbitrary exponent, and with $|t|<1$:

$$(1+t)^p=\sum_{k=0}^\infty\binom{p}{k}t^k$$

To be more precise, this formula, which generalizes the usual binomial formula, holds indeed due to the Taylor formula, with the binomial coefficients being given by:

$$\binom{p}{k}=\frac{p(p-1)\ldots(p-k+1)}{k!}$$

(7) For the exponent $p=1/2$, the generalized binomial coefficients are:

$$\begin{aligned}\binom{1/2}{k} &= \frac{1/2(-1/2)(-3/2)\ldots(3/2-k)}{k!}\\ &= (-1)^{k-1}\frac{1\cdot 3\cdot 5\ldots(2k-3)}{2^k k!}\\ &= (-1)^{k-1}\frac{(2k-2)!}{2^{k-1}(k-1)!2^k k!}\\ &= \frac{(-1)^{k-1}}{2^{2k-1}}\cdot\frac{1}{k}\binom{2k-2}{k-1}\\ &= -2\left(\frac{-1}{4}\right)^k\cdot\frac{1}{k}\binom{2k-2}{k-1}\end{aligned}$$

(8) Thus the generalized binomial formula at exponent $p=1/2$ reads:

$$\sqrt{1+t}=1-2\sum_{k=1}^\infty\frac{1}{k}\binom{2k-2}{k-1}\left(\frac{-t}{4}\right)^k$$

But with $t=-4z$ we obtain from this the following formula:

$$\sqrt{1-4z}=1-2\sum_{k=1}^\infty\frac{1}{k}\binom{2k-2}{k-1}z^k$$

(9) Now back to our series f, we obtain the following formula for it:

$$\begin{aligned}f(z) &= \frac{1-\sqrt{1-4z}}{2z}\\ &= \sum_{k=1}^\infty\frac{1}{k}\binom{2k-2}{k-1}z^{k-1}\\ &= \sum_{k=0}^\infty\frac{1}{k+1}\binom{2k}{k}z^k\end{aligned}$$

(10) Thus the Catalan numbers are given by the formula the statement, namely:

$$C_k=\frac{1}{k+1}\binom{2k}{k}$$

So done, we have now proof for everything claimed in the statement. □

The above was quite exciting, but the occurrence of heavy calculus at the end can be interpreted as good or bad news, depending on your mathematical knowledge and philosophy, and mood. Personally I tend to take such things as good news, whenever I see calculus showing up in abstract algebra questions, I say to myself “calculus, saved”.

But this is of course something subjective, assuming that calculus is indeed the foundation of mathematics, which, while most likely true, remains something debatable.

So, here is as well a bijective proof for the formula of C_k, that I sort of love too, while considering however that this is no match for our previous $\sqrt{1-4z}$ beauties:

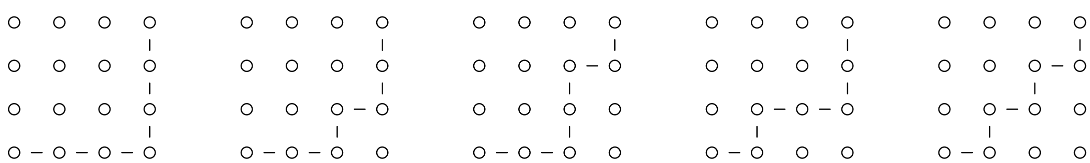
THEOREM 1.10. *The Catalan numbers are given by the formula*

$$C_k = \frac{1}{k+1}\binom{2k}{k}$$

with this being seen also by counting the length $2k$ Dyck paths in the plane.

PROOF. This is something quite tricky, the idea being as follows:

(1) To start with, the length $2k$ Dyck paths in the plane are by definition the paths from $(0,0)$ to (k,k), marching North-East over the integer lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$, by staying inside the square $[0,k] \times [0,k]$, and staying as well under the diagonal of this square. As an example, here are the 5 possible Dyck paths at $k = 3$:



(2) In practice, counting a bit, and we will leave this as an exercise, shows that the number of such paths is as follows, exactly as the Catalan numbers:

$$1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430,\ 4862,\ 16796,\ 58786,\ \ldots$$

Now forgetting all the math that we know, from Theorem 1.9, let us denote by C_k' the number of such length $2k$ Dyck paths. We have to do two things, namely prove that these numbers C_k' equal indeed the Catalan numbers $C_k = |NC(k)|$, and then, do some sort of direct counting, as to reach to the formula for $C_k = C_k'$ in the statement.

(3) In what concerns the first question, this is easy settled. Indeed, when looking at the point where our Dyck path last intersects the diagonal, we are led to the following recurrence relation for the number of such paths, exactly as for the Catalan numbers:

$$C_{k+1}' = \sum_{a+b=k} C_a' C_b'$$

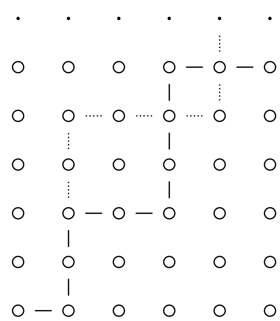
Moreover, the initial data being $C_1' = 1$, $C_2' = 2$, we conclude that we have:

$$C_k' = C_k$$

(4) Let us count now the Dyck paths in the plane. For this purpose, we use a trick. If we ignore the assumption that our path must stay under the diagonal of the square, we

have $\binom{2k}{k}$ such paths. And among these, we have the "good" ones, those that we want to count, and then the "bad" ones, those that we want to ignore.

(5) So, let us count the bad paths, those crossing the diagonal of the square, and reaching the higher diagonal next to it, the one joining $(0,1)$ and $(k,k+1)$. In order to count these, the trick is to "flip" their bad part over that higher diagonal, as follows:



(6) Now observe that, as it is obvious on the above picture, due to the flipping, the flipped bad path will no longer end in (k,k), but rather in $(k-1,k+1)$. Moreover, more is true, in the sense that, by thinking a bit, we see that the flipped bad paths are precisely those ending in $(k-1,k+1)$. Thus, we can count these flipped bad paths, and so the bad paths, and so the good paths too, and so good news, we are done.

(7) To finish now, by putting everything together, we have:

$$\begin{aligned} C_k' &= \binom{2k}{k}-\binom{2k}{k-1}\\ &= \binom{2k}{k}-\frac{k}{k+1}\binom{2k}{k}\\ &= \frac{1}{k+1}\binom{2k}{k} \end{aligned}$$

Thus, we are led to the formula in the statement. □

To conclude now this opening section, good work done for the day, with some interesting theorems, and as a summary of our findings so far, let us record:

Conclusion 1.11. *The following happen, in relation with partitions:*

(1) *The noncrossing partitions $\pi\in NC(k)$ are simpler objects than the arbitrary partitions $\pi\in P(k)$, potentially leading to interesting mathematics.*
(2) *And with this coming from the fact that, as we can see when practically drawing partitions, there is less topology involved in $NC(k)$ than in $P(k)$.*

To be more precise, all this is of course quite subjective, with (1) coming by comparing Proposition 1.4, which is ugly, with Theorem 1.9, and with Theorem 1.10 too, which are both beautiful, and with (2), which certainly contradicts our instant intuition, but go draw some partitions first, coming from Conclusion 1.6 and Thought 1.7.

Of course, I can hear you screaming, what is the point with all these subjective comments. In answer, despite the formal simplicity of both $P(k)$ and $NC(k)$, we are in fact into uncharted territory, not far from quantum mechanics. And love, hate, and subjectiveness in general can only help, a bit in the same way as in quantum mechanics.

But probably the best here, in connection with formal mathematics vs subjectivity, is to quote Hermann Weyl, one of the best mathematicans and physicists ever:

WEYL 1.12. *Among the correct and the beautiful, I always chose the beautiful.*

Finally, also in relation with this, a big, modern conjecture in physics is that at very small scales, somewhere between quarks and the Planck scale, with both ends not excluded, free geometry, coming from $NC(k)$, rules, and produces via thermodynamic limits the higher theories, including our usual, continuous geometry, coming from $P(k)$.

In short, Conclusion 1.11 is something quite deep, and if looking for a good prize in mathematics or physics, simply work some more on that. But more on all this later.

