A guide to quantum algebra

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Abstract. The classical spaces X, such as the Lie groups, homogeneous spaces, or various algebraic or differential manifolds, can be described by various associated algebras A, which can be usually defined over an arbitrary field k. In quantum algebra the situation is a bit different, with different fields k leading, via different algebras A, to different classes of quantum spaces X. We review here a number of well-known quantum algebra theories, including Hopf algebras, quantum groups, tensor categories, and so on, and their noncommutative geometry extensions, with the aim of looking for quantum spaces X which do not depend on the choice of the ground field k.

Preface

The story goes that abstract algebra was invented by the Devil. And any fine mathematician, including the author of the present book, can only agree with this. Normally mathematics is about geometry, be it differential, algebraic, or of any other kind. With the healthy goal of understanding how things continuously evolve in time.

Algebra basically attempts to grab a part of this beautiful mathematics which is geometry, by using all sorts of clever space/algebra correspondences, as to reformulate everything in an opaque, but powerful language: the language of abstract algebra. With the power coming for instance from the tools of linear algebra. Isn't it satisfying, although a bit sad, to reformulate your nice problem into a technical eigenvalue computation.

Further creations of the Devil include the Bourbaki group, who elevated these embarassing algebraic techniques to the status of religion. Then probability theory, which is opposite to geometry too, assuming that God plays dice or something. And finally quantum mechanics, or at least quantum mechanics as we know it, which appears to be a mixture of opaque algebra and probability, reminding a bit a Bourbaki invention.

The goal of the present book is to talk about all these bad things. Part I is an introduction to abstract algebra, with motivations coming from number theory, and especially, from classical geometry. Then in Part II we will go into the study of quantum algebra, with motivation coming from quantum mechanics, and by adding some probability into the picture, too. As a continuation of this, Part III comes as a more advanced study of quantum algebra, from a geometric and analytic viewpoint, putting together ideas from Parts I and II, and adding some more. Finally, Part IV deals with the question of deforming the various quantum algebras and spaces discussed in Parts II and III, with once again motivations coming from various branches of modern theoretical physics.

All in all, the present book covers what can be taught during a one-year graduate course, on the theme "Introduction to quantum algebra". For a one-semester course, this could follow the material in Parts II and III, which are the core of the book, with a quick exposition of Part I, to start with, and with a quick look into Part IV as well.

Quantum algebra is of course something quite wide, and there are many other ways of getting introduced to it, and books dedicated to this. Our idea here, coming from the various stories discussed above, is that the classical spaces X, such as the Lie groups, homogeneous spaces, or various algebraic or differential manifolds, can be described by various associated algebras A, which can be usually defined over an arbitrary field k. In quantum algebra the situation is a bit different, with different fields k leading, via different algebras A, to different classes of quantum spaces X. We will review here a number of well-known quantum algebra theories, including Hopf algebras, quantum groups, tensor categories, and so on, and their noncommutative geometry extensions, with the aim of looking for quantum spaces X which do not depend on the choice of the ground field k. With the idea, or rather hope in mind, that these should be the quantum spaces which are really revelant to physics, and to common sense science in general.

In practice now, no surprise, our conclusion will be somehow that such "absolute" quantum spaces X are well-defined over the good old complex numbers, $k = \mathbb{C}$, and that their further study basically lies here, at $k = \mathbb{C}$. However, remember that we believe in Algebra, and so having all sorts of versions of these "absolute spaces", coming either arithmetically, via other fields k, or via deformation and so on, is a useful thing. That is, we would like to have these various quantum spaces X carefully arranged in our lab, a bit like chemical products in Gargamel's, for quick and efficient use when needed. And the present book is here for that, carefully arranging all these quantum spaces X, with their hierarchy and ordering coming from the above ideas and philosophy.

Most of this book is based on lecture notes from various classes at Cergy, and I would like to thank my students, always eager to learn concrete and understandable mathematics. This book contains, besides the basics of the quantum algebra theory, some personal contributions as well, and I would like to thank Julien Bichon, Benoît Collins, Ion Nechita and the others, for our joint work. Many thanks go as well to Nicolás Andruskiewitsch and Sonia Natale, for further algebraic discussions. Finally, it is a pleasure to thank my cats, for precious help in the preparation of the present book.

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Part I

Abstract algebra

Love is a stranger
In an open car
To tempt you in
And drive you far away

CHAPTER 1

Number theory

1a. Numbers and fields

You are probably familiar with the rational numbers $\mathbb{Q}$, and with the real numbers $\mathbb{R}$. Although these are both God-given, and do not need any formal definition, it is customary for any algebra book to start with their definition, out of nothing, as if the reader was an alien or something. So, let us first embark on the clarification of this, of the obvious.

Regarding the rational numbers, things here are quickly settled, as follows:

Definition 1.1. *The rational numbers are the quotients of type*

$$q = \frac{a}{b}$$

with $a, b \in \mathbb{Z}$, and $b \neq 0$, identified according to the usual rule for quotients, namely:

$$\frac{a}{b} = \frac{c}{d} \iff ad = bc$$

These numbers add according to the usual rule for quotients, namely:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Also, they multiply according to the usual rule for quotients, namely:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

We denote the set of rational numbers, with these sum and product operations, by $\mathbb{Q}$.

Observe that we have an inclusion $\mathbb{Z} \subset \mathbb{Q}$, because we can write any integer $a \in \mathbb{Z}$ as a rational, as follows, and with the identifications for integers/rationals matching:

$$a = \frac{a}{1}$$

Moreover, this inclusion "preserves the algebra", in the sense that the usual sum and product in $\mathbb{Z}$ correspond in this way to the above sum and product in $\mathbb{Q}$.

Regarding now the real numbers, there are several ways of introducing them, out of "nothing", which in our case means out of the rational numbers. First, we have:

Definition 1.2. *The real numbers are the usual numbers, from the real life surrounding us, written as usual in decimal form*

$$x = \pm a_1 \ldots a_n . b_1 b_2 b_3 \ldots\ldots$$

with $a_i, b_i \in \{0, 1, \ldots, 9\}$, *with the usual convention for this writing, namely:*

$$\ldots a999 \ldots\ldots = \ldots (a+1)000 \ldots\ldots$$

These numbers add and multiply according to the usual formulae for numbers written in decimal form. We denote the set of real numbers, with this sum and product, by $\mathbb{R}$.

Observe that we have an inclusion $\mathbb{Q} \subset \mathbb{R}$, obtained by writing any rational number in decimal form, using the well-known algorithm for doing so. Also, the usual sum and product in $\mathbb{Q}$ correspond in this way to the usual sum and product in $\mathbb{R}$.

Normally we should stop here, but there are some bugs with Definition 1.2. First of all, not that we are aliens, but if we are trying for instance to establish communication with aliens by using our mathematics and real numbers, these fellow aliens will probably not understand anything, because the 10 in the above is definitely something human-made, not making any sense at the galactic level. So, let us replace Definition 1.2 with:

Definition 1.3. *The real numbers are the numbers written in binary form*

$$x = \pm a_1 \ldots a_n . b_1 b_2 b_3 \ldots\ldots$$

with $a_i, b_i \in \{0, 1\}$, *with the usual convention for this binary writing, namely:*

$$\ldots a111 \ldots\ldots = \ldots (a+1)000 \ldots\ldots$$

These numbers add and multiply according to the usual formulae for numbers written in binary form. We denote the set of real numbers, with this sum and product, by $\mathbb{R}$.

