Quantum permutations and quantum reflections

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"Introduction to quantum groups", 5/6

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Tannaka

Theorem. We have a Tannakian duality correspondence

$$A \longleftrightarrow C$$

between Woronowicz algebras and tensor categories, given by

$$C_A = (Hom(u^{\otimes k}, u^{\otimes l}))_{kl}$$

in one sense, from algebras to categories, and by

$$A_C = C(U_N^+)/ < C \subset C_A >$$

in the other sense, from categories to algebras.

Easiness

Theorem. Any category of partitions $D = (D(k,l))$ produces a family of quantum groups $G = (G_N)$ via the formula

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

where the linear maps T_π associated to partitions are given by

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with $\{e_i\}$ being the basis of $\mathbb{C}^N$, and $\delta_\pi \in \{0,1\}$ being Kronecker symbols. These quantum groups G_N are called easy.

Plan

(1) Quantum permutation groups

(2) Easiness: algebra and analysis

(3) Quantum reflection groups

(4) Transitivity, planar algebras

$\Longrightarrow$ next lecture: tori, models

Quantum permutations

The coordinates of $S_N \subset O_N$, permutation matrices, are:

$$u_{ij} = \chi\left(\sigma \in S_N \middle| \sigma(j) = i\right)$$

A quick study of u suggests the following definition:

<u>Definition</u>. The quantum permutation group S_N^+ is defined via

$$C(S_N^+) = C^*\left((u_{ij}) \middle| u = N \times N \text{ magic}\right)$$

where "magic" = made of projections, sum 1 on rows/columns.

[the verification of the CQG axioms is routine: Wang 98]

Alternative definition

Theorem. S_N^+ is the biggest quantum group acting on

$$X = \{1, \ldots, N\}$$

by keeping the counting measure invariant.

Proof. In order to have a quantum group action

$$G \times X \to X \quad , \quad (\sigma, i) \to \sigma(i)$$

we need a coaction map $\Phi : C(X) \to C(G) \otimes C(X)$. With

$$\Phi(\delta_i) = \sum_j u_{ij} \otimes \delta_j$$

the matrix $u = (u_{ij})$ must be magic. Thus $G_{max} = S_N^+$.

Basic properties 1/4

We have a quotient map $C(S_N^+) \to C(S_N)$, given by:

$$u_{ij} \to \chi\left(\sigma \in S_N \middle| \sigma(j) = i\right)$$

Thus we have an embedding $S_N \subset S_N^+$. Study:

<u>$N = 2$</u>: We have $S_2^+ = S_2$, because the 2×2 magics are

$$u = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

and their entries commute. Thus $C(S_2^+)$ is commutative.

<u>$N = 3$</u>: We have $S_3^+ = S_3$, by similar arguments.

Basic properties 2/4

We know $S_N \subset S_N^+$ isomorphism at $N = 2, 3$. Continuation:

<u>$N = 4$</u>: Here S_4^+ is non-classical and infinite, because

$$u = \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

with $p, q \in B(H)$ shows that $C(S_4^+)$ is NC and ∞D.

<u>$N \geq 5$</u>: Here S_N^+ stays non-classical and infinite (clear).

Basic properties 3/4

$\Longrightarrow$ Can we understand better <u>why $S_4^+ \neq S_4$</u>?

Recall that given $\Gamma =< g_1, \ldots, g_N >$ discrete group, $A = C^*(\Gamma)$ is a Woronowicz algebra, written $A = C(\widehat{\Gamma})$, with:

$$u = diag(g_1, \ldots, g_N)$$

Now observe that we have, trivially by Fourier transform:

$$\widehat{\mathbb{Z}_2} = \mathbb{Z}_2 = S_2 = S_2^+$$

Thus <u>our concatenation trick at $N = 4$</u> amounts in saying that:

$$\widehat{D_\infty} = \widehat{\mathbb{Z}_2 * \mathbb{Z}_2} \subset S_4^+$$

Even better, we have $\widehat{D_\infty} \subset G^+(\square)$. More on this later.