1b. Noncrossing pairings

Looking at what we did so far with the Catalan numbers, and looking for more occurrences of these numbers, we are led to some sort of combinatorial wonderland. Many things can be said here, and for the purposes of our present book, let us record:

THEOREM 1.13. *The Catalan numbers C_k count:*

(1) *The noncrossing partitions of* $1,\ldots,k$.
(2) *The noncrossing pairings of* $1,\ldots,2k$.
(3) *The length* $2k$ *loops on* $\mathbb{N}$*, based at* 0.
(4) *The length* $2k$ *Dyck paths in the plane.*

PROOF. All this is standard combinatorics, the idea being as follows:

(1) This is something that we know, standing as a definition for C_k.

(2) This is something surprising, having no crossing counterpart, in the sense that the pairings $P_2(2k)$ of the set $\{1,\ldots,2k\}$ are by no means related to the partitions $P(k)$ of the set $\{1,\ldots,k\}$. However, by some kind of magic, when restricting the attention to the noncrossing partitions, all this works. In order to understand this, let us begin with some examples. If we set $C_k'=|NC_2(2k)|$, then we have $C_1'=1$, $C_2'=2$, coming from:

At $k=3$ now, we have $C_3'=5$, the noncrossing pairings being as follows:

$$\cap\cap\cap\quad,\quad\cap\Cap\quad,\quad\Cap\cap,\quad,$$  , 

And then $C_4'=14$, $C_5'=42$ so on, we obtain the Catalan numbers. In order now to prove this, we have two choices. First, we can try to establish a bijection as follows:

$$NC(k)\simeq NC_2(2k)$$

However, we will leave this for later, because this bijection will be in fact so important for us, that it is worth a separate treatment, with a dedicated theorem, coming with full details, comments, examples, pictures and so on. In the meantime, we can establish as well $C_k'=C_k$ by recurrence, as follows. In order to construct a noncrossing pairing of $\{1,\ldots,2k+2\}$ we must choose a partner x for the first number, 1, and then pair in a noncrossing way the $2k$ elements left, by avoiding the string $1-x$. Thus, we have:

$$C_k'=\sum_{a+b=k}C_a'C_b'$$

Since the initial data is $C_1'=1$, $C_2'=2$, we conclude that we have, as claimed:

$$C_k'=C_k$$

(3) This is something very interesting too, which will end up in clarifying our probability work, started with Theorem 1.5. To begin with, some examples. If we denote by C_k'' the number of $2k$ loops on $\mathbb{N}$, based at 0, we first have $C_1''=1$, the only loop here being $0-1-0$. Then we have $C_2''=2$, due to two possible loops, namely:

$$0-1-0-1-0$$

$$0-1-2-1-0$$

Then we have $C_3''=5$, the possible loops here being as follows:

$$0-1-0-1-0-1-0$$

$$0-1-0-1-2-1-0$$

$$0-1-2-1-0-1-0$$

$$0-1-2-1-2-1-0$$

$$0-1-2-3-2-1-0$$

And then $C_4''=14$, $C_5''=42$ so on, we obtain the Catalan numbers. In order now to formally prove this, we can either establish a bijection with the partitions in (1), or with the pairings in (2), or pull out a formal proof, by showing that our numbers satisfy:

$$C_k''=\sum_{a+b=k}C_a''C_b''$$

But all three proofs work, and we will leave them as an instructive exercise.

(4) In what regards the Dyck paths, we already know from Theorem 1.10 that these are counted by the Catalan numbers, so done. However, if looking for some good exercises in combinatorics, prove that these Dyck paths are in bijection with the partitions in (1), and also with the pairings in (2), and also with the paths on $\mathbb{N}$ in (3). Enjoy. $\square$

Getting back now to our philosophical considerations, regarding the creation of sets and mathematics, starting with $\emptyset$, what we have in Theorem 1.13 is quite exciting, suggesting a rivalry between noncrossing partitions and pairings. So, let us formulate:

QUESTION 1.14. *What is the correct object among:*

(1) *The set $NC(k)$ of noncrossing partitions of $\{1,\ldots,k\}$.*
(2) *The set $NC_2(2k)$ of noncrossing pairings of $\{1,\ldots,2k\}$.*

Here the term "correct" should be taken in the sense of Weyl 1.12, meaning potentially more conceptual, potentially more useful, and in a word, since we cannot rely on mathematics that we don't have yet, simply meaning more beautiful.

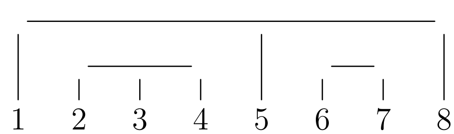
In order to deal with this question, let us first understand the bijection between our sets, which was something left open in the proof of Theorem 1.13. We have here:

THEOREM 1.15. *We have a bijection as follows,*

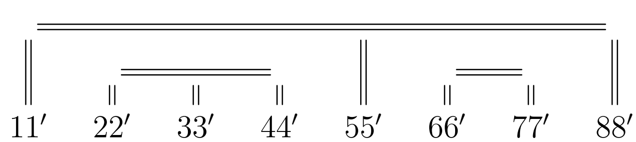
$$NC(k)\simeq NC_2(2k)$$

obtained by fattening the partitions, and by shrinking the pairings.

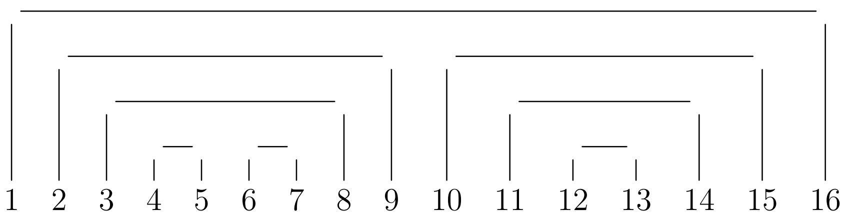
PROOF. This is something self-explanatory, and in order to see how this works, let us discuss an example. Consider a noncrossing partition, say the one in Definition 1.8:



Now let us "fatten" this partition, by doubling everything, as follows:



We can see emerging here a noncrossing pairing, and by relabeling the points $1,\ldots,16$, and properly redrawing the picture, what we have is indeed a noncrossing pairing:



As for the reverse operation, that is obviously obtained by "shrinking" our pairing, by collapsing pairs of consecutive neighbors, that is, by identifying $1=2$, then $3=4$, then $5=6$, and so on. Thus, we are led to the conclusion in the statement. □

With this done, let us get back to Question 1.14, which remains to be answered. Not an easy choice, but remembering from Conclusion 1.6 and Thought 1.7 that we hate crossings, which after all appear when drawing any partition $\pi\in NC(k)-NC_{12}(k)$, with 12 standing here for "singletons and pairings", we have a naive answer, as follows:

ANSWER 1.16. *Pairings are better than partitions, because they are easier to draw, therefore suggesting that they contain less complex information.*

However, all this remains subjective, and since switching from partitions to pairings can amount in an earthquake, hitting all the mathematics that we did so far in this book, let us doublecheck our answer, by some alternative means. And here, thinking a bit, the best is to go to the usual, crossing partitions. And good news, we have here:

THEOREM 1.17. *The number of pairings of $\{1,\ldots,k\}$ is zero when k is odd, and is*

$$|P_2(k)|=k!!$$

when k is even, with $k!!=(k-1)(k-3)\ldots$ Also, the moments of the normal law are

$$g_1=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx\quad:\quad M_k(g_1)=|P_2(k)|$$

and more generally, the moments of the normal law of parameter $t>0$ are

$$g_t=\frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}dx\quad:\quad M_k(g_t)=\sum_{\pi\in P_2(k)}t^{|\pi|}$$

with $|.|$ standing as usual for the number of blocks.

PROOF. There are several things going on here, the idea being as follows:

(1) First, in what regards the count, assuming that k is even, in order to construct a pairing of $\{1,\ldots,k\}$ we must choose a partner for 1, and use a pairing of the $k-2$ elements left. Thus, we are led by recurrence to the formula in the statement, namely:

$$|P_2(k)|=(k-1)(k-3)(k-5)\ldots$$

(2) Regarding the moments of the standard normal law g_1, the odd ones vanish because the density is even, and the even ones can be computed as follows:

$$\begin{aligned}
M_k &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x^k e^{-x^2/2}dx\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} (x^{k-1})\left(-e^{-x^2/2}\right)'dx\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} (k-1)x^{k-2}e^{-x^2/2}dx\\
&= (k-1)\times\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x^{k-2}e^{-x^2/2}dx\\
&= (k-1)M_{k-2}
\end{aligned}$$

Thus by recurrence, we are led to the formula in the statement.

