Tannakian duality, diagrams and easiness

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"Introduction to quantum groups", 4/6

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Plan

(1) Tensor categories

(2) Tannakian duality

(3) Diagrams, easiness

(4) Free quantum groups

⇒ next lecture: quantum permutations
Representations

(1) A corepresentation of a Woronowicz algebra $A$ is a biunitary matrix $v \in M_n(A)$ satisfying:

- $\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$
- $\varepsilon(v_{ij}) = \delta_{ij}$
- $S(v_{ij}) = v_{ji}^*$

(2) Basic example: the fundamental corepresentation $u$. In fact, the axioms state that $u$ must be a faithful corepresentation.

(3) With $A = C(G)$, the corepresentations of $A$ correspond to the FD unitary smooth representations of the quantum group $G$.

(4) We have a full Peter-Weyl theory for them, the main result stating that $A$ decomposes as an orthogonal direct sum.
Definition. The Tannakian category of a Woronowicz algebra \((A, u)\) is the collection \(C = (C(k, l))\) of vector spaces

\[ C(k, l) = \text{Hom}(u^\otimes k, u^\otimes l) \]

where the corepresentations \(u^\otimes k\) with \(k = \circ \bullet \bullet \circ \ldots\) colored integer are defined by \(u^\otimes \circ = u, u^\otimes \bullet = \bar{u}\) and multiplicativity.

Remark 1. We already know that \(C\) is a tensor \(*\)-category, the verification of all conditions being elementary.

Remark 2. In fact, \(C\) appears by definition as subcategory of the tensor \(*\)-category \(E(k, l) = \mathcal{L}(H^\otimes k, H^\otimes l)\), where \(H = \mathbb{C}^N\).
Our purpose will be that of reconstructing \((A, u)\) in terms of \(C = (C(k, l))\). Here is a useful preliminary result:

**Theorem.** Given a morphism \(\pi : (A, u) \to (B, v)\) we have

\[
\text{Hom}(u^\otimes k, u^\otimes l) \subset \text{Hom}(v^\otimes k, v^\otimes l)
\]

and if these inclusions are all equalities, \(\pi\) is an isomorphism.

**Proof.** Follows from Peter-Weyl, by contradiction, because each irreducible corepresentation is contained in some \(u^\otimes k\).
In order to exploit the fact that \( u \) is biunitary, we can use:

**Theorem.** An matrix \( u \in M_N(A) \) is biunitary if and only if

\[
R \in \text{Hom}(1, u \otimes \bar{u}) \quad , \quad R \in \text{Hom}(1, \bar{u} \otimes u)
\]

\[
R^* \in \text{Hom}(u \otimes \bar{u}, 1) \quad , \quad R^* \in \text{Hom}(\bar{u} \otimes u, 1)
\]

where \( R : \mathbb{C} \to \mathbb{C}^N \otimes \mathbb{C}^N \) is given by \( R(1) = \sum_i e_i \otimes e_i \).

**Proof.** This follows from some elementary computations.
Definition. Let $H$ be a finite dimensional Hilbert space. A tensor category over $H$ is a collection $C = (C(k, l))$ of subspaces

$$C(k, l) \subset \mathcal{L}(H^\otimes k, H^\otimes l)$$

satisfying the following conditions:

1. $S, T \in C$ implies $S \otimes T \in C$.
2. If $S, T \in C$ are composable, then $ST \in C$.
3. $T \in C$ implies $T^* \in C$.
4. Each $C(k, k)$ contains the identity operator.
5. $C(\emptyset, \circ \bullet)$ and $C(\emptyset, \bullet \circ)$ contain the map $R : 1 \to \sum_i e_i \otimes e_i$. 
Theorem. Let \((A, u)\) be a Woronowicz algebra, with fundamental corepresentation \(u \in M_N(A)\). The associated Tannakian category

\[ C(k, l) = \text{Hom}(u \otimes^k, u \otimes^l) \]

is then a tensor category over the Hilbert space \(H = \mathbb{C}^N\).

Proof. We already know that axioms (1-4) hold indeed, this being elementary, and (5) is something that we just did, clear too.
Theorem. Given a tensor category $C = (C(k, l))$, the following algebra is a Woronowicz algebra:

$$A_C = \frac{C(U^+_N)}{\left\langle T \in \text{Hom}(u \otimes^k, u \otimes^l) \big| \forall k, l, \forall T \in C(k, l) \right\rangle}$$

In the case where $C$ comes from a Woronowicz algebra $(A, \nu)$, we have a quotient map $A_C \to A$.

Proof. We have indeed a Woronowicz algebra, because the relations $T \in \text{Hom}(u \otimes^k, u \otimes^l)$ are of "Hopf type", i.e. $\Delta, \varepsilon, S$ factorize.