This sounds more reasonable, but there are still some bugs with this, coming from the formulae for addition and multiplication. To be more precise, these addition and multiplication formulae are in fact quite complicated algorithms, and we would like these algorithms not to be involved in something as fundamental as the definition of $\mathbb{R}$.

The solution to this comes from the following clever definition, due to Dedekind:

Definition 1.4. *The real numbers are the formal cuts in the set of rational numbers*

$$x = \left\{ \mathbb{Q} = \mathbb{Q}_{\leq x} \sqcup \mathbb{Q}_{>x} \right\}$$

with such a cut being by definition subject to the following condition:

$$q \in \mathbb{Q}_{\leq x}, r \in \mathbb{Q}_{>x} \implies q \leq r$$

These numbers add and multiply by adding and multiplying the corresponding cuts, in the obvious way. We denote the set of real numbers, with this sum and product, by $\mathbb{R}$.

As before, we have an inclusion $\mathbb{Q}\subset\mathbb{R}$, obtained by identifying each rational number $q\in\mathbb{Q}$ with the obvious cut that it produces in the set of rational numbers, namely:

$$\mathbb{Q}_{\leq q}=\left\{r\in\mathbb{Q}\Big|r\leq q\right\}\quad,\quad\mathbb{Q}_{>q}=\left\{r\in\mathbb{Q}\Big|r>q\right\}$$

Moreover, this inclusion "preserves the algebra", in the sense that the usual sum and product in $\mathbb{Q}$ correspond in this way to the above sum and product in $\mathbb{R}$.

The above definition is something fairly simple, and rock-solid, and so time to stop our discussion about the real numbers. Let us point out, however, that before stopping we still have to decide about the fate of the previous Definition 1.2 and Definition 1.3.

These are certainly true things, that we would like to have as statements, in our bag of tricks, and since we cannot call them any longer "definitions", we must upgrade, or rather downgrade them, to "theorems". So, the remaining technical problem is that, based on Definition 1.4 above, to establish Theorem 1.2 and Theorem 1.3. But this can be done indeed, with some work, and you probably already know all this, so time to stop.

Moving ahead now, let us take a closer look at the sets $\mathbb{Q}$ and $\mathbb{R}$, as constructed in Definition 1.1 and Definition 1.4 above. Despite their obvious differences, these sets of numbers have a lot of common features, coming from the addition and the multiplication. To be more precise, these sets are both fields, in the following sense:

DEFINITION 1.5. *A field is a set k, whose elements are usually called "numbers", with two binary operations as follows,*

$$(x,y)\to x+y$$

$$(x,y)\to xy$$

satisfying the usual conditions for an addition and a multiplication, namely commutativity, associativity, distributivity, and existence of units and of inverses.

Getting now to examples of fields, we certainly have $\mathbb{Q}$ and $\mathbb{R}$, both useful, but are there some more. Certainly yes. The idea is that of solving equations in k: if there are solutions, fine, and if not, we should "enlarge" k, as to contain these solutions.

As a basic example here, consider the following equation:

$$X^2+1=0$$

This equation does not have solutions in $\mathbb{Q}$, and nor does it have in the larger field $\mathbb{R}$. This suggests extending $\mathbb{R}$ with the help of a mysterious number i, satisfying:

$$i^2=-1$$

We are led in this way to the following definition, that you certainly know too:

Definition 1.6. *The complex numbers are variables of the form*

$$x = a + ib \quad , \quad y = c + id \quad , \quad \ldots$$

with $a, b, c, d, \ldots \in \mathbb{R}$*, which add in the obvious way, namely*

$$x + y = (a + c) + i(b + d)$$

and which multiply according to the rule $i^2 = -1$*, which gives in practice:*

$$xy = (ac - bd) + i(ad + bc)$$

These complex numbers form a field, extending $\mathbb{R}$*, which is denoted* $\mathbb{C}$*.*

With this notion in hand, the equation $X^2 + 1 = 0$ has now two complex solutions, $X = \pm i$. We will see in a moment that any degree 2 equation has in fact 2 complex solutions. And that, more generally, any degree N equation has N complex solutions.

Before doing this, we must gain some familiarity with the complex numbers. It is convenient to represent these numbers in the plane, with $x = a + ib$ corresponding to:

$$x = \begin{pmatrix} a \\ b \end{pmatrix}$$

In this picture, the real numbers correspond to the numbers on the Ox axis. As for the purely imaginary numbers, these lie on the Oy axis, with:

$$i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

With this convention, the sum of complex numbers is the usual sum of vectors. However, in order to understand now the multiplication, in a geometric way, we must do something more complicated, namely using polar coordinates, as follows:

Definition 1.7. *The complex numbers* $x = a + ib$ *can be written in polar coordinates,*

$$x = r(\cos t + i \sin t)$$

with the connecting formulae being

$$a = r \cos t \quad , \quad b = r \sin t$$

and in the other sense being

$$r = \sqrt{a^2 + b^2} \quad , \quad \tan t = b/a$$

and with r, t *being called modulus, and argument.*

The point now is that in polar coordinates, the multiplication formula for the complex numbers, which was so far something quite opaque, takes a very simple form:

PROPOSITION 1.8. *Two complex numbers written in polar coordinates,*

$$x = r(\cos s + i \sin s)$$

$$y = p(\cos t + i \sin t)$$

multiply according to the following formula:

$$xy = rp(\cos(s+t) + i \sin(s+t))$$

In other words, the moduli multiply, and the arguments sum up.

PROOF. We can assume that we have $r = p = 1$, and we have:

$$\begin{aligned} xy &= (\cos s + i \sin s)(\cos t + i \sin t) \\ &= (\cos s \cos t - \sin s \sin t) + i(\cos s \sin t + \sin s \cos t) \\ &= \cos(s+t) + i \sin(s+t) \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

In order to further advance on all this, we will need a deep result, as follows:

THEOREM 1.9. *We have the following formula, valid for any $t \in \mathbb{R}$,*

$$e^{it} = \cos t + i \sin t$$

where $e = 2.7182\ldots$ is the usual constant from analysis.

PROOF. We have the following computation, based on the formula of e^x:

$$\begin{aligned} e^{it} &= \sum_k \frac{(it)^k}{k!} \\ &= \sum_{k=2l} \frac{(it)^k}{k!} + \sum_{k=2l+1} \frac{(it)^k}{k!} \\ &= \sum_l (-1)^l \frac{t^{2l}}{(2l)!} + i \sum_l (-1)^l \frac{t^{2l+1}}{(2l+1)!} \end{aligned}$$

Our claim now, which will complete the proof, is that we have:

$$\cos t = \sum_l (-1)^l \frac{t^{2l}}{(2l)!}$$

$$\sin t = \sum_l (-1)^l \frac{t^{2l+1}}{(2l+1)!}$$

But this follows by computing the Taylor series of cos and sin. Indeed, by using the formulae for sums of angles, used in the proof of Proposition 1.8, we have:

$$\sin' = \cos \quad , \quad \cos' = -\sin$$

Thus, we know how to differentiate sin and cos, as many times as we want to, and so we can compute the corresponding Taylor series, and we obtain the formulae above. □

We can now improve the writing in Definition 1.7, as follows:

THEOREM 1.10. *The complex numbers can be written in polar coordinates,*

$$x = re^{it}$$

the connecting formulae being those in Definition 1.7. The multiplication formula is

$$re^{is} \cdot pe^{it} = rp\, e^{i(s+t)}$$

with the arguments s, t being taken modulo 2π.