(3) Finally, regarding the moments of the normal law g_t with $t>0$, we can get them either from (2) via a change of variable, or by redoing the computation, which gives:

$$M_k = t(k-1)M_{k-2}$$

Thus, we are led to the following formula for these moments:

$$M_k = t^{k/2}|P_2(k)|$$

But this can be reformulated more fancily as in the statement, as to make the link with what we have in Theorem 1.5, the point being that the number of blocks of any $\pi\in P_2(k)$ is of course $|\pi| = k/2$. Thus, we are led to the conclusions in the statement. □

All the above is quite exciting, and time now to face the truth. We got it all wrong with our partitions, be them crossing or noncrossing, the good objects are obviously the pairings, and more specifically, the noncrossing pairings. Let us record this, as follows:

Conclusion 1.18. *The correct hierarchy of the various sets of partitions is*

$$\begin{array}{ccccc}
 & & NC_2(k) & & \\
 & \swarrow & & \searrow & \\
NC(k) & & & & P_2(k)\\
 & \searrow & & \swarrow & \\
 & & P(k) & &
\end{array}$$

with $NC_2(k)$ being the king, for a multitude of reasons, explained above.

To be more precise here, the "multitude of reasons" evoked above include the primary school drawing of our partitions, the mathematical count of these partitions, and also the probabilistic aspects of these partitions, with in each case $NC_2(k)$, sometimes helped by its close subordinates, namely $NC(k)$ and $P_2(k)$, clearly beating $P(k)$.

What to do now, in view of all this? As always when it comes to discovering new things, blowing up previous mathematics that you did, with sweat and tears, relax and enjoy. There are actually 3 things to be done, in relation with all this, namely:

(1) Rewrite what we know about Catalan numbers, with $NC_2(k)$ coming first.

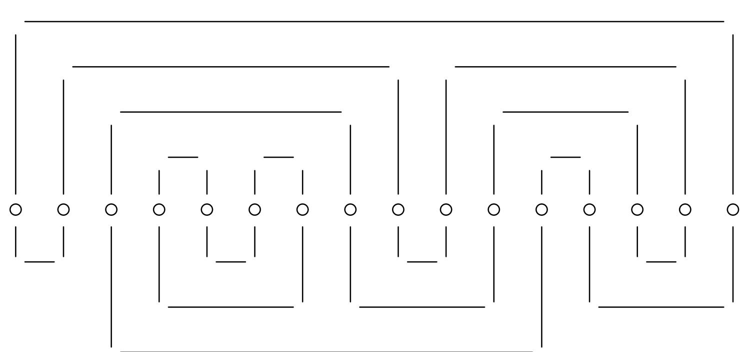
(2) Explore further algebraic properties of $NC_2(k)$, by playing with pairings.

(3) Have done as well the probabillistic aspects of $NC_2(k)$, and of $NC(k)$ too.

In what follows we will leave (1) as a thought exercise, with this being just a matter of meditating a bit at what we did in this chapter, and how this reorganizes with $NC_2(k)$ coming first. Regarding (2), we will certainly jump on this, and develop this next. As for (3), no hurry here, and we will leave this for the end of this chapter. With the good news, coming in advance, that we will reach in this way to the central laws in random matrix theory, namely those of Wigner and Marchenko-Pastur, and with this providing us with some solid evidence that we are on our way in doing some good physics, with all this.

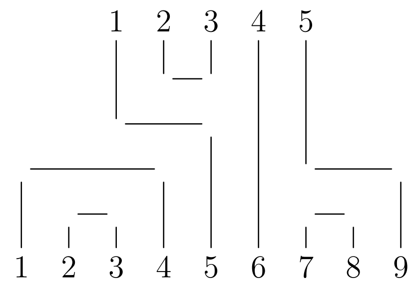
1c. Temperley-Lieb

In order to further advance, the idea is to use the obvious algebraic operation on the pairings in $NC_2(k)$, obtained by superposing such pairings. This leads to some interesting diagrams, known as "meanders", and here is an illustrating example:



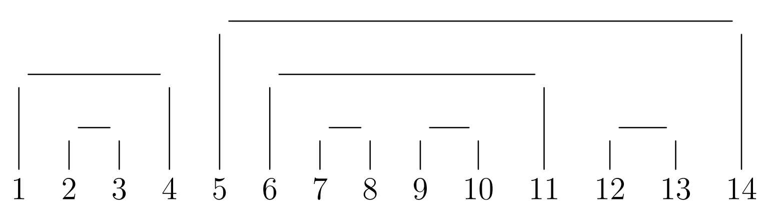
However, we can in fact do better than this. Remember category theory, telling us that for conceptual mathematics, we need objects, and arrows between them? We can do this in our context, by formulating first the following definition:

Definition 1.19. *We denote by $NC_2(k,l)$ the set of noncrossing pairings between an upper row of k points, and a lower row of l points, with for instance*



being an element of $NC_2(5,9)$. With the remark that at $k=0$ we obtain the former $NC_2(l)$, and that at $l=0$ we obtain the former $NC_2(k)$, written upside down.

Observe that we have $NC_2(k,l)=\emptyset$ when $k+l$ is odd. As another key remark, the above definition brings in fact nothing new, combinatorially speaking, because we can always rotate the upper legs, say via $\curvearrowright$, as to reach to a diagram in $NC_2(k+l)$. As an illustration, the rotated version of the pairing in Definition 1.19 looks as follows:



Thus, no need for new counting results of anything, we are ready to go with more algebra. Now with the above definition in hand, we can formulate:

Definition 1.20. *The Temperley-Lieb category TL_N° has the positive integers $\mathbb{N}$ as objects, with the space of arrows $k\to l$ being the formal span*

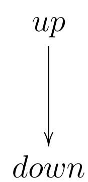
$$TL_N^\circ(k,l)=span(NC_2(k,l))$$

and with the composition of arrows appearing by composing the pairings, in the obvious way, with the rule $\bigcirc=N$, for the closed loops that might appear.

This definition is something quite subtle, hiding several non-trivial things, and is worth a detailed discussion, our comments about it being as follows:

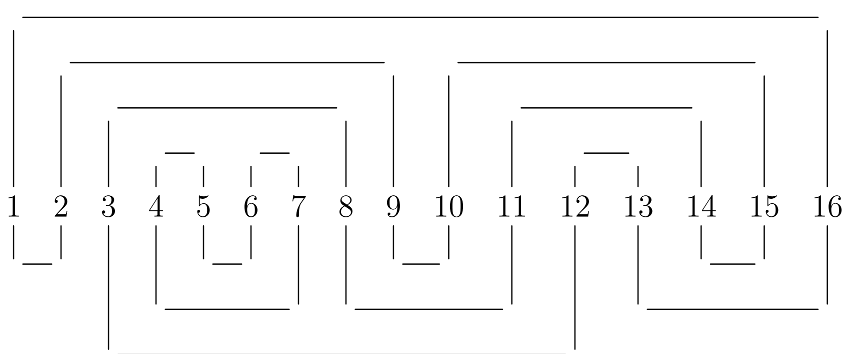
(1) First of all, our scalars in this book will be complex numbers, $\lambda\in\mathbb{C}$, and the "formal span" in the above must be understood in this sense, namely abstract complex vector space spanned by the elements of $NC_2(k,l)$. Of course it is possible to use an arbitrary field, at least at this stage of things, but remember that we are interested in quantum mechanics, and related mathematics, where the field of scalars is $\mathbb{C}$.

(2) Regarding the composition of arrows, this is by obvious vertical concatenation, with the convention, for here and for the rest of this book, that things go “from up to down”. And with this convention coming from pure laziness, why pushing things from left to right, when we can have gravity work for us, pulling them from up to down:



(3) Less poetically, this “from up to down” convention is also useful for purely mathematical purposes, because the left-right direction will be reserved for the intervention of sums Σ and scalars $\lambda \in \mathbb{C}$, while the up-down direction will be reserved for “action”. But of course, you might argue that this is a bit poetical, too. To which I will answer, give up with your cool and poetry, and your math will soon become some total garbage.