The fact that we have a quotient map $A_C \to A$ is clear, because the relations defining $A_C$ are satisfied inside $A$. 
We have so far:

(1) **Axioms for** $A$: $N \times N$ Woronowicz algebra

(2) **Axioms for** $C$: tensor category over $\mathbb{C}^N$

(3) **Correspondence** $A \rightarrow C$: set $C_A = (\text{Hom}(u \otimes^k, u \otimes^l))_{kl}$

(4) **Correspondence** $C \rightarrow A$: set $A_C = C(U_N^+)/ < C \subset C_A >$

$\implies$ we want to prove that we have a bijection $A \leftrightarrow C$
Step 1

**Theorem.** Consider the following conditions:

1. \( C = C_{AC} \), for any Tannakian category \( C \).
2. \( A = A_{CA} \), for any Woronowicz algebra \((A, u)\).

We have then \((1) \implies (2)\). Also, \( C \subseteq C_{AC} \) is automatic.

**Proof.** We know that we have an arrow as follows:

\[
A_{CA} \rightarrow A
\]

On the other hand, assuming (1), with \( C = C_A \) we get:

\[
C_A = C_{AC_{CA}}
\]

Thus, we can use our quotient map criterion from before, and we get \( A_{CA} = A \), as desired. Finally, the last assertion is clear.
Definition. Given a tensor category $C$ over $H$, we set:

$$E_C = \bigoplus_{k,l} C(k, l) \subset \bigoplus_{k,l} B(H^\otimes k, H^\otimes l) \subset B \left( \bigoplus_k H^\otimes k \right)$$

Also, for any $s \in \mathbb{N}$, we consider the following truncation:

$$E^{(s)}_C = \bigoplus_{|k|,|l| \leq s} C(k, l) \subset \bigoplus_{|k|,|l| \leq s} B(H^\otimes k, H^\otimes l) = B \left( \bigoplus_{|k| \leq s} H^\otimes k \right)$$

Remark. We obtain in this way certain $\ast$-algebras.
Step 3

Theorem. For any $C^*$-algebra $B \subset M_n(\mathbb{C})$ we have

$$B = B''$$

where prime denotes the commutant, taken inside $M_n(\mathbb{C})$.

Proof. Let us decompose $B$ as a direct sum of matrix algebras:

$$B = M_{r_1}(\mathbb{C}) \oplus \ldots \oplus M_{r_k}(\mathbb{C})$$

The commutant of this algebra is then as follows:

$$B' = \mathbb{C} \oplus \ldots \oplus \mathbb{C}$$

By taking once again the commutant we obtain $B$ itself.

(This is a particular case of von Neumann’s bicommutant theorem)
Step 4

Theorem. Given a category \( C \), the following are equivalent:

1. \( C = C_{A_C} \).
2. \( E_C = E_{C_{A_C}} \).
3. \( E_C^{(s)} = E_{C_{A_C}}^{(s)} \), for any \( s \in \mathbb{N} \).
4. \( E_C^{(s)'} = E_{C_{A_C}}^{(s)'} \), for any \( s \in \mathbb{N} \).

In addition, \( \subset, \subset, \subset, \supset \) respectively are automatically satisfied.

Proof. Here (1) \( \iff \) (2) is clear from definitions, (2) \( \iff \) (3) is clear from definitions as well, and (3) \( \iff \) (4) comes from the bicommutant theorem. As for the last assertion, we have indeed \( C \subset C_{A_C} \), and the other inclusions follow from this.
Step 5

**Theorem.** Given a Woronowicz algebra \((A, u)\), we have

\[
E^{(s)}_{CA} = \text{End} \left( \bigoplus_{|k| \leq s} u^\otimes k \right)
\]

as subalgebras of \(B(\bigoplus_{|k| \leq s} H^\otimes k)\).

**Proof.** The algebra \(E^{(s)}_{CA}\) appears by definition as follows:

\[
E^{(s)}_{CA} = \bigoplus_{|k|, |l| \leq s} \text{Hom}(u^\otimes k, u^\otimes l) \subset B \left( \bigoplus_{|k| \leq s} H^\otimes k \right)
\]

But this is precisely the algebra of intertwiners of \(\bigoplus_{|k| \leq s} u^\otimes k\).
Step 6

**Theorem.** For any corepresentation \( \nu \in M_n(A) \), the map

\[
\pi_\nu : A^* \to M_n(\mathbb{C}) \ , \ \varphi \to (\varphi(\nu_{ij}))_{ij}
\]

is a representation, having as image \( \text{Im}(\pi_\nu) = \text{End}(\nu)' \).