PROOF. This is a reformulation of Definition 1.7 and Proposition 1.8, by using the formula $e^{it} = \cos t + i \sin t$, from Theorem 1.9 above. □

The above result is quite powerful, and opens us a whole new perspective on the complex numbers, and what can be done with them. As a first application here, let us go back now to $X^2 + 1 = 0$, and to other degree 2 equations. We can now prove:

PROPOSITION 1.11. *Any degree 2 equation over the real or the complex numbers,*

$$aX^2 + bX + c = 0$$

has two complex solutions, given by the formula

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with the square root of the complex numbers being defined as

$$\sqrt{re^{it}} = \pm\sqrt{r}e^{it/2}$$

in polar coordinate writing.

PROOF. By doing some simple algebraic manipulations, exactly as in the familiar case where $a, b, c \in \mathbb{R}$ and $\Delta > 0$, our degree 2 equation can be written as follows:

$$\left(X + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Now since we know, as a consequence of Theorem 1.10, that any complex number has a square root, given by the formula in the statement, we obtain the result. □

More generally now, we can prove that any polynomial equation, of arbitrary degree $N \in \mathbb{N}$, has exactly N complex solutions, counted with multiplicities:

THEOREM 1.12. *Any polynomial $P \in \mathbb{C}[X]$ decomposes as*

$$P = c(X - a_1) \ldots (X - a_N)$$

with $c \in \mathbb{C}$ and with $a_1, \ldots, a_N \in \mathbb{C}$.

PROOF. The problem is that of proving that our polynomial has at least one root, because afterwards we can proceed by recurrence. We prove this by contradiction. So, assume that P has no roots, and pick a number $z \in \mathbb{C}$ where $|P|$ attains its minimum:

$$|P(z)| = \min_{x \in \mathbb{C}} |P(x)| > 0$$

Since $Q(t) = P(z+t) - P(z)$ is a polynomial which vanishes at $t = 0$, this polynomial must be of the form ct^k + higher terms, with $c \neq 0$, and with $k \geq 1$ being an integer. We obtain from this that, with $t \in \mathbb{C}$ small, we have the following estimate:

$$P(z+t) \simeq P(z) + ct^k$$

Now let us write $t = rw$, with $r > 0$ small, and with $|w| = 1$. Our estimate becomes:

$$P(z+rw) \simeq P(z) + cr^k w^k$$

Now recall that we have assumed $P(z) \neq 0$. We can therefore choose $w \in \mathbb{T}$ such that cw^k points in the opposite direction to that of $P(z)$, and we obtain in this way:

$$\begin{aligned} |P(z+rw)| &\simeq |P(z) + cr^k w^k| \\ &= |P(z)|(1 - |c|r^k) \end{aligned}$$

Now by choosing $r > 0$ small enough, as for the error in the first estimate to be small, and overcame by the negative quantity $-|c|r^k$, we obtain from this:

$$|P(z+rw)| < |P(z)|$$

But this contradicts our definition of $z \in \mathbb{C}$, as a point where $|P|$ attains its minimum. Thus P has a root, and by recurrence it has N roots, as stated. $\square$

As an illustration for the above result, which is something quite theoretical, and once again in relation with our favorite equation, namely $X^2 + 1 = 0$, we have:

PROPOSITION 1.13. *The equation $X^N = 1$ has N complex solutions, namely*

$$\left\{ w^k \middle| k = 0, 1, \ldots, N-1 \right\} \quad , \quad w = e^{2\pi i/N}$$

which are called roots of unity of order N.

PROOF. This follows from Theorem 1.10. Indeed, with $X = re^{it}$ our equation reads:

$$r^N e^{itN} = 1$$

Thus $r = 1$, and $t \in [0, 2\pi)$ must be a multiple of $2\pi/N$, as stated. $\square$

More generally, it follows from this that the equation $X^N = c$, with $c \in \mathbb{C}$, has N solutions over the complex numbers, obtained by computing one such solution, using the multiplication rule in Theorem 1.10, and then multiplying by the N-th roots of unity.

1b. Galois theory

We have so far three fields, namely $\mathbb{Q},\mathbb{R},\mathbb{C}$, with each one having its own advantages and disadvantages. The problem with $\mathbb{C}$, which looks as the "smartest" field in our collection, and this due to Theorem 1.12, allowing us to factorize arbitrary polynomials over $\mathbb{C}$, comes precisely from this Theorem 1.12, because the factorization there is not explicit. To be more precise, unlike we are in some special cases, such as the degree 2 case, or the root of unity case, worked out in Proposition 1.11 and Proposition 1.13, the roots of polynomials are not explicitely computable, and all this remains quite theoretical.

So, let us get back to the basics, namely $\mathbb{Q}$, and perhaps $\mathbb{R}$ too. The field $\mathbb{C}$ was obtained from $\mathbb{R}$ by adding the formal root i of the equation $X^2+1=0$, and we can use the same idea for constructing a whole menagery of fields. For instance, we can set:

$$\mathbb{Q}(i)=\left\{a+ib\Big|a,b\in\mathbb{Q}\right\}$$

Also, and perhaps more concretely now, having nothing to do with the complex numbers, and with all the complications coming with them, we can set:

$$\mathbb{Q}(\sqrt{2})=\left\{a+b\sqrt{2}\Big|a,b\in\mathbb{Q}\right\}$$

To be more precise, this latter field is certainly something quite interesting, and if we want to understand the solutions of $X^2-2=0$, which is itself a very interesting question, ignoring calculators and modern technology, this is the right field, to be in.

To summarize, given a field k and a polynomial $P\in k[X]$, we would like to construct a bigger field $k\subset K$, taken as small as possible, where the equation $P(x)=0$ has solutions. Thus, we are led to the study of the field extensions $k\subset K$, and the answers to the various questions that appear here are provided by a clever theory, called Galois theory.

1c. Number theory

More concretely now, and still talking about fields, we have several interesting fields coming from the prime numbers, such as the finite fields, or the p-adic numbers.

1d. Advanced topics

We discuss here some more advanced questions, in relation with various number theory problems, one theorem that we would like to understand, whose philosophy will be of use as well later on, when doing quantum algebra, being the Hasse-Minkowski theorem.

1e. Exercises

CHAPTER 2

Algebraic geometry

2a. Basic manifolds

There is nothing more basic in mathematics than the conics. These has been observed since ancient times, as being the trajectories of planets and comets around the Sun. So, let us start with some physics, following Kepler, Newton and others.

Physics has "laws", exactly as mathematics has axioms. These laws, exactly as the mathematical axioms, are simple, true statements, which cannot be abstractly proved, and fall into the "definition" category. In relation with our questions, we have:

Definition 2.1. *The force of attraction between two bodies of masses m_1, m_2 is*

$$F = g \cdot \frac{m_1 m_2}{d^2}$$

where d is the distance between them, and g is a constant.

Obviously, it is this definition which governs classical mechanics, and in particular the movement of planets and comets around the Sun. To be more precise, in order to solve these latter questions, we must examine and solve the "two-body problem", asking for the computation of the trajectories of two objects, of masses m_1, m_2 as above, known to be at time $t = 0$ at distance d apart, with given velocity vectors $v_1, v_2 \in \mathbb{R}^3$.