(4) More seriously now, let us discuss what happens with the closed circles, when concatenating. As an example here, let us consider the meander pictured before:



According to our conventions, this meander appears as the product $\pi\sigma \in NC_2(0,0)$ between the upper pairing $\sigma \in NC_2(0,16)$ and the lower pairing $\pi \in NC_2(16,0)$. But, what is the value of this product? We have two loops appearing, namely:

$$1 - 2 - 9 - 10 - 15 - 14 - 11 - 8 - 3 - 12 - 13 - 16$$

$$4 - 5 - 6 - 7$$

Thus, according to Definition 1.20, the value of this meander is N^2, with one N for each of the above loops, and with these two values of N multiplying each other.

(5) The same discussion applies to an arbitrary composition $\pi\sigma \in NC_2(k,m)$ between an upper pairing $\sigma \in NC_2(k,l)$ and a lower pairing $\pi \in NC_2(l,m)$, with a certain number of loops appearing in this way, each contributing with a multiplicative factor N.

(6) Finally, in Definition 1.20 the value of the circle $N = \bigcirc$ can be pretty much anything, but due to some positivity reasons to become clear later, we will assume in what follows $N \in [1,\infty)$. Also, we will call this parameter N the "index", with the precise reasons for calling this index to become clear later, too, as this books develops.

With all this discussed, what is next? More category theory I guess, and matter of having a theorem formulated too, instead of definitions only, let us formulate:

THEOREM 1.21. *The Temperley-Lieb category TL_N° is a tensor $*$-category, with:*

(1) *Composition of arrows: by vertical concatenation.*
(2) *Tensoring of arrows: by horizontal concatenation.*
(3) *Star operation: by turning the arrows upside-down.*

PROOF. This is more of a definition, disguised as a theorem. To be more precise, we already know about (1), from Definition 1.20, and we can talk as well about (2) and (3), constructed as above, with (2) using of course multiplicativity with respect to the scalars, and with (3) using antimultiplicativity with respect to the scalars:

$$\left(\sum_i \lambda_i\pi_i\right)\otimes\left(\sum_j \mu_j\sigma_j\right)=\sum_{ij}\lambda_i\mu_j\pi_i\otimes\sigma_j$$

$$\left(\sum_i \lambda_i\pi_i\right)^*=\sum_i\bar{\lambda}_i\pi_i^*$$

And the point now is that our three operations are compatible with each other via all sorts of compatibility formulae, which are all clear from definitions, with the conclusion being that what we have a tensor $*$-category, as stated. We will leave the details here, basically amounting in figuring out what a tensor $*$-category exactly is, as an exercise. □

In order to further understand the category TL_N°, let us focus on its diagonal part, formed by the End spaces of various objects. With the convention that these End spaces embed into each other by adding bars at right, this is a graded algebra, as follows:

$$TL_N=\bigcup_{k\geq 0}TL_N^\circ(k,k)$$

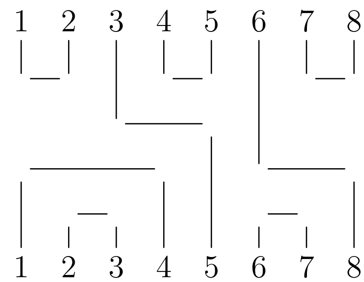
Moreover, for further fine-tuning our study, let us actually focus on the individual components of this graded algebra. These components will play a key role in what follows, and they are worth a dedicated definition, and new notation and name, as follows:

DEFINITION 1.22. *The Temperley-Lieb algebra $TL_N(k)$ is the formal span*

$$TL_N(k)=span(NC_2(k,k))$$

with multiplication coming by concatenating, with the rule $\bigcirc = N$.

In other words, $TL_N(k)$ appears as the formal span of the noncrossing pairings between an upper row of k points, and a lower row of k points, with multiplication coming by concatenating, with $\bigcirc = N$. As an example, here is a basis element of $TL_N(8)$:



Getting back now to what we know about TL_N°, from Theorem 1.21, the tensor product operation makes sense in the context of the diagonal algebra TL_N, but does not apply to its individual components $TL_N(k)$. However, the involution is useful, and we have:

THEOREM 1.23. *The Temperley-Lieb algebra $TL_N(k)$ is a $*$-algebra, with involution coming by turning the diagrams upside-down.*

PROOF. This is something trivial, which follows from Theorem 1.21, and can be verified as well directly, and we will leave this as an instructive exercise. □

And good news, we have here all the needed definitions, in our bag. Looking now a bit retrospectively, first came the algebra $TL_N(k)$, discovered by Temperley and Lieb in [81], in the context of certain questions from statistical mechanics. Then comes the inductive limit TL_N of these algebras, that we will see in chapter 2 below to be the simplest example of a planar algebra in the sense of Jones [49]. And finally, the full Temperley-Lieb category TL_N° is the full categorical object, behind all this.

1d. Analytic aspects

There are many things that we can do, as a continuation of the above. First, we can further study the Temperley-Lieb algebra $TL_N(k)$, for instance with a multimatrix decomposition for it, and also with a study of its natural trace $tr : TL_N(k) \to \mathbb{C}$, obtained by "closing" the diagrams in the obvious way. We will leave all this for chapter 2.

Also, we can do many algebraic and topological things with $TL_N(k)$, such as working out a number of selected Brauer theorems, for groups or quantum groups, or constructing some selected knot invariants. Again, we will leave all this for later in this book.

For the end of this opening chapter, however, let us do something analytic, that was left open in the above. We would like to solve the following question:

QUESTION 1.24. *We know that $P_2(k)$ corresponds to the normal laws g_t, and that $P(k)$ corresponds to the Poisson laws p_t. What about $NC_2(k)$ and $NC(k)$?*

Observe that this is related indeed to the Temperley-Lieb algebra TL_N, because we can define the Poincaré series of this graded algebra as follows:

$$\begin{aligned} f(z) &= \sum_{k\geq 0}\dim(TL_N(k))z^k \\ &= \sum_{k\geq 0}|NC_2(k,k)|z^k \\ &= \sum_{k\geq 0}|NC_2(2k)|z^k \\ &= \sum_{k\geq 0}|NC(k)|z^k \end{aligned}$$

Thus, we can reformulate Question 1.24 in a more fancy way, as follows:

QUESTION 1.25. *What are the measures π_1,γ_1 having the Poincaré series*

$$f(z)=\sum_{k\geq 0}\dim(TL_N(k))z^k$$

and its version $g(z)=f(z^2)$ as Stieltjes transforms? What about π_t,γ_t, with $t>0$?

Here we are assuming a bit of familiarity with advanced algebra and probability, but clarifying this fancy blurb being not a pressing issue, we can always do this later in this book, no worries for that, let us get back now to work, and do some computations. We have the following result, in the spirit of Theorem 1.5 and Theorem 1.17:

THEOREM 1.26. *The moments of the Wigner semicircle law are*

$$\gamma_1=\frac{1}{2\pi}\sqrt{4-x^2}dx\quad:\quad M_k(\gamma_1)=|NC_2(k)|$$

and the moments of the Marchenko-Pastur law are

$$\pi_1=\frac{1}{2\pi}\sqrt{4x^{-1}-1}\,dx\quad:\quad M_k(\pi_1)=|NC(k)|$$

and in addition, we have suitable $t>0$ analogues of both these results.

PROOF. This follows as usual, via calculus, the idea being as follows:

(1) Regarding the two moment formulae in the statement, these both follow by doing some standard calculus, which shows that the moments in question satisfy the needed recurrence formulae, and we will leave the proofs here as an instructive exercise.