Basic properties 4/4

$\Longrightarrow$ Can we understand what this S_4^+ beast is?

Algebra $C(SO_3^{-1})$, with orthogonal coordinates a_{ij}, satisfying:
– $a_{ij}a_{kl} = \pm a_{kl}a_{ij}$, with $+$ if $i \neq k, j \neq l$, and $-$ otherwise
– twisted determinant condition: $\sum_{\sigma \in S_3} a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)} = 1$

The point is that the following matrix must be magic:

$$\frac{1}{4}\begin{pmatrix}1&1&1&1\\1&-1&-1&1\\1&-1&1&-1\\1&1&-1&-1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&a_{11}&a_{12}&a_{13}\\0&a_{21}&a_{22}&a_{23}\\0&a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}1&1&1&1\\1&-1&-1&1\\1&-1&1&-1\\1&1&-1&-1\end{pmatrix}$$

Thus $S_4^+ = SO_3^{-1}$, via the Fourier transform over $K = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Representations 1/4

Theorem. The Tannakian category of S_N is given by

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in P(k,l)\right)$$

where the linear maps associated to partitions are:

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1,\ldots,j_l} \delta_\pi \begin{pmatrix} i_1 \ldots i_k \\ j_1 \ldots j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

Regarding now S_N^+, the situation is quite similar:

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in NC(k,l)\right)$$

In other words, S_N, S_N^+ are easy, coming from P, NC.

Representations 2/4

Proof for S_N. Consider the one-block partition $\mu \in P(2,1)$. We have $T_\mu(e_i \otimes e_j) = \delta_{ij} e_i$, and a computation gives:

$$T_\mu \in Hom(u^{\otimes 2}, u) \iff u_{ij} u_{ik} = \delta_{jk} u_{ij}, \forall i, j, k$$

On the right we have the magic condition. We conclude that:

$$C(S_N) = C(O_N) \Big/ \Big\langle T_\mu \in Hom(u^{\otimes 2}, u) \Big\rangle$$

Now since P is generated by $\mu \in P(2,1)$, we are done.

Proof for S_N^+. Similar, by using the Brauer theorem for O_N^+.

Representations 3/4

Theorem. The fusion rules for S_N^+ are the same as for SO_3,

$$r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+1} + \ldots + r_{k+l}$$

with $\dim(r_k) = \frac{q^{k+1} - q^{-k}}{q-1}$, where $q^2 - (N-2)q + 1 = 0$.

Proof. We know from easiness that we have:

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in NC(k,l)\right)$$

Thus, the main character χ is squared-semicircular:

$$\int_{S_N^+} \chi^p = |NC(0,p)| = \frac{1}{p+1}\binom{2p}{p}$$

But this gives the result, using $S^3_{\mathbb{R}} \simeq SU_2 \to SO_3$.

Representations 4/4

Comment: the above proof is valid in fact only with $N >> 0$, where the maps $\{T_\pi\}$ are linearly independent.

However, things are in fact fine as long as $N \geq 4$.

Why 4? Because this is a "Jones index". We have indeed

$$NC(0,p) \simeq NC_2(0,2p) \simeq NC_2(p,p) = \{basis\ of\ \ TL(p)\}$$

and according to Jones, we must have $N \geq 4$ for things to work.

$\implies$ note that all this is simpler than for S_N (!)

Analysis 1/4

Let $S_N \subset O_N$ as usual. The main character is then:

$$\chi(\sigma) = \sum_i u_{ii}(\sigma) = \sum_i \delta_{\sigma(i)i} = \#\left\{i \middle| \sigma(i) = i\right\}$$

By using the inclusion-exclusion principle, we obtain:

$$\mathbb{P}(\chi = 0) = 1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + (-1)^{N-1}\frac{1}{(N-1)!} + (-1)^N \frac{1}{N!}$$

Thus, we have the following asymptotic formula:

$$\lim_{N\to\infty} \mathbb{P}(\chi = 0) = \frac{1}{e}$$

In fact, the character χ becomes Poisson with $N \to \infty$.