The answer to this question was first found by Kepler. Later on Newton came with Definition 2.1, and with the math going with it, leading to the following result:

Theorem 2.2. *When two bodies move freely, subject to the gravitation force between them only, the trajectory of a body with respect to the other is a conic.*

Proof. This is something very classical, which can be done as follows:

(1) Consider indeed two objects, of masses m_1, m_2, which at time $t = 0$ are at given positions $x_1, x_2 \in \mathbb{R}^3$, with given velocity vectors $v_1, v_2 \in \mathbb{R}^3$. Since things are relative, we can assume, for the purposes of our question, that the first object is fixed, say at the origin, and also that its initial velocity is zero:

$$x_1 = 0 \quad , \quad v_1 = 0$$

(2) The second observation is that, assuming that the first object stays at 0, as indicated above, the second object can only move in the plane passing through 0, through the

point of the initial position $x_2 \in \mathbb{R}^3$, and through the point corresponding to the initial velocity vector $v_2 \in \mathbb{R}^3$. Indeed, in view of Definition 2.1 above, which fully describes the forces acting on this second object, which will eventually produce its trajectory, there is nothing that can pull this second object out of this plane, upwards or downwards.

(3) Summing up, just with some abstract thinking, we have reduced the two-body problem in $\mathbb{R}^3$ to a sort of one-body problem in the plane $\mathbb{R}^2$. To be more precise, we have now just one object, lying intitially at a point $x \in \mathbb{R}^2$, with a given initial velocity $v \in \mathbb{R}^2$, and we would like to compute the trajectory of this object in the plane, knowing that the force which is acts upon it is given by the following formula, where d is the distance from our object to the origin, and G is a constant:

$$F = \frac{G}{d^2}$$

(4) But this latter question can be solved by doing some calculus, with this calculus being in fact invented by Newton, precisely for solving such questions, and we are led to the conclusion that the trajectory is a conic, as stated. □

All this is very nice, and we have now the choice of either staying with classical mechanics, perhaps by following Einstein, who later on pointed out that all the above is not exactly true, and needs some corrections, or by doing some relaxed mathematics, in relation in the conics. We will choose this latter way, leaving Einstein for later.

Regarding now the conics, regarded purely mathematically, this is in fact a very old topic as well, once again going back to ancient times, and more specifically to the Greeks, and we have the following result, summarizing their mathematics:

THEOREM 2.3. *The conics are exactly the degree 2 algebraic curves, and they can be as well characterized as being the curves which appear when cutting a cone. Moreover, they can be fully classified, as being elliptic, parabolic or hyperbolic.*

PROOF. This follows indeed by doing some mathematics, the computations being quite simple, and the only issue being that of taking care of various degenerate cases.

To be more precise, the situation is as follows:

(1) First of all, a degree 2 equation is what we got in Theorem 2.2, coming from the Newton equation, and so this will be our definition for the conics.

(2) Regarding now the cone cutting, this can be either done abstractly, leading to the abstract conclusion that the cut is described by a degree 2 equation, or more concretely, by examining the slope of the cut, and reaching to a classification of the cuts, which coincides with the classification of the conics, which is done below.

(3) Finally, regarding the classification of the conics, this can be done either directly, with some linear algebra involved, or by using the cone, as indicated above. □

As an interesting remark here, the classification of conics in basically 3 classes, namely elliptic, parabolic or hyperbolic, is something fundamental, and we will meet many other classifications of this type, in three classes, throughout this book.

For more complicated questions now, we are led into:

DEFINITION 2.4. *Algebraic curves.*

We can talk about surfaces as well, in about the same way:

DEFINITION 2.5. *Algebraic surfaces.*

More generally, we can talk about algebraic manifolds, as follows:

DEFINITION 2.6. *Algebraic manifolds.*

Our goal in what follows will be that of understanding how to deal with the algebraic manifolds. One big issue is that these are not necessarily smooth.

We will be actually interested in algebraic manifolds over arbitrary fields:

DEFINITION 2.7. *Algebraic manifolds over arbitrary fields.*

Here the motivation comes on one hand from the need to replace $k = \mathbb{R}$ by the "smarter" field $k = \mathbb{C}$, in connection with certain questions, and on the other hand from number theory, because many of the questions there can be formulated geometrically, in terms of certain algebraic manifolds, defined over certain fields of numbers.

2b. Commutative algebra

We have been so far into physics and geometry, and it is of course possible to keep going in this direction. But let us pull now some algebraic tricks.

The question that we would like to understand is how an algebraic manifold is determined by the ideal of polynomials vanishing on it.

Thus, we must do some commutative algebra.

As we will see, this best works when the ground field k is algebrically closed.

2c. Algebraic theory

As a continuation of the above, we can do now some advanced theory for the algebraic manifolds. There are many things that can be said here. As before, all this best works when $k = \mathbb{C}$, or more generally when k is algebrically closed.

2d. Projective geometry

An interesting feature of algebraic geometry is that this best works in the projective setting. We explore here this idea, with basic results about the projective manifolds. As before, all this best works when $k=\mathbb{C}$, or more generally when k is algebrically closed.

2e. Exercises

CHAPTER 3

Differential geometry

3a. Smooth manifolds

We have seen in the previous chapter that we have all sorts of interesting algebraic curves, surfaces and other manifolds. In the cases $k=\mathbb R$ or $k=\mathbb C$ these manifolds can be smooth or not, and we will focus here on the study in the smooth case.

The basic study of the smooth manifolds is in fact independent from algebraic geometry, and at the level of motivations, there are plenty of them, coming from physics. But more on this later. Let us start with a very simple definition, as follows:

Definition 3.1. *A smooth manifold is a space X which is locally isomorphic to $\mathbb R^N$. To be more precise, this space X must be covered by charts, bijectively mapping open pieces of it to open pieces of $\mathbb R^N$, with the changes of charts being C^∞ functions.*

As a basic example, we have $\mathbb R^N$ itself, or any open subset $X\subset\mathbb R^N$. Another example is the circle, or things like ellipses and so on, for obvious reasons. To be more precise, the unit circle can be covered by 2 charts as above, by using polar coordinates, in the obvious way, and then by applying dilations, translations and other such transformations, namely bijections which are smooth, we obtain a whole menagery of circle-looking manifolds.

More generally now, we have as example the unit sphere in $\mathbb R^N$, and smooth deformations of it, once again, somehow by obvious reasons. In case you are wondering on how to construct explicit charts for the sphere, the answer comes from:

Theorem 3.2. *We have spherical coordinates in N dimensions,*

$$\begin{cases}x_1&=\ r\cos t_1\\ x_2&=\ r\sin t_1\cos t_2\\ \vdots\\ x_{N-1}&=\ r\sin t_1\sin t_2\ldots\sin t_{N-2}\cos t_{N-1}\\ x_N&=\ r\sin t_1\sin t_2\ldots\sin t_{N-2}\sin t_{N-1}\end{cases}$$

the corresponding Jacobian being given by the following formula:

$$J(r,t)=r^{N-1}\sin^{N-2}t_1\sin^{N-3}t_2\,\ldots\,\sin^2t_{N-3}\sin t_{N-2}$$

Proof. Here the fact that we have indeed spherical coordinates is clear, with the only point to be clarified being the identification of the precise ranges of the angles, which follows from some geometric thinking, first at $N = 2, 3$, and then in general.