(2) Alternatively, and answering a question that you surely have in mind, you can also come upon the measures in the statement via the Stieltjes inversion formula, which states

that the density of a real probability measure μ can be recaptured from its sequence of moments $\{M_k\}_{k\geq 0}$ by setting $G(\xi) = \xi^{-1} + M_1\xi^{-2} + M_2\xi^{-3} + \ldots$, and then:

$$d\mu(x) = \lim_{t\searrow 0} -\frac{1}{\pi}\, Im\,(G(x+it)) \cdot dx$$

(3) So, exercise for you to work out all this, Stieltjes inversion at $t = 1$, as to reach to γ_1, π_1, and then at general $t > 0$ too, with the desired moment formula for γ_t, π_t being the usual one, namely $M_k = \sum_{\pi\in D(k)} t^{|\pi|}$, with $D = NC_2, NC$ respectively. □

All the above is quite interesting, and will regularly reappear, throughout this book. Let us mention here, coming a bit in advance, that the Marchenko-Pastur and Wigner laws π_t, γ_t, originally found in [**57**], [**98**], are the main laws in random matrix theory, that they are the free analogues of the Poisson and Gaussian laws p_t, g_t in the sense of Voiculescu's free probability theory [**86**], and more specifically, of the Bercovici-Pata bijection [**9**], and finally that there is a lot of quantum algebra as well, involving the groups S_N, O_N and their free analogues S_N^+, O_N^+, behind all this. We will be back to this.

1e. Exercises

We had a lot of exciting combinatorics in this chapter, sometimes spilling into abstract algebra, or probability, or even philosophy, and as exercises on all this, we have:

EXERCISE 1.27. *Find other interesting formulae for the Bell numbers B_k.*

EXERCISE 1.28. *Find other interpretations of the Catalan numbers C_k.*

EXERCISE 1.29. *Try counting loops, on graphs more complicated than $\mathbb{N}$.*

EXERCISE 1.30. *Find bijections between our various objects counted by C_k.*

EXERCISE 1.31. *Learn about CLT and PLT, and about Stieltjes inversion too.*

EXERCISE 1.32. *Rewrite all our combinatorics, with $NC_2(k)$ coming first.*

EXERCISE 1.33. *Rewrite all our category theory, with $TL_N(k)$ coming first.*

EXERCISE 1.34. *Have a look at the trace $tr : TL_N(k) \to \mathbb{C}$, and its positivity.*

As bonus exercise, learn more probability. We already met some, and that is just the tip of the iceberg. But also because of quantum and statistical mechanics.

CHAPTER 2

Planar algebras

2a. Planar algebras

Following Jones [49], we have the following definition:

Definition 2.1. *The planar algebras are defined as follows:*

(1) *We consider rectangles in the plane, with the sides parallel to the coordinate axes, and taken up to planar isotopy, and we call such rectangles boxes.*
(2) *A labelled box is a box with $2n$ marked points on its boundary, n on its upper side, and n on its lower side, for some integer $n \in \mathbb{N}$.*
(3) *A tangle is labelled box, containing a number of labelled boxes, with all marked points, on the big and small boxes, being connected by noncrossing strings.*
(4) *A planar algebra is a sequence of finite dimensional vector spaces $P = (P_n)$, together with linear maps $P_{n_1} \otimes \ldots \otimes P_{n_k} \to P_n$, one for each tangle, such that the gluing of tangles corresponds to the composition of linear maps.*

In this definition we are using rectangles, but everything being up to isotopy, we could have used instead circles with marked points, as in [49]. Our choice for using rectangles comes from the main examples that we have in mind, to be discussed below, where the planar algebra structure is best viewed by using rectangles, as above.

Let us also mention that Definition 2.1 is something quite simplified. As explained in [49], in order for subfactors to produce planar algebras and vice versa, there are quite a number of supplementary axioms that must be added. More on this later.

As a basic example of a planar algebra, we have the Temperley-Lieb algebra:

Theorem 2.2. *The Temperley-Lieb algebra TL_N, viewed as graded algebra*

$$TL_N = (TL_N(n))_{n \in \mathbb{N}}$$

is a planar algebra, with the corresponding linear maps associated to the planar tangles

$$TL_N(n_1) \otimes \ldots \otimes TL_N(n_k) \to TL_N(n)$$

appearing by putting the various $TL_N(n_i)$ diagrams into the small boxes of the given tangle, which produces a $TL_N(n)$ diagram.

PROOF. This is something trivial, which follows from definitions:

(1) Assume indeed that we are given a planar tangle π, as in Definition 2.1, consisting of a box having $2n$ marked points on its boundary, and containing k small boxes, having respectively $2n_1,\ldots,2n_k$ marked points on their boundaries, and then a total of $n+\Sigma n_i$ noncrossing strings, connecting the various $2n+\Sigma 2n_i$ marked points.

(2) We want to associate to this tangle π a linear map as follows:

$$T_\pi : TL_N(n_1)\otimes\ldots\otimes TL_N(n_k)\to TL_N(n)$$

For this purpose, by linearity, it is enough to construct elements as follows, for any choice of Temperley-Lieb diagrams $\sigma_i\in TL_N(n_i)$, with $i=1,\ldots,k$:

$$T_\pi(\sigma_1\otimes\ldots\otimes\sigma_k)\in TL_N(n)$$

(3) But constructing such an element is obvious, just by putting the various diagrams $\sigma_i\in TL_N(n_i)$ into the small boxes the given tangle π. Indeed, this procedure produces a certain diagram in $TL_N(n)$, that we can call $T_\pi(\sigma_1\otimes\ldots\otimes\sigma_k)$, as above.

(4) Finally, we have to check that everything is well-defined up to planar isotopy, and that the gluing of tangles corresponds to the composition of linear maps. But both these checks are trivial, coming from the definition of TL_N, and we are done. □

As a conclusion to all this, $P=TL_N$ is indeed a planar algebra, but of somewhat "trivial" type, with the triviality coming from the fact that, in this case, the elements of P are planar diagrams themselves, and so the planar structure appears trivially.

The Temperley-Lieb planar algebra TL_N is however an important planar algebra, because it is the "smallest" one, appearing inside the planar algebra of any subfactor. But more on this later, when talking about planar algebras and subfactors.

2b. Fuss-Catalan

Moving ahead now, here is our second basic example of a planar algebra, which is also "trivial" in the above sense, with the elements of the planar algebra being planar diagrams themselves, but which appears in a bit more complicated way:

THEOREM 2.3. *The Fuss-Catalan algebra $FC_{N,M}$, obtained by coloring the Temperley-Lieb diagrams with black and white colors, clockwise, as follows*

$$\circ\bullet\bullet\circ\circ\bullet\bullet\circ\ldots\ldots\ldots\circ\bullet\bullet\circ$$

and keeping those diagrams whose strings connect either $\circ-\circ$ or $\bullet-\bullet$, is a planar algebra, with again the corresponding linear maps associated to the planar tangles

$$FC_{N,M}(n_1)\otimes\ldots\otimes FC_{N,M}(n_k)\to FC_{N,M}(n)$$

appearing by putting the various $FC_{N,M}(n_i)$ diagrams into the small boxes of the given tangle, which produces a $FC_{N,M}(n)$ diagram.

PROOF. The proof here is nearly identical to the proof of Theorem 2.2, with the only change appearing at the level of the colors. To be more precise:

(1) Forgetting about upper and lower sequences of points, which must be joined by strings, a Temperley-Lieb diagram can be thought of as being a collection of strings, say black strings, which compose in the obvious way, with the rule that the value of the circle, which is now a black circle, is N. And it is this obvious composition rule that gives the planar algebra structure, as explained in the proof of Theorem 2.2.

(2) Similarly, forgetting about points, a Fuss-Catalan diagram can be thought of as being a collection of strings, which come now in two colors, black and white. These Fuss-Catalan diagrams compose then in the obvious way, with the rule that the value of the black circle is N, and the value of the white circle is M. And it is this obvious composition rule that gives the planar algebra structure, as before for TL_N. □

Getting back now to generalities, and to Definition 2.1, that of a general planar algebra, we have so far two illustrations for it, which, while both important, are both "trivial", with the planar structure simply coming from the fact that, in both these cases, the elements of the planar algebra are planar diagrams themselves.

In general, the planar algebras can be more complicated than this, and we will see some further examples in a moment. However, the idea is very simple, namely "the elements of a planar algebra are not necessarily diagrams, but they behave like diagrams".