**Proof.** The first assertion is clear, coming from:

\[
(\pi_\nu(\varphi \ast \psi))_{ij} = (\varphi \otimes \psi)\Delta(\nu_{ij}) = \sum_k \varphi(\nu_{ik})\psi(\nu_{kj}) = \sum_k (\pi_\nu(\varphi))_{ik}(\pi_\nu(\psi))_{kj} = (\pi_\nu(\varphi)\pi_\nu(\psi))_{ij}
\]

As for the second assertion, this comes by double inclusion.
We want to prove Tannakian duality, \( A \leftrightarrow C \). Passed a few trivialities, this amounts in proving that:

\[ C_{A_C} \subset C \]

By using the \( C \rightarrow E_C \) construction, truncated at \( s \in \mathbb{N} \), and then a bicommutant trick, this is the same as proving that:

\[ E_C^{(s)'} \subset E_{C_{A_C}}^{(s)'} \]

We know that for any \( A \) we have \( E_{C_A}^{(s)'} = \text{Im}(\pi_v) \), where

\[ v = \bigoplus_{|k| \leq s} u^\otimes k \]

and where \( \pi_v : A^* \rightarrow M_n(\mathbb{C}) \) is given by \( \varphi \rightarrow (\varphi(v_{ij}))_{ij} \).
In order to model $A_C$, and to fine-tune the results that we have, consider the following pair of dual vector spaces:

\[ F = \bigoplus_k B \left( H \otimes^k \right), \quad F^* = \bigoplus_k B \left( H \otimes^k \right)^* \]

Let $f_{ij}, f_{ij}^* \in F^*$ be the standard generators of $B(H)^*, B(\bar{H})^*$.

(1) $F^*$ is a $\ast$-algebra, with multiplication $\otimes$ and involution $f_{ij} \leftrightarrow f_{ij}^*$.

(2) $F^*$ is a $\ast$-bialgebra, with $\Delta(f_{ij}) = \sum_k f_{ik} \otimes f_{kj}$ and $\varepsilon(f_{ij}) = \delta_{ij}$.

(3) We have a $\ast$-bialgebra isomorphism $\langle u_{ij} \rangle \simeq F^*$, $u_{ij} \rightarrow f_{ij}$. 

Theorem. The smooth part of the algebra $A_C$ is given by

$$A_C \simeq F^*/J$$

where $J \subset F^*$ is the ideal coming from the following relations,

$$\sum_{p_1,\ldots,p_k} T_{i_1\ldots i_l,p_1\ldots p_k} f_{p_1 j_1} \otimes \ldots \otimes f_{p_k j_k}$$

$$= \sum_{q_1,\ldots,q_l} T_{q_1\ldots q_l,j_1\ldots j_k} f_{i_1 q_1} \otimes \ldots \otimes f_{i_l q_l} , \quad \forall i,j$$

one for each pair of colored integers $k, l$, and each $T \in C(k, l)$.

Proof. This is indeed clear from definitions.
Theorem. The linear space $A_C^*$ is given by the formula

$$ A_C^* = \left\{ a \in F \mid Ta_k = a_l T, \forall T \in C(k, l) \right\} $$

and its representation constructed before, namely

$$ \pi_v : A_C^* \rightarrow B(\oplus_{|k| \leq s} H^{\otimes k}) $$

appears diagonally, by truncating, $\pi_v : a \rightarrow (a_k)_{kk}$.

Proof. Once again, this an elementary computation.
In order to conclude, consider the following spaces:

\[ F_s = \bigoplus_{|k| \leq s} B \left( H^{\otimes k} \right) , \quad F_s^* = \bigoplus_{|k| \leq s} B \left( H^{\otimes k} \right)^* \]

We denote by \( a \rightarrow a_s \) the truncation \( F \rightarrow F_s \). We have:

1. \( E_C^{(s)'} \subset F_s \).
2. \( E_C' \subset F \).
3. \( \mathcal{A}_C^* = E_C' \).
4. \( \text{Im}(\pi_v) = (E_C')_s \).

Indeed, all this follows from the above interpretation of \( \mathcal{A}_C^* \).
Duality

**Theorem.** We have a Tannakian duality correspondence

\[ A \leftrightarrow C \]

between Woronowicz algebras and tensor categories, given by

\[ C_A = (\text{Hom}(u \otimes^k, u \otimes^l))_{kl} \]

in one sense, from algebras to categories, and by

\[ A_C = C(U^+_N)/ < C \subset C_A > \]

in the other sense, from categories to algebras.
Proof 1/2

We have to prove that, for any category $C$, and any $s \in \mathbb{N}$:

$$E_C^{(s)'} = (E'_C)_s$$

By taking duals, this is the same as proving that:

$$\left\{ f \in F^*_s \mid f\big|_{(E'_C)_s} = 0 \right\} = \left\{ f \in F^*_s \mid f\big|_{E'^{(s)'}_C} = 0 \right\}$$