As for the formula of the Jacobian, that we will not exactly need at this point, let us prove this as well, because, as per analysis philosophy, no change of coordinates should ever be formulated without computing its Jacobian, and death penalty otherwise. By developing the Jacobian determinant over the last column, we have:

$$\begin{aligned} J_N &= r\sin t_1 \ldots \sin t_{N-2} \sin t_{N-1} \times \sin t_{N-1} J_{N-1} \\ &+ r\sin t_1 \ldots \sin t_{N-2} \cos t_{N-1} \times \cos t_{N-1} J_{N-1} \\ &= r\sin t_1 \ldots \sin t_{N-2} (\sin^2 t_{N-1} + \cos^2 t_{N-1}) J_{N-1} \\ &= r\sin t_1 \ldots \sin t_{N-2} J_{N-1} \end{aligned}$$

Thus, we obtain the formula in the statement, by recurrence. □

In relation with these interesting questions, namely parametrizing the spheres, we have the stereographic projection as well, which works as follows:

Theorem 3.3. *The stereographic projection is given by inverse maps*

$$\Phi : \mathbb{R}^N \to S^N_{\mathbb{R}} - \{\infty\}$$
$$\Psi : S^N_{\mathbb{R}} - \{\infty\} \to \mathbb{R}^N$$

which are given by the following formulae,

$$\Phi(v) = (1, 0) + \frac{2}{1 + ||v||^2}(-1, v)$$
$$\Psi(c, x) = \frac{x}{1 - c}$$

with the convention $\mathbb{R}^{N+1} = \mathbb{R} \times \mathbb{R}^N$, and with the coordinate of $\mathbb{R}$ denoted x_0, and with the coordinates of $\mathbb{R}^N$ denoted $x_1, \ldots, x_N$.

Proof. We are looking for the formulae of the isomorphism $\mathbb{R}^N \simeq S^N_{\mathbb{R}} - \{\infty\}$, obtained by identifying $\mathbb{R}^N = \mathbb{R}^N \times \{0\} \subset \mathbb{R}^{N+1}$ with the unit sphere $S^N_{\mathbb{R}} \subset \mathbb{R}^{N+1}$, with the convention that the point which is added is $\infty = (1, 0, \ldots, 0)$, via the stereographic projection. That is, we need the precise formulae of two inverse maps, as follows:

$$\Phi : \mathbb{R}^N \to S^N_{\mathbb{R}} - \{\infty\}$$
$$\Psi : S^N_{\mathbb{R}} - \{\infty\} \to \mathbb{R}^N$$

In one sense we must have $\Phi(v) = t(0, v) + (1 - t)(1, 0)$, with $t \in (0, 1)$ being such that $||\Phi(v)|| = 1$. The equation here is $(1 - t)^2 + t^2||v||^2 = 1$, which simplifies to $t^2(1 + ||v||^2) = 2t$, with solution $t = \frac{2}{1+||v||^2}$, and so the formula of Φ is as follows:

$$\Phi(v) = (1, 0) + \frac{2}{1 + ||v||^2}(-1, v)$$

In the other sense we must have $(0, \Psi(c, x)) = \alpha(c, x) + (1 - \alpha)(1, 0)$ for a certain $\alpha \in \mathbb{R}$, and from $\alpha c + 1 - \alpha = 0$ we get $\alpha = \frac{1}{1-c}$, so the formula of Ψ is as follows:

$$\Psi(c, x) = \frac{x}{1 - c}$$

Here, as before, we use the convention in the statement, namely $\mathbb{R}^{N+1} = \mathbb{R} \times \mathbb{R}^N$, with the coordinate of $\mathbb{R}$ denoted x_0, and with the coordinates of $\mathbb{R}^N$ denoted $x_1, \ldots, x_N$. □

Leaving aside now the spheres, or rather keeping them for later, let us systematically study the abstract smooth manifolds X, as appearing in Definition 3.1. Most of these examples appear as submanifolds of $\mathbb{R}^N$, and we have here the following useful result:

THEOREM 3.4. *Smooth submanifolds $X \subset \mathbb{R}^N$.*

PROOF. This follows from the basic theorems of multivariable calculus. □

Next in line, let us discuss the computation of lengths, geodesics, curvature, and other such things. In order to do so, we must upgrade Definition 3.1, as follows:

DEFINITION 3.5. *Riemannian manifolds X.*

As an important philosophical question, we must understand if the Riemannian manifolds X, as axiomatized above, appear or not as smooth submanifolds $X \subset \mathbb{R}^N$. The answer here is yes, due to a famous theorem of Nash, and more on this later.

3b. Algebraic structure

We discuss here various abstract algebraic techniques in the study of the smooth and Riemannian manifolds. Among others, we will need, for some use later on, the construction of the Hodge Laplacian for a Riemannian manifold, which is as follows:

DEFINITION 3.6. *Given a compact Riemannian manifold X, we denote by $\Omega^1(X)$ the space of smooth 1-forms on X, with scalar product given by*

$$< \omega, \eta > = \int_X < \omega(x), \eta(x) > dx$$

and we construct the Hodge Laplacian $\Delta : L^2(X) \to L^2(X)$ by setting

$$\Delta = d^* d$$

where $d : C^\infty(X) \to \Omega^1(X)$ is the usual differential map, and d^* is its adjoint.

According to a standard differential geometry result, whose proof is elementary, the classical isometry group $\mathcal{G}(X)$ of our Riemannian manifold X is then the group of diffeomorphisms $\varphi : X \to X$ whose induced action on $C^\infty(X)$ commutes with Δ. This is something that we will use later on in this book, when doing quantum algebra.

Finally, among the questions which are left, we still have to clarify the relation between the abstract Riemannian manifolds X and the Riemannian submanifolds $X \subset \mathbb{R}^N$, and we have to discuss as well the relation between the geometry developed in this chapter and the algebraic geometry developed in the previous chapter, over $k = \mathbb{R}$ or $k = \mathbb{C}$. There are many interesting theorems here, due to Nash, Serre and others.

3c. Space and time

Let us go back to the Newton formula for gravitation from chapter 2, namely:

$$F = g \cdot \frac{m_1 m_2}{d^2}$$

In order to improve the Newton theory, one possible starting point is the obvious fact that a feather falls slower than a rock. However, we would like talk here about something else, namely Einstein's discovery that, even when avoiding feathers and talking about rocks, there are still some corrections needed. As crazy as this might seem.

In order to discuss this, let us make first a detour through electricity. Here the basic equation, due to Coulomb, is almost identical to the Newton equation, namely:

$$F = k \cdot \frac{q_1 q_2}{d^2}$$

Thus, save for the signs of the charges, which can be now arbitrary real numbers, $q_1, q_2 \in \mathbb{R}$, as opposed to the previous masses which must be positive, $m_1, m_2 \in \mathbb{R}_+$, we should normally expect a similar mathematical theory here. However, this is false, because unlike what happens with gravitation, moving electric charges lead to magnetism, which considerably complicates things, and in the end, we obtain something quite different.

To be more precise, we obtain, as correct analogue of the Newton equation for gravitation, the Maxwell equations for electromagnetism. However, and comes now the true exciting point, the time variable t, which was something dull and linear in the Newton gravitation theory, becomes something quite subtle, mixing somehow with the space, in the Maxwell electricity theory. So, who is right about time, Newton or Maxwell?

Looking back retrospectively, probably Maxwell. Indeed, what do we really know about things travelling at very fast speeds, such as the speed of light: if there is a weak point in our intuition and knowledge, this is precisely about things travelling at fast speeds. And the point is that Maxwell's equations are precisely about this, charges travelling at fast speeds. So these equations are probably correct, and Newton's, probably not.

So this is the story, and it is of course easy to comment now on all this, from the confort of an early 21th century Ikea sofa. However, things were not at all like this during the end of the 19th century, with Newton gravity being at that time a well-established theory, and the pride of physics, and with electromagnetism being something quite modern. The full credit for discovering all this, namely that Newton was in fact wrong, goes to Einstein.