2c. Tensor and spin

In relation with groups, all this machinery is interesting for us. We will need the construction of the tensor and spin planar algebras $\mathcal{T}_N, \mathcal{S}_N$. Let us start with:

DEFINITION 2.4. *The tensor planar algebra $\mathcal{T}_N$ is the sequence of vector spaces*

$$P_k = M_N(\mathbb{C})^{\otimes k}$$

with the multilinear maps $T_\pi : P_{k_1} \otimes \ldots \otimes P_{k_r} \to P_k$ being given by the formula

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_r}) = \sum_j \delta_\pi(i_1, \ldots, i_r : j) e_j$$

with the Kronecker symbols δ_π being 1 if the indices fit, and being 0 otherwise.

In other words, we put the indices of the basic tensors on the marked points of the small boxes, in the obvious way, and the coefficients of the output tensor are then given by Kronecker symbols, exactly as in the easy group case.

The fact that we have indeed a planar algebra, in the sense that the gluing of tangles corresponds to the composition of linear maps, as required by Definition 2.1, is something elementary, in the same spirit as the verification of the functoriality properties of the correspondence $\pi \to T_\pi$, from easiness, and we refer here to Jones [49].

Let us discuss now a second planar algebra of the same type, which is important as well for various reasons, namely the spin planar algebra $\mathcal{S}_N$. This planar algebra appears somehow as the "square root" of the tensor planar algebra $\mathcal{T}_N$. Let us start with:

Definition 2.5. *We write the standard basis of $(\mathbb{C}^N)^{\otimes k}$ in $2 \times k$ matrix form,*

$$e_{i_1\ldots i_k} = \begin{pmatrix} i_1 & i_1 & i_2 & i_2 & i_3 & \ldots & \ldots \\ i_k & i_k & i_{k-1} & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

by duplicating the indices, and then writing them clockwise, starting from top left.

Now with this convention in hand for the tensors, we can formulate the construction of the spin planar algebra $\mathcal{S}_N$, also from [49], as follows:

Definition 2.6. *The spin planar algebra $\mathcal{S}_N$ is the sequence of vector spaces*

$$P_k = (\mathbb{C}^N)^{\otimes k}$$

written as above, with the multiplinear maps $T_\pi : P_{k_1} \otimes \ldots \otimes P_{k_r} \to P_k$ being given by

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_r}) = \sum_j \delta_\pi(i_1, \ldots, i_r : j) e_j$$

with the Kronecker symbols δ_π being 1 if the indices fit, and being 0 otherwise.

Here are some illustrating examples for the spin planar algebra calculus:

(1) The identity 1_k is the (k,k)-tangle having vertical strings only. The solutions of $\delta_{1_k}(x,y) = 1$ being the pairs of the form (x,x), this tangle 1_k acts by the identity:

$$1_k \begin{pmatrix} j_1 & \ldots & j_k \\ i_1 & \ldots & i_k \end{pmatrix} = \begin{pmatrix} j_1 & \ldots & j_k \\ i_1 & \ldots & i_k \end{pmatrix}$$

(2) The multiplication M_k is the (k,k,k)-tangle having 2 input boxes, one on top of the other, and vertical strings only. It acts in the following way:

$$M_k \left(\begin{pmatrix} j_1 & \ldots & j_k \\ i_1 & \ldots & i_k \end{pmatrix} \otimes \begin{pmatrix} l_1 & \ldots & l_k \\ m_1 & \ldots & m_k \end{pmatrix} \right) = \delta_{j_1 m_1} \ldots \delta_{j_k m_k} \begin{pmatrix} l_1 & \ldots & l_k \\ i_1 & \ldots & i_k \end{pmatrix}$$

(3) The inclusion I_k is the $(k, k+1)$-tangle which looks like 1_k, but has one more vertical string, at right of the input box. Given x, the solutions of $\delta_{I_k}(x,y) = 1$ are the elements y obtained from x by adding to the right a vector of the form $\binom{l}{l}$, and so:

$$I_k \begin{pmatrix} j_1 & \ldots & j_k \\ i_1 & \ldots & i_k \end{pmatrix} = \sum_l \begin{pmatrix} j_1 & \ldots & j_k & l \\ i_1 & \ldots & i_k & l \end{pmatrix}$$

(4) The expectation U_k is the $(k+1,k)$-tangle which looks like 1_k, but has one more string, connecting the extra 2 input points, both at right of the input box:

$$U_k\begin{pmatrix}j_1&\ldots&j_k&j_{k+1}\\ i_1&\ldots&i_k&i_{k+1}\end{pmatrix}=\delta_{i_{k+1}j_{k+1}}\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}$$

(5) The Jones projection E_k is a $(0,k+2)$-tangle, having no input box. There are k vertical strings joining the first k upper points to the first k lower points, counting from left to right. The remaining upper 2 points are connected by a semicircle, and the remaining lower 2 points are also connected by a semicircle. We have:

$$E_k(1)=\sum_{ijl}\begin{pmatrix}i_1&\ldots&i_k&j&j\\ i_1&\ldots&i_k&l&l\end{pmatrix}$$

The elements $e_k=N^{-1}E_k(1)$ are then projections, and define a representation of the infinite Temperley-Lieb algebra of index N inside the inductive limit algebra $\mathcal{S}_N$.

(6) The rotation R_k is the (k,k)-tangle which looks like 1_k, but the first 2 input points are connected to the last 2 output points, and the same happens at right:

$$R_k=\begin{matrix}\Cap&|&|&|&\|\\ \|&&&&\|\\ \|&|&|&|&\Cup\end{matrix}$$

The action of R_k on the standard basis is by rotation of the indices, as follows:

$$R_k(e_{i_1i_2\ldots i_k})=e_{i_2\ldots i_ki_1}$$

There are many other interesting examples of k-tangles, but in view of our present purposes, we can actually stop here, due to the following fact:

THEOREM 2.7. *The multiplications, inclusions, expectations, Jones projections and rotations generate the set of all tangles, via the gluing operation.*

PROOF. This is something well-known and elementary, obtained by "chopping" the various planar tangles into small pieces, as in the above list. See [49]. □

Finally, in order for our discussion to be complete, we must talk as well about the $*$-structure of the spin planar algebra. Once again this is constructed as in the easy quantum group calculus, by turning upside-down the diagrams, as follows:

$$\begin{pmatrix}j_1&\ldots&j_k\\ i_1&\ldots&i_k\end{pmatrix}^*=\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_k\end{pmatrix}$$

As before, we refer to Jones' paper [49] for more on all this.

In relation with groups, following [49], we have the following result:

Theorem 2.8. *Given $G \subset S_N$, consider the tensor powers of the associated coaction map on $C(X)$, where $X = \{1, \ldots, N\}$, which are the folowing linear maps:*

$$\Phi^k : C(X^k) \to C(X^k) \otimes C(G)$$

$$e_{i_1 \ldots i_k} \to \sum_{j_1 \ldots j_k} e_{j_1 \ldots j_k} \otimes u_{j_1 i_1} \ldots u_{j_k i_k}$$

The fixed point spaces of these coactions, which are by definition the spaces

$$P_k = \left\{ x \in C(X^k) \Big| \Phi^k(x) = 1 \otimes x \right\}$$

are given by $P_k = Fix(u^{\otimes k})$, and form a subalgebra of the spin planar algebra $\mathcal{S}_N$.