We use $A^*_C = E'_C$. Since we have $A_C = F^*/J$, we conclude that the ideal $J \subset F^*$ previously constructed is given by:

$$J = \left\{ f \in F^* \mid f\big|_{E'_C} = 0 \right\}$$
Proof 2/2

The point now is that we have, for any $s \in \mathbb{N}$:

$$J \cap F_s^* = \left\{ f \in F_s^* \mid f_{|E_C}'(s) = 0 \right\}$$

On the other hand, we have as well:

$$J \cap F_s^* = \left\{ f \in F_s^* \mid f_{|E_C}' = 0 \right\} \cap F_s^*$$

$$= \left\{ f \in F_s^* \mid f_{|E_C}' = 0 \right\}$$

$$= \left\{ f \in F_s^* \mid f_{|(E_C')_{s}} = 0 \right\}$$

Thus, we are led to the equality that we wanted to prove.
Many applications, and to begin with, we have as plan:

1. The biggest quantum group, namely $U^+_N$, must correspond to the smallest tensor category, namely $< R >$.

2. It is well-known that $R : 1 \rightarrow \sum_i e_i \otimes e_i$ can be pictured as a semicircle $\cap$, so we have to get into diagrams.

3. We will reach in this way to a notion of "easy quantum group", covering $O_N, O^+_N, U_N, U^+_N$, and many other examples.

4. As a main application, we will solve the problem of computing the law of the main character for $O_N, O^+_N, U_N, U^+_N$. 
Let $P(k, l)$ be the set of partitions between an upper colored integer $k$, and a lower colored integer $l$.

**Definition.** A collection of subsets $D(k, l) \subset P(k, l)$ is called a category of partitions when it satisfies:

1. Stability under the horizontal concatenation, $(\pi, \sigma) \rightarrow [\pi \sigma]$.
2. Stability under vertical concatenation $(\pi, \sigma) \rightarrow [\sigma \pi]$ (matching).
3. Stability under the upside-down turning $*$, with $\circ \leftrightarrow \bullet$.
4. Each $P(k, k)$ contains the identity partition $\| \ldots \|$.
5. Both $P(\emptyset, \circ\bullet)$ and $P(\emptyset, \bullet\circ)$ contain the semicircle $\cap$. 
Definition. A closed subgroup $G \subset U^+_N$ is called easy when

$$\text{Hom}(u^\otimes k, u^\otimes l) = \text{span} \left( T_\pi \big| \pi \in D(k, l) \right)$$

for a certain category of partitions $D \subset P$, where

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with $\delta_\pi \in \{0, 1\}$ depending on whether the indices fit or not.
Theorem. The basic unitary quantum groups, namely

\[
\begin{array}{c}
U_N \longrightarrow U_N^+ \\
\uparrow \quad \quad \quad \quad \quad \uparrow \\
O_N \longrightarrow O_N^+
\end{array}
\]

are all easy, coming from the following categories of pairings:

\[
\begin{array}{c}
\mathcal{P}_2 \leftarrow \mathcal{N}C_2 \\
\downarrow \quad \quad \quad \quad \quad \downarrow \\
P_2 \leftarrow NC_2
\end{array}
\]

Proof. This comes from Tannaka (classical case: Brauer).
Theorem. We have the following free complexification formula,

\[ \tilde{O}_N^+ = U_N^+ \]

and for projective versions we have the following isomorphism,

\[ PO_N^+ = PU_N^+ \]

by identifying as usual the full and reduced versions.

Proof. We know that we have \( \tilde{O}_N^+ \subset U_N^+ \), and since the Tannakian categories coincide, this is an isomorphism.
Applications 2/3

**Theorem.** The moments of the main characters for

$$U_N \rightarrow U_N^+$$

$$O_N \rightarrow O_N^+$$

are, in the $N \rightarrow \infty$ limit, as follows:

1. On the bottom, with $k = 2l$, we have $(2l)!!$ and $\frac{1}{l+1} \binom{2l}{l}$.
2. On top we have similar numbers, with $k$ being now colored.

**Proof.** This follows by counting the pairings, with $N \rightarrow \infty$ being needed as for $\{T_\pi\}$ to be linearly independent.
Theorem. The asymptotic laws of the main characters for $U_N \rightarrow U_N^+$ are the basic measures in probability and free probability:

- Complex Gaussian $\rightarrow$ Voiculescu circular
- Real Gaussian $\rightarrow$ Wigner semicircular

Proof. Calculus if we guess the answer, Stieltjes inversion otherwise.
Conclusion

We have a theory of easy quantum groups, featuring:

(1) Simple axioms: "C must come from partitions".

(2) The quantum groups $O_N, O_N^+, U_N, U_N^+$ as main examples.

(3) Many other potential examples, e.g. coming from $P, NC$.

(4) Interesting connections with probability/free probability.

⇒ next lecture: quantum permutations