Based on this, and on some exciting astronomy data as well, Einstein proposed a series of corrections to Newton's mechanics. Quite remarkably, from an abstract, and slightly retrospective viewpoint, all these corrections come from one single axiom, namely:

Definition 3.7. *The speed of light is a constant c, to all observers.*

Based on this new axiom, it is possible to change and clarify everything, in a somewhat uniquely determined manner, and with the consequences going far beyond the above-mentioned space-time "mixing", and including the following well-known formula:

$$E = mc^2$$

Now back to mechanics, mathematically we are now in curved space-time, and the tools for understanding this come from differential geometry, as developed above.

3d. Electricity and heat

Now that we fully clarified classical mechanics, let us discuss some further physics topics, such as waves, electricity and heat.

As before, we will attempt to present things axiomatically. First, in relation with the waves, we have a rock-solid result, as follows:

Theorem 3.8. *Wave equation.*

Proof. This is well-known and elementary, coming by modelling the space with a network of small balls connected by springs, and taking the $N \to \infty$ limit of this. □

The main idea in the above proof, namely considering a "lattice model", and then taking the $N \to \infty$ limit of what we found, is something very fruitful, that we will meet on several occasions later on, when talking about quantum algebra.

In relation now with electricity, we have:

Theorem 3.9. *Laplace equation.*

Proof. This can be deduced axiomatically too, starting from the Coulomb law and its consequences, by using some geometry, and more specifically, integration theory. □

Regarding now heat, we have here:

Theorem 3.10. *Heat equation.*

Proof. Again, this can be somewhat deduced axiomatically too, by using some mathematical tools, such as geometry and probability. □

Summarizing, we are led into questions involving the Laplacian. And these questions lead us into harmonic functions, and related mathematical questions.

3e. Exercises

CHAPTER 4

Topology, K-theory

4a. Topological spaces

We have seen so far that some interesting algebraic theory can be developed for the algebraic manifolds, and for the smooth manifolds as well. In this chapter we present some further algebraic results, this time about "manifolds" taken in a very large sense, basically meaning topological spaces, without further assumptions on them.

As a starting point, now that we do not care anymore about algebraic or differential equations and structure, we are free to study the "shape" of our spaces X, from a purely topological viewpoint. A first natural construction here is as follows:

Definition 4.1. *The homotopy group $\pi_1(X)$ of a connected space X is the group of loops based at a given point $* \in X$, with the following conventions,*

1. *Two such loops are identified when one can pass continuously from one loop to the other, via a family of loops indexed by $t \in [0,1]$,*
2. *The composition of two such loops is the obvious one, namely is the loop obtaining by following the first loop, then the second loop,*
3. *The unit loop is the null loop at $*$, which stays there, and the inverse of a given loop is the loop itself, followed backwards,*

with the remark that the group $\pi_1(X)$ defined in this way does not depend on the choice of the given point $* \in X$, where the loops are based.

Here the fact that $\pi_1(X)$ defined in this way is indeed a group is obvious, and obvious as well is the fact that, since X is assumed to be connected, this group does not depend on the choice of the given point $* \in X$, where the loops are based.

As basic examples, for spaces having "no holes", such as $\mathbb{R}$ itself, or $\mathbb{R}^N$, and so on, we have $\pi_1 = \{1\}$. In fact, having no holes can only mean, by definition, that $\pi_1 = \{1\}$.

As further illustrations, here are now a few basic computations:

Proposition 4.2. *We have the following computations of homotopy groups:*

1. *For the circle, we obtain $\pi_1 = \mathbb{Z}$.*
2. *For the torus, we obtain $\pi_1 = \mathbb{Z} \times \mathbb{Z}$.*

PROOF. These results are all standard, as follows:

(1) The first assertion is clear, because a loop on the circle must wind $n \in \mathbb{Z}$ times around the center, and this parameter $n \in \mathbb{Z}$ uniquely determines the loop, up to the identification in Definition 4.1. Thus, the homotopy group of the circle is the group of such parameters $n \in \mathbb{Z}$, which is of course the group $\mathbb{Z}$ itself.

(2) In what regards now the second assertion, the torus being a product of two circles, we are led to the conclusion that its homotopy group must be some kind of product of $\mathbb{Z}$ with itself. But pictures show that the two standard generators of $\mathbb{Z}$, and so the two copies of $\mathbb{Z}$ themselves, commute, $gh = hg$, and so we obtain the product of $\mathbb{Z}$ with itself, subject to commutation, which is the usual product $\mathbb{Z} \times \mathbb{Z}$. □

There are many interesting things that can be said about homotopy groups.

4b. K-theory groups

Another thing that can be done with the arbitrary spaces X, again in relation with studying their "shape", is that of looking at the fiber bundles over them, again up to continuous deformation. We are led in this way into a group, called $K_0(X)$.

We can construct as well, along the same lines, but in a bit more complicated way, a higher K-theory group, called $K_1(X)$, and in fact higher K-theory groups $K_i(X)$ too, for any $i \in \mathbb{N}$. But these latter groups are in fact periodic, due to a subtle result, called Bott periodicty, and so we end up with two main groups, namely $K_0(X)$ and $K_1(X)$.

4c. Advanced algebra

All the above, and especially K-theory and its mysteries, has led to a lot of further algebraic work, along the same lines, but much more technical, due to Atiyah, Connes, Kasparov and many others.

4d. Knots and beyond

Leaving general manifolds aside, let us focus now on the simplest objects of topology, namely the knots. Given a closed curve, say via its equations, is is tied or not, and if tied, how complicated is it tied, and how to untie it? Difficult questions.

Perhaps simpler now, experience with cables and ropes shows that a random closed curve is usually tied. But can we really prove this? Once again, difficult question.

More modestly now, let us try to construct some knot invariants. A natural idea is that of defining the invariant on the 2D picture of the knot, that is, on a plane projection of the knot, and then proving that the invariant is indeed independent on the chosen plane. This method rests on the following technical result:

PROPOSITION 4.3. *Reidemeister moves.*

PROOF. This is somewhat clear from definitions. □

In order to construct invariants, we will need:

DEFINITION 4.4. *Braid group.*

We can throw in a field k, and we have here:

DEFINITION 4.5. *The Temperley-Lieb algebra of index $N \in [1, \infty)$ is defined as*

$$TL_N(k) = span(NC_2(k,k))$$

with product given by vertical concatenation, with the rule

$$\bigcirc = N$$

for the closed circles that might appear when concatenating.

In other words, the algebra $TL_N(k)$, depending on parameters $k \in \mathbb{N}$ and $N \in [1, \infty)$, is the formal linear span of the pairings $\pi \in NC_2(k,k)$. The product operation is obtained by linearity, for the pairings which span $TL_N(k)$ this being the usual vertical concatenation, with the conventions that things go "from top to bottom", and that each circle that might appear when concatenating is replaced by a scalar factor, equal to N.

As a side comment here, such diagram algebras are commonplace in mathematics, with the multiplication convention being however different, depending on the source. Here we use the simplest such multiplication convention, vertical concatenation with things going "from top to bottom", via natural gravitation, and therefore not heating our planet.

This algebra was discovered by Temperley and Lieb in the context of general statistical mechanics, and we refer here to [86], and subsequent work.

With such algebraic technology in hand, we can construct lots of interesting knot invariants, by using the above-mentioned method, namely projecting the knot on a plane, doing 2D mathematics, and then proving that this 2D mathematics lifts well into 3D.

To be more precise, we have the Alexander polynomial, then the Jones polynomial, and then more complicated invariants.