Proof. Since the map Φ is a coaction, its tensor powers Φ^k are coactions too, and at the level of fixed point algebras we have the following formula:

$$P_k = Fix(u^{\otimes k})$$

In order to prove now the planar algebra assertion, we will use Theorem 2.7. Consider the rotation R_k. Rotating, then applying Φ^k, and rotating backwards by R_k^{-1} is the same as applying Φ^k, then rotating each k-fold product of coefficients of Φ. Thus the elements obtained by rotating, then applying Φ^k, or by applying Φ^k, then rotating, differ by a sum of Dirac masses tensored with commutators in $A = C(G)$:

$$\Phi^k R_k(x) - (R_k \otimes id)\Phi^k(x) \in C(X^k) \otimes [A, A]$$

Now let $\int_A$ be the Haar functional of A, and consider the conditional expectation onto the fixed point algebra P_k, which is given by the following formula:

$$\phi_k = \left(id \otimes \int_A \right) \Phi^k$$

Since $\int_A$ is a trace, it vanishes on commutators. Thus R_k commutes with ϕ_k:

$$\phi_k R_k = R_k \phi_k$$

The commutation relation $\phi_k T = T\phi_l$ holds in fact for any (l, k)-tangle T. These tangles are called annular, and the proof is by verification on generators of the annular category. In particular we obtain, for any annular tangle T:

$$\phi_k T \phi_l = T \phi_l$$

We conclude from this that the annular category is contained in the suboperad $\mathcal{P}' \subset \mathcal{P}$ of the planar operad consisting of tangles T satisfying the following condition, where $\phi = (\phi_k)$, and where $i(.)$ is the number of input boxes:

$$\phi T \phi^{\otimes i(T)} = T \phi^{\otimes i(T)}$$

On the other hand the multiplicativity of Φ^k gives $M_k \in \mathcal{P}'$. Now since the planar operad $\mathcal{P}$ is generated by multiplications and annular tangles, it follows that we have

$\mathcal{P}' = P$. Thus for any tangle T the corresponding multilinear map between spaces $P_k(X)$ restricts to a multilinear map between spaces P_k. In other words, the action of the planar operad $\mathcal{P}$ restricts to P, and makes it a subalgebra of $\mathcal{S}_N$, as claimed. $\square$

2d. Bipartite graphs

Let us start with a technical definition, from [50], as follows:

Definition 2.9. *Associated to a bipartite graph Γ is the vector space P_k spanned by the $2k$-loops based at points of Γ_a. The basis elements of P_k will be denoted*

$$x = \begin{pmatrix} e_1 & e_2 & \ldots & e_k \\ e_{2k} & e_{2k-1} & \ldots & e_{k+1} \end{pmatrix}$$

where $e_1, e_2, \ldots, e_{2k}$ is the sequence of edges of the corresponding $2k$-loop.

Consider now the adjacency matrix of Γ, which is of the following type:

$$M = \begin{pmatrix} 0 & m \\ m^t & 0 \end{pmatrix}$$

We pick an M-eigenvalue $\gamma \neq 0$, and then a γ-eigenvector, as follows:

$$\eta : \Gamma_a \cup \Gamma_b \to \mathbb{C} - \{0\}$$

With this data in hand, we have the following construction, due to Jones [50]:

Definition 2.10. *Associated to any tangle is the multilinear map*

$$T(x_1 \otimes \ldots \otimes x_r) = \gamma^c \sum_x \delta(x_1, \ldots, x_r, x) \prod_m \mu(e_m)^{\pm 1} x$$

where the objects on the right are as follows:

1. *The sum is over the basis of P_k, and c is the number of circles of T.*
2. *$\delta = 1$ if all strings of T join pairs of identical edges, and $\delta = 0$ if not.*
3. *The product is over all local maxima and minima of the strings of T.*
4. *e_m is the edge of Γ labelling the string passing through m (when $\delta = 1$).*
5. *$\mu(e) = \sqrt{\eta(e_f)/\eta(e_i)}$, where e_i, e_f are the initial and final vertex of e.*
6. *The $\pm$ sign is $+$ for a local maximum, and $-$ for a local minimum.*

This looks quite similar to the calculus for the tensor and spin planar algebras. Let us work out the precise formula of the action, for 6 carefully chosen tangles:

(1) Let us look first at the identity 1_k. This tangle acts by the identity:

$$1_k \begin{pmatrix} f_1 & \ldots & f_k \\ e_1 & \ldots & e_k \end{pmatrix} = \begin{pmatrix} f_1 & \ldots & f_k \\ e_1 & \ldots & e_k \end{pmatrix}$$

(2) The multiplication tangle M_k acts as follows:

$$M_k\left(\begin{pmatrix}f_1&\ldots&f_k\\e_1&\ldots&e_k\end{pmatrix}\otimes\begin{pmatrix}h_1&\ldots&h_k\\g_1&\ldots&g_k\end{pmatrix}\right)=\delta_{f_1g_1}\ldots\delta_{f_kg_k}\begin{pmatrix}h_1&\ldots&h_k\\e_1&\ldots&e_k\end{pmatrix}$$

(3) Regarding now the inclusion I_k, the formula here is:

$$I_k\begin{pmatrix}f_1&\ldots&f_k\\e_1&\ldots&e_k\end{pmatrix}=\sum_g\begin{pmatrix}f_1&\ldots&f_k&g\\e_1&\ldots&e_k&g\end{pmatrix}$$

(4) The expectation tangle U_k acts with a spin factor, as follows:

$$U_k\begin{pmatrix}f_1&\ldots&f_k&h\\e_1&\ldots&e_k&g\end{pmatrix}=\delta_{gh}\mu(g)^2\begin{pmatrix}f_1&\ldots&f_k\\e_1&\ldots&e_k\end{pmatrix}$$

(5) For the Jones projection E_k, the formula is as follows:

$$E_k(1)=\sum_{egh}\mu(g)\mu(h)\begin{pmatrix}e_1&\ldots&e_k&h&h\\e_1&\ldots&e_k&g&g\end{pmatrix}$$

(6) As for the shift J_k, its action is given by:

$$J_k\begin{pmatrix}f_1&\ldots&f_k\\e_1&\ldots&e_k\end{pmatrix}=\sum_{gh}\begin{pmatrix}g&h&f_1&\ldots&f_k\\g&h&e_1&\ldots&e_k\end{pmatrix}$$

Summarizing, we have here formulae which are quite similar to those for the tensor and spin planar algebras. We have the following result, from Jones' paper [50]:

THEOREM 2.11. *The graded linear space $P=(P_k)$, together with the action of the planar tangles given above, is a planar algebra.*

PROOF. This is something which is quite routine, starting from the above study of the main tangles, which can be proved by using Theorem 2.7. Also, let us mention that all this generalizes the previous constructions of the spin and tensor planar algebras $\mathcal{S}_N,\mathcal{T}_N$, which appear rfrom the Bratteli diagrams of $\mathbb{C}\subset\mathbb{C}^N$ and $\mathbb{C}\subset M_N(\mathbb{C})$. □

Let us go now towards the Markov inclusions of algebras $A\subset B$. We have here the following result, regarding such inclusions, also from Jones' paper [50]:

THEOREM 2.12. *The planar algebra associated to the graph of $A\subset B$, with eigenvalue $\gamma=\sqrt{r}$ and eigenvector $\eta(i)=a_i/\sqrt{\dim A}$, $\eta(j)=b_j/\sqrt{\dim B}$, is as follows:*

(1) *The graded algebra structure is given by $P_{2k}=A'\cap A_k$, $P_{2k+1}=A'\cap B_k$.*
(2) *The elements e_k are the Jones projections for $A\subset B\subset A_1\subset B_1\subset\ldots$*
(3) *The expectation and shift are given by the above formulae.*

PROOF. Indeed, the Jones tower algebras A_k,B_k are simply the span of the $4k$-paths, respectively $4k+2$-paths on Γ, starting at points of Γ_a. With this description in hand, when taking commutants with A we have to just have to restrict attention from paths to loops, and we obtain the above spaces P_{2k},P_{2k+1}. For details here, see [50]. □

In the case of the inclusions satisfying $[A,B]=0$, called abelian, we have:

PROPOSITION 2.13. *The "bipartite graph" planar algebra $P(A\subset B)$ associated to an abelian inclusion $A\subset B$ can be described as follows:*

(1) *As a graded algebra, this is the Jones tower $A\subset B\subset A_1\subset B_1\subset\ldots$*
(2) *The Jones projections and expectations are the usual ones for this tower.*
(3) *The shifts correspond to the canonical identifications $A_1'\cap P_{k+2}=P_k$.*

PROOF. The first assertion is a reformulation of Theorem 2.12 in the abelian case. The assertion on Jones projections follows as well from Theorem 2.12, and the assertion on expectations follows from the fact that their composition is the usual trace. Regarding now the third assertion, we have $A_1'\cap A_{k+1}=A_k$ and $A_1'\cap B_{k+1}=B_k$, and by using the path model for these algebras, as in the proof of Theorem 2.12, we obtain the result. □

In order to formulate now our main result, regarding the subfactors associated to the compact groups G, we will need a few abstract notions. Let us start with:

DEFINITION 2.14. *Let P_1,P_2 be two finite dimensional algebras, coming with coactions $\alpha_i:P_i\to P_i\otimes L^\infty(G)$, and let $T:P_1\to P_2$ be a linear map.*

(1) *We say that T is G-equivariant if $(T\otimes id)\alpha_1=\alpha_2T$.*
(2) *We say that T is weakly G-equivariant if $T(P_1^G)\subset P_2^G$.*

Consider now a planar algebra $P=(P_k)$. The annular category over P is the collection of maps $T:P_k\to P_l$ coming from the "annular" tangles, having at most one input box. These maps form sets $Hom(k,l)$, and these sets form a category. We have:

DEFINITION 2.15. *A coaction of $L^\infty(G)$ on P is a graded algebra coaction*

$$\alpha:P\to P\otimes L^\infty(G)$$

such that the annular tangles are weakly G-equivariant.