Analysis 2/4

Theorem. If G is easy, coming from a category of partitions D,

$$\int_G u_{i_1j_1}\ldots u_{i_kj_k}=\sum_{\pi,\sigma\in D(k)}\delta_\pi(i)\delta_\sigma(j)W_{kN}(\pi,\sigma)$$

where $W_{kN}=G_{kN}^{-1}$ is the inverse of $G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$.

Proof. This is the Weingarten formula, coming from the fact that the above integrals form the projection onto $Fix(u^{\otimes k})$.

In the unitary case we must use colored integers.
Works too in the symplectic case, and other cases.

Analysis 3/4

Theorem. The main character $\chi = \sum_{i=1}^N u_{ii}$ for the quantum groups S_N, S_N^+ follows with $N \to \infty$ the laws

$$p_1 = \frac{1}{e} \sum_k \frac{\delta_k}{k!}$$

$$\pi_1 = \frac{1}{2\pi} \sqrt{4x^{-1} - 1}dx$$

called Poisson and Marchenko-Pastur (or free Poisson) of parameter 1, and appearing via the PLT and FPLT.

Proof. Here we do not really need Weingarten, because:

$$\int_G \chi^k \simeq |D(k)|$$

By using standard calculus (e.g. cumulants) we can conclude.

Analysis 4/4

Theorem. The truncated characters $\chi_t = \sum_{i=1}^{[tN]} u_{ii}$ for the quantum groups S_N, S_N^+ follow with $N \to \infty$ the laws

$$p_t = e^{-t} \sum_k \frac{t^k}{k!} \delta_k$$

$$\pi_t = \max(1-t, 0)\delta_0 + \frac{\sqrt{4t - (x-1-t)^2}}{2\pi x} \, dx$$

called Poisson and Marchenko-Pastur (or free Poisson) of parameter t, and appearing via the PLT and FPLT.

Proof. Here, by using the Weingarten formula, we have:

$$\int_G \chi_t^k \simeq \sum_{\pi \in D(k)} t^{|\pi|}$$

By using standard calculus (e.g. cumulants) we can conclude.

Summary

(1) The analogy between S_N, S_N^+ is best understood via easiness

$$\begin{array}{ccccc}
S_N^+ & \cdots\cdots & NC & \cdots\cdots & \pi_t \\
\updownarrow & & \updownarrow & & \updownarrow \\
S_N & \cdots\cdots & P & \cdots\cdots & p_t
\end{array}$$

with N generic, for algebra, and with $N \to \infty$, for analysis.

(2) When N is fixed things collapse for both S_N, S_N^+, the collapsing being worse for S_N in algebra, and worse for S_N^+ in analysis.

(3) All this is just the "tip of the iceberg". Many advanced results, both algebra and analysis (planar algebras, Diaconis type).

Graphs 1/4

Let X be a finite graph, $|X| = N < \infty$, with adjacency matrix $d \in M_N(0,1)$. Its <u>quantum symmetry group</u> is given by:

$$G^+(X) = C(S_N^+)\Big/\Big\langle du = ud \Big\rangle$$

We have then a diagram of inclusions, as follows:

$$\begin{array}{ccc} G^+(X) & \longrightarrow & S_N^+ \\ \uparrow & & \uparrow \\ G(X) & \longrightarrow & S_N \end{array}$$

<u>Trivial example</u>: no edges (or complete graph) $\implies$ get S_N^+.