There is a relation with statistical mechanics here, following Jones, happening in 2D as well, the idea being that "interactions happen at crossings", and it is these interactions that produce the knot invariant, as a kind of partition function.

There are as well interesting some connections with fluid mechanics.

These invariants can be directly understood in 3D as well, in a purely geometric way, with elegance, and no need for 2D reduction. But this is a far more complicated story, due to Witten and others, to be left for later on, when discussing quantum algebra.

4e. Exercises

Part II

Quantum algebra

Join me for a ride
Speed up the music
Join me for a ride
Maximum overdrive

CHAPTER 5

Group theory

5a. Finite groups

We have seen so far that we have several interesting correspondences between spaces X and algebras A, coming from classical geometry, and with motivation from classical physics. These latter algebras A are usually commutative, $ab = ba$, by construction.

In what follows we study the same types of algebras A, but this time in the general, noncommutative situation, $ab \neq ba$. With the hope of reaching in this way to a nice theory of "quantum spaces" X, and with the extra hope that all these quantum algebras A and quantum spaces X can be of help with quantum mechanics, and related topics.

All this is of course a bit speculatory, scientifically speaking. Mathematically, things are quite delicate as well, due to our lack of intuition, and of clear motivations. The starting point for everything is traditionally the group case, $X = G$, where many things can be done. We will dedicate this whole second part of the present book to the group case, and leave quotient spaces $X = G/H$ and more general manifolds X for later.

In order to get started now, we are in need of a crash course on group theory. As usual in this book, we will adopt a rather algebraic viewpoint on all this, with the idea in mind that, afterwards, all our algebraic techniques will extend well. And in order to reach to algebra, we will attempt to understand the groups G via their representation theory.

Let us first discuss the finite group case. The starting definition is as follows:

Definition 5.1. *A representation of a finite group G is a group morphism, as follows:*

$$u : G \to U_N$$

The character of such a representation is the function $\chi : G \to \mathbb{C}$ given by

$$g \to Tr(u_g)$$

where Tr is the usual trace of the $N \times N$ matrices, $Tr(M) = \sum_i M_{ii}$.

As examples, for any group we have available the trivial representation $u : G \to U_1$, mapping $g \to (1)$, as well as the null representation $u : G \to U_1$, mapping $g \to (0)$.

At the level of non-trivial examples now, most of the known groups naturally appear as closed subgroups $G\subset U_N$. In this case, the embedding $G\subset U_N$ is of course a representation, called fundamental representation:

$$u:G\subset U_N\quad,\quad g\to g$$

In this situation, there are many other representations of G, which are equally interesting. For instance, we can define the representation conjugate to u, as being:

$$\bar{u}:G\subset U_N\quad,\quad g\to\bar{g}$$

In order to clarify all this, and see which representations are available, let us first discuss the various operations on the representations. The result here is as follows:

PROPOSITION 5.2. *The representations of a given group G are subject to the following operations:*

(1) *Making sums. Given a N-dimensional representation u and a M-dimensional representation v, their sum is the $N+M$-dimensional representation:*
$$u+v=diag(u,v)$$
(2) *Making products. Given a N-dimensional representation u and a M-dimensional representation v, their tensor product is the NM-dimensional representation:*
$$(u\otimes v)_{ia,jb}=u_{ij}v_{ab}$$
(3) *Taking conjugates. Given a N-dimensional representation u, its conjugate is following the N-dimensional representation:*
$$(\bar{u})_{ij}=\bar{u}_{ij}$$
(4) *Spinning by unitaries. Given a N-dimensional representation u, and a unitary $V\in U_N$, we can spin u by this unitary V, as follows:*
$$u\to VuV^*$$

PROOF. The fact that the operations in the statement are indeed well-defined, among maps from G to unitary groups, is standard, and this leads to the above conclusions. □

In relation now with characters, we have the following result:

PROPOSITION 5.3. *We have the following formulae, regarding characters*

$$\chi_{u+v}=\chi_u+\chi_v$$
$$\chi_{u\otimes v}=\chi_u\chi_v$$
$$\chi_{\bar{u}}=\bar{\chi}_u$$
$$\chi_{VuV^*}=\chi_u$$

in relation with the basic operations for the representations.

PROOF. All these assertions are elementary, by using the following well-known trace formulae, valid for any two square matrices g, h, and any unitary V:

$$Tr(diag(g,h)) = Tr(g) + Tr(h)$$

$$Tr(g \otimes h) = Tr(g)Tr(h)$$

$$Tr(\bar{g}) = \overline{Tr(g)}$$

$$Tr(VgV^*) = Tr(g)$$

Thus, we are led to the conclusions in the statement. □

Assume now that we are given a finite subgroup $G \subset U_N$. By using the above operations, we can construct a whole family of representations of G, as follows:

DEFINITION 5.4. *Given a finite subgroup $G \subset U_N$, its Peter-Weyl representations are the tensor products between the fundamental representation and its conjugate:*

$$u : G \subset U_N$$

$$\bar{u} : G \subset U_N$$

We denote these tensor products $u^{\otimes k}$, with $k = \circ \bullet \bullet \circ \ldots$ being a colored integer, with the colored tensor powers being defined according to the rules

$$u^{\otimes \circ} = u$$

$$u^{\otimes \bullet} = \bar{u}$$

$$u^{\otimes kl} = u^{\otimes k} \otimes u^{\otimes l}$$

and with the convention that $u^{\otimes \emptyset}$ is the trivial representation $1 : G \to U_1$.

Here are a few examples of such Peter-Weyl representations, namely those coming from the colored integers of length 2, to be often used in what follows:

$$u^{\otimes \circ\circ} = u \otimes u \quad , \quad u^{\otimes \circ\bullet} = u \otimes \bar{u}$$

$$u^{\otimes \bullet\circ} = \bar{u} \otimes u \quad , \quad u^{\otimes \bullet\bullet} = \bar{u} \otimes \bar{u}$$

In relation now with characters, we have the following result:

PROPOSITION 5.5. *The characters of the Peter-Weyl representations are given by*

$$\chi_{u^{\otimes k}} = (\chi_u)^k$$

with the colored powers of a variable χ being by definition given by

$$\chi^\circ = \chi$$

$$\chi^\bullet = \bar{\chi}$$

$$\chi^{kl} = \chi^k \chi^l$$

and with the convention that $\chi^\emptyset$ equals by definition 1.

Proof. This follows indeed from the additivity, multiplicativity and conjugation formulae established in Proposition 5.3 above, via the conventions in Definition 5.4. □

In order to advance, we must develop some general theory. Let us start with:

Definition 5.6. *Given a finite group G, and two of its representations,*

$$u : G \to U_N$$

$$v : G \to U_M$$

we define the linear space of intertwiners between these representations as being

$$Hom(u,v) = \left\{ T \in M_{M\times N}(\mathbb{C}) \Big| Tu(g) = v(g)T, \forall g \in G \right\}$$

and we use the following conventions:

(1) *We use the notations $Fix(u) = Hom(1,u)$, and $End(u) = Hom(u,u)$.*
(2) *We write $u \sim v$ when $Hom(u,v)$ contains an invertible element.*
(3) *We say that u is irreducible, and write $u \in Irr(G)$, when $End(u) = \mathbb{C}1$.*

The terminology here is standard, with Hom and End standing for "homomorphisms" and "endomorphisms" between the representations in question, and with Fix standing for "fixed points". It is useful to think of the representations of G as being the "objects" of some kind of abstract picture of G, of rather combinatorial nature, and of the intertwiners between these representations as being the "arrows" between these objects.