This is something a bit technical, coming out of the known examples that we have. In fact, as we will show below, the examples are basically those coming from actions of quantum groups on Markov inclusions $A\subset B$, under the assumption $[A,B]=0$. For the moment, at the generality level of Definition 2.15, we have:

PROPOSITION 2.16. *If G acts on a planar algebra P, then P^G is a planar algebra.*

PROOF. The weak equivariance condition tells us that the annular category is contained in the suboperad $\mathcal{P}'\subset\mathcal{P}$ consisting of tangles which leave invariant P^G. On the other hand the multiplicativity of α gives $M_k\in\mathcal{P}'$, for any k. Now since $\mathcal{P}$ is generated by multiplications and annular tangles, we get $\mathcal{P}'=\mathcal{P}$, and we are done. □

We are now in position of stating and proving a main result, as follows:

THEOREM 2.17. *In the abelian case, we have a fixed point algebra*

$$P(A \subset B)^G \subset P(A \subset B)$$

for any group action on a Markov inclusion, $G \curvearrowright (A \subset B)$.

PROOF. This basically follows from what we have, as follows:

(1) Let $P = P(A \subset B)$, and $Q = P^G$. We have to prove that the planar algebra structure on Q agrees with the planar algebra structure of P, from Proposition 2.13.

(2) Since $\mathcal{P}$ is generated by the annular category $\mathcal{A}$ and by the multiplication tangles M_k, we just have to check that the annular tangles agree on P, Q. Moreover, since $\mathcal{A}$ is generated by I_k, E_k, U_k, J_k, we just have to check that these tangles agree on P, Q.

(3) We know that $Q \subset P$ is an inclusion of graded algebras, that all the Jones projections for P are contained in Q, and that the conditional expectations agree. Thus the tangles I_k, E_k, U_k agree on P, Q, and the only verification left is that for the shift J_k.

(4) Thus, it is enough to show that the image of the subfactor shift J_k' coincides with that of the planar shift J_k. But this follows as for the tensor and spin algebras. $\square$

2e. Exercises

Exercises:

EXERCISE 2.18.

EXERCISE 2.19.

EXERCISE 2.20.

EXERCISE 2.21.

EXERCISE 2.22.

EXERCISE 2.23.

EXERCISE 2.24.

EXERCISE 2.25.

Bonus exercise.

CHAPTER 3

General theory

3a. Basic algebra

Basic algebra.

3b. Trace, positivity

Trace, positivity.

3c. Meander determinants

Meander determinants.

3d. Structure, invariants

Structure, invariants.

3e. Exercises

Exercises:

Exercise 3.1.

Exercise 3.2.

Exercise 3.3.

Exercise 3.4.

Exercise 3.5.

Exercise 3.6.

Exercise 3.7.

Exercise 3.8.

Bonus exercise.

CHAPTER 4

Knot invariants

4a. Braid group

Braid group.

4b. Jones polynomial

Jones polynomial.

4c. Mechanical aspects

Mechanical aspects.

4d. Three dimensions

Three dimensions.

4e. Exercises

Exercises:

Exercise 4.1.

Exercise 4.2.

Exercise 4.3.

Exercise 4.4.

Exercise 4.5.

Exercise 4.6.

Exercise 4.7.

Exercise 4.8.

Bonus exercise.

Part II

Quantum algebra

Flashing for the warriors
Whose strength is not to fight
Flashing for the refugees
On the unarmed road of flight

CHAPTER 5

Subfactor theory

5a. Operator algebras

5b. Subfactor theory

5c. Index theorem

5d. Planar algebras

5e. Exercises

Exercises:

Exercise 5.1.

Exercise 5.2.

Exercise 5.3.

Exercise 5.4.

Exercise 5.5.

Exercise 5.6.

Exercise 5.7.

Exercise 5.8.

Bonus exercise.

CHAPTER 6

Quantum groups

6a. Quantum groups

6b. Actions, invariants

6c. Tannakian results

6d. Arithmetic versions

6e. Exercises

Exercises:

Exercise 6.1.

Exercise 6.2.

Exercise 6.3.

Exercise 6.4.

Exercise 6.5.

Exercise 6.6.

Exercise 6.7.

Exercise 6.8.

Bonus exercise.

CHAPTER 7

Heavy theorems

7a. Axiomatization

7b. Universal models

7c. Hyperfinite factors

7d. Commuting squares

7e. Exercises

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Exercise 7.2.

Exercise 7.3.

Exercise 7.4.

Exercise 7.5.

Exercise 7.6.

Exercise 7.7.

Exercise 7.8.

Bonus exercise.

CHAPTER 8

Small index

8a. Index 4

8b. Below 4

8c. Above 4

8d. Fun at 6

8e. Exercises

Exercises:

Exercise 8.1.

Exercise 8.2.

Exercise 8.3.

Exercise 8.4.

Exercise 8.5.

Exercise 8.6.

Exercise 8.7.

Exercise 8.8.

Bonus exercise.

Part III

Quantum fields

Hear the mighty engines roar
See the silver bird on high
She's away and westward bound
Far above the clouds she'll fly

CHAPTER 9

Quantum mechanics

9a. Maxwell and Einstein

9b. Heisenberg, Schrödinger

9c. Elementary particles

9d. Collisions, scattering

9e. Exercises

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Exercise 9.1.

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Exercise 9.3.

Exercise 9.4.

Exercise 9.5.

Exercise 9.6.

Exercise 9.7.

Exercise 9.8.

Bonus exercise.

CHAPTER 10

Feynman diagrams

10a.

10b.

10c.

10d.

10e. Exercises

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Exercise 10.3.

Exercise 10.4.

Exercise 10.5.

Exercise 10.6.

Exercise 10.7.

Exercise 10.8.

Bonus exercise.

CHAPTER 11

Quantum fields

11a.

11b.

11c.

11d.

11e. Exercises

Exercises:

EXERCISE 11.1.

EXERCISE 11.2.

EXERCISE 11.3.

EXERCISE 11.4.

EXERCISE 11.5.

EXERCISE 11.6.

EXERCISE 11.7.

EXERCISE 11.8.

Bonus exercise.

CHAPTER 12

String theory

12a.

12b.

12c.

12d.

12e. Exercises

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Exercise 12.3.

Exercise 12.4.

Exercise 12.5.

Exercise 12.6.

Exercise 12.7.

Exercise 12.8.

Bonus exercise.

Part IV

Analytic aspects

She smiles, walks the other way
As the last ship sails
And the moon fades away
From Black Diamond Bay

CHAPTER 13

Spectral measures

13a.

13b.

13c.

13d.

13e. Exercises

Exercises:

EXERCISE 13.1.

EXERCISE 13.2.

EXERCISE 13.3.

EXERCISE 13.4.

EXERCISE 13.5.

EXERCISE 13.6.

EXERCISE 13.7.

EXERCISE 13.8.

Bonus exercise.

CHAPTER 14

Integration theory

14a.

14b.

14c.

14d.

14e. Exercises

Exercises:

EXERCISE 14.1.

EXERCISE 14.2.

EXERCISE 14.3.

EXERCISE 14.4.

EXERCISE 14.5.

EXERCISE 14.6.

EXERCISE 14.7.

EXERCISE 14.8.

Bonus exercise.

CHAPTER 15

Fourier analysis

15a.

15b.

15c.

15d.

15e. Exercises

Exercises:

Exercise 15.1.

Exercise 15.2.

Exercise 15.3.

Exercise 15.4.

Exercise 15.5.

Exercise 15.6.

Exercise 15.7.

Exercise 15.8.

Bonus exercise.

CHAPTER 16

Large index

16a.

16b.

16c.

16d.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter.

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