Graphs 2/4

Cycle graph C_N. Here generically we have, by algebra,

$$G^+(C_N) = G(C_N) = D_N$$

unless at $N = 4$, where the following thing happens:

$$G^+(C_4) = G^+(\square) = G^+(|\ |) \supset \widehat{\mathbb{Z}_2 * \mathbb{Z}_2} = \widehat{D_\infty}$$

$\implies$ Question: what is $G^+(\square)$?

Looking at hypercube graphs $\square_N$. Here we have:

$$G^+(\square_N) = O_N^{-1}$$

$\implies$ In particular, we obtain $G^+(\square) = O_2^{-1}$.

Graphs 3/4

This is still not ok, because $H_N \to O_N^{-1}$ cannot be a "true liberation", for analytic reasons (same law as for O_N).

$\implies$ Question: what is H_N^+?

Answer. Consider the graph $|| \ldots ||$ consisting of N segments (the $[-1,1]$ segments on the N coordinate axes). Then:

$$G(|| \ldots ||) = \mathbb{Z}_2 \wr S_N = H_N \longleftrightarrow P_{even}$$

We can therefore define H_N^+ as follows, and we are done:

$$G^+(|| \ldots ||) = \mathbb{Z}_2 \wr_* S_N^+ = H_N^+ \longleftrightarrow NC_{even}$$

Graphs 4/4

More generally, for any $s \in \{1, 2, \ldots, \infty\}$ we have:

$$G(\triangle_s \ldots \triangle_s) = \mathbb{Z}_s \wr S_N = H_N^s \longleftrightarrow P^s$$

We can liberate this reflection group as follows:

$$G^+(\triangle_s \ldots \triangle_s) = \mathbb{Z}_s \wr_* S_N^+ = H_N^{s+} \longleftrightarrow NC^s$$

(the "s" at right mean $\#\circ = \#\bullet\,(s)$, signed, in each block)

– at $s = 1$ we recover S_N, S_N^+
– at $s = 2$ we recover H_N, H_N^+
⋮
– at $s = \infty$ non-QPG, called K_N, K_N^+

Many other interesting results here.

Orbits 1/4

Recall that for $G \subset S_N$ the coordinates via $S_N \subset O_N$ are:

$$u_{ij} = \chi\left(\sigma \in G \middle| \sigma(j) = i\right)$$

<u>Definition</u>. A quantum permutation group $G \subset S_N^+$ is called transitive when $u_{ij} \neq 0$, for any i, j.

As basic examples, all QPG that we met so far:

– we have $G^+(X)$ with X transitive (i.e. with $G(X)$ transitive)

– in particular we have H_N^s, H_N^{s+}, for any $s \in \mathbb{N}$

– also in particular, we have $O_N^{-1} = G^+(\square_N)$

Orbits 2/4

Orbits. Given a closed subgroup $G \subset S_N^+$, let us set:

$$i \sim j \iff u_{ij} \neq 0$$

This is an equivalence relation. Indeed (using positivity):

$$\Delta(u_{ik}) = \sum_j u_{ij} \otimes u_{jk} \implies [i \sim j, j \sim k \implies i \sim k]$$

$$\varepsilon(u_{ii}) = 1 \implies i \sim i$$

$$S(u_{ij}) = u_{ji} \implies [i \sim j \implies j \sim i]$$

In the classical case, $G \subset S_N$, we recover the usual orbits.

$\implies$ what to do with this notion? (no examples so far)

Orbits 3/4

Consider a quotient group of type $\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$, with $N = N_1 + \ldots + N_k$. We have then, by Fourier:

$$\begin{aligned}
\widehat{\Gamma} &\subset \widehat{\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k}} = \widehat{\mathbb{Z}_{N_1}} \,\hat{*}\, \ldots \,\hat{*}\, \widehat{\mathbb{Z}_{N_k}} \\
&\simeq \mathbb{Z}_{N_1} \,\hat{*}\, \ldots \,\hat{*}\, \mathbb{Z}_{N_k} \subset S_{N_1} \,\hat{*}\, \ldots \,\hat{*}\, S_{N_k} \\
&\subset S_{N_1}^+ \,\hat{*}\, \ldots \,\hat{*}\, S_{N_k}^+ \subset S_N^+
\end{aligned}$$

Theorem. Any group dual subgroup $\widehat{\Gamma} \subset S_N^+$ appears in this way, for a certain partition $N = N_1 + \ldots + N_k$.

Proof. Orbit decomposition $N = N_1 + \ldots + N_k$.

Orbits 4/4

Orbitals. Let $G \subset S_N^+$, and $k \in \mathbb{N}$. The relation

$$(i_1, \ldots, i_k) \sim (j_1, \ldots, j_k) \iff u_{i_1 j_1} \ldots u_{i_k j_k} \neq 0$$

is then reflexive and symmetric (proof as before, at $k = 1$).

Transitivity holds at $k = 1$. Also at $k = 2$, the trick being:

$$\begin{aligned}
& (u_{i_1 j_1} \otimes u_{j_1 l_1}) \Delta(u_{i_1 l_1} u_{i_2 l_2})(u_{i_2 j_2} \otimes u_{j_2 l_2}) \\
= & \sum_{s_1 s_2} u_{i_1 j_1} u_{i_1 s_1} u_{i_2 s_2} u_{i_2 j_2} \otimes u_{j_1 l_1} u_{s_1 l_1} u_{s_2 l_2} u_{j_2 l_2} \\
= & \; u_{i_1 j_1} u_{i_2 j_2} \otimes u_{j_1 l_1} u_{j_2 l_2}
\end{aligned}$$

At $k \geq 3$ this fails (but few things still hold), at $k \geq 4$ totally fails.

Algebra 1/4

What can be said about the arbitrary subgroups $G \subset S_N^+$?

(in addition to the orbit/orbital theory explained above)

Theorem. Quantum Cayley fails.

Recall indeed the Cayley theorem, stating that, for classical groups:

$$|G| = N \implies G \subset S_N$$

This does not work for quantum groups. There are finite quantum groups which are not quantum permutation groups (!)

Algebra 2/4

What can be said (good) about the subgroups $G \subset S_N^+$?

Theorem. The collection of vector spaces

$$P_k = Fix(u^{\otimes k})$$

is a planar algebra in the sense of Jones. More precisely, we have an inclusion as follows, where Q_N is the "spin" planar algebra,

$$P \subset Q_N$$

and any planar subalgebra $P \subset Q_N$ appears in this way.

Proof. Tannakian duality, applied in this setting, "rotated".

Algebra 3/4

Planar algebras, more. The correspondence established above

$$G \subset S_N^+ \longleftrightarrow P \subset Q_N$$

makes correspond the following objects and constructions,

$$\{1\} \longleftrightarrow Q_N$$

$$S_N^+ \longleftrightarrow TL_N$$

$$H_N^+ \longleftrightarrow FC_N$$

$$G^+(X) \longleftrightarrow < \square_X >$$

where $\square_X$ is the Laplacian (adjacency matrix) viewed as 2-box.

$\implies$ Bisch-Jones, "Laplacian in the box" philosophy

Algebra 4/4

A difficult conjecture states that $S_N \subset S_N^+$ is maximal, in the sense that there is no object in between. Status:

(1) Trivial: no groups, no group duals.

(2) Elementary: no easy solutions.

(3) Advanced: OK at $N = 4$, cf. ADE classification of the subgroups $G \subset S_4^+ = SO_3^{-1}$.

(4) Difficult: OK at $N = 5$, due to the classification of index 5 subfactors. No known QPG proof.

Conclusion

We have a theory of quantum permutations, featuring:

(1) General theory, orbits, easiness.

(2) $S_N, S_N^+, H_N, H_N^+, K_N, K_N^+$ as main examples.

(3) Many other examples, e.g. coming from graphs.

(4) Interesting connections with probability/free probability.

$\implies$ next lecture: tori, models