Here are a few basic results, regarding the above intertwiner spaces:

Proposition 5.7. *We have the following results:*

(1) *The intertwiners are stable under composition:*
$$T \in Hom(u,v)\ ,\ S \in Hom(v,w) \implies ST \in Hom(u,w)$$
(2) *The intertwiners are stable under taking tensor products:*
$$S \in Hom(u,v)\ ,\ T \in Hom(w,t) \implies S \otimes T \in Hom(u \otimes w, v \otimes t)$$
(3) *The intertwiners are stable under taking adjoints:*
$$T \in Hom(u,v) \implies T^* \in Hom(v,u)$$

In abstract terms, we say that the Hom spaces form a tensor $*$-category.

Proof. All the formulae in the statement are clear from definitions, or rather follow from some elementary computations, based on the main definition, namely:

$$Hom(u,v) = \left\{ T \in M_{M\times N}(\mathbb{C}) \Big| Tu(g) = v(g)T, \forall g \in G \right\}$$

As for the last assertion, this is something coming from (1,2,3). We will be back to tensor categories later on, with more details on all this. □

As a main consequence of the above result, we have:

PROPOSITION 5.8. *Given a representation $u : G \to U_N$, the corresponding End space*

$$End(u) \subset M_N(\mathbb{C})$$

is a $*$-algebra, with respect to the usual involution of the matrices.

PROOF. By definition, $End(u)$ is a linear subspace of $M_N(\mathbb{C})$. We know from Proposition 5.7 (1) that this subspace $End(u)$ is a subalgebra of $M_N(\mathbb{C})$, and then we know as well from Proposition 5.7 (3) that this subalgebra is stable under the involution $*$. Thus, what we have here is a $*$-subalgebra of $M_N(\mathbb{C})$, as claimed. □

In order to exploit this fact, we will need a basic result, as follows:

PROPOSITION 5.9. *Let $A \subset M_N(\mathbb{C})$ be a $*$-algebra.*

(1) *The unit decomposes as follows, with $p_i \in A$ being central minimal projections:*

$$1 = p_1 + \ldots + p_k$$

(2) *Each of the following linear spaces is a non-unital $*$-subalgebra of A:*

$$A_i = p_i A p_i$$

(3) *We have a non-unital $*$-algebra sum decomposition, as follows:*

$$A = A_1 \oplus \ldots \oplus A_k$$

(4) *We have unital $*$-algebra isomorphisms as follows, with $N_i = rank(p_i)$:*

$$A_i \simeq M_{N_i}(\mathbb{C})$$

(5) *Thus, we have a $*$-algebra isomorphism as follows:*

$$A \simeq M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$$

PROOF. Consider indeed an arbitrary $*$-algebra of the $N \times N$ matrices, $A \subset M_N(\mathbb{C})$. Let us first look at the center of this algebra, which given by:

$$Z(A) = A \cap A'$$

It is elementary to prove that this center, as an algebra, is of the following form:

$$Z(A) \simeq \mathbb{C}^k$$

Consider now the standard basis $e_1, \ldots, e_k \in \mathbb{C}^k$, and let $p_1, \ldots, p_k \in Z(A)$ be the images of these vectors via the above identification. In other words, these elements $p_1, \ldots, p_k \in A$ are central minimal projections, summing up to 1:

$$p_1 + \ldots + p_k = 1$$

The idea is then that this partition of the unity will eventually lead to the block decomposition of A, as in the statement. We prove this in 4 steps, as follows:

Step 1. We first construct the matrix blocks, our claim here being that each of the following linear subspaces of A are non-unital $*$-subalgebras of A:

$$A_i = p_i A p_i$$

But this is clear, with the fact that each A_i is closed under the various non-unital $*$-subalgebra operations coming from the projection equations $p_i^2 = p_i = p_i^*$.

Step 2. We prove now that the above algebras $A_i \subset A$ are in a direct sum position, in the sense that we have a non-unital $*$-algebra sum decomposition, as follows:

$$A = A_1 \oplus \ldots \oplus A_k$$

As with any direct sum question, we have two things to be proved here. First, by using the formula $p_1 + \ldots + p_k = 1$ and the projection equations $p_i^2 = p_i = p_i^*$, we conclude that we have the needed generation property, namely:

$$A_1 + \ldots + A_k = A$$

As for the fact that the sum is indeed direct, this follows as well from the formula $p_1 + \ldots + p_k = 1$, and from the projection equations $p_i^2 = p_i = p_i^*$.

Step 3. Our claim now, which will finish the proof, is that each of the $*$-subalgebras $A_i = p_i A p_i$ constructed above is a full matrix algebra. To be more precise here, with $r_i = rank(p_i)$, our claim is that we have isomorphisms, as follows:

$$A_i \simeq M_{r_i}(\mathbb{C})$$

In order to prove this claim, recall that the projections $p_i \in A$ were chosen central and minimal. Thus, the center of each of the algebras A_i reduces to the scalars:

$$Z(A_i) = \mathbb{C}$$

But this shows, either via a direct computation, or via the bicommutant theorem, that the each of the algebras A_i is a full matrix algebra, as claimed.

Step 4. We can now obtain the result, by putting together what we have. Indeed, by using the results from Step 2 and Step 3, we obtain an isomorphism as follows:

$$A \simeq M_{r_1}(\mathbb{C}) \oplus \ldots \oplus M_{r_k}(\mathbb{C})$$

Moreover, a careful look at the isomorphisms established in Step 3 shows that at the global level, of the algebra A itself, the above isomorphism simply comes by twisting the following standard multimatrix embedding, discussed in the beginning of the proof, (1) above, by a certain unitary matrix $U \in U_N$:

$$M_{r_1}(\mathbb{C}) \oplus \ldots \oplus M_{r_k}(\mathbb{C}) \subset M_N(\mathbb{C})$$

Now by putting everything together, we obtain the result. □

We can now formulate our first Peter-Weyl type theorem, as follows:

Theorem 5.10 (PW1). *Let $u : G \to U_N$ be a group representation, consider the algebra $A = End(u)$, and write its unit as above, as follows:*

$$1 = p_1 + \ldots + p_k$$

We have then a decomoposition result for u, as follows,

$$u = u_1 + \ldots + u_k$$

with each u_i being an irreducible representation, obtained by restricting u to $Im(p_i)$.

Proof. This follows from Proposition 5.8 and Proposition 5.9, as follows:

(1) We first associate to our representation $u : G \to U_N$ the corresponding action map on $\mathbb{C}^N$. If a linear subspace $V \subset \mathbb{C}^N$ is invariant, the restriction of the action map to V is an action map too, which must come from a subrepresentation $v \subset u$.

(2) Consider now a projection $p \in End(u)$. From $pu = up$ we obtain that the linear space $V = Im(p)$ is invariant under u, and so this space must come from a subrepresentation $v \subset u$. It is routine to check that the operation $p \to v$ maps subprojections to subrepresentations, and minimal projections to irreducible representations.

(3) With these preliminaries in hand, let us decompose the algebra $End(u)$ as in Proposition 5.9, by using the decomposition $1 = p_1 + \ldots + p_k$ into minimal projections. If we denote by $u_i \subset u$ the subrepresentation coming from the vector space $V_i = Im(p_i)$, then we obtain in this way a decomposition $u = u_1 + \ldots + u_k$, as in the statement. □

Here is now our second Peter-Weyl theorem, complementing Theorem 5.10:

Theorem 5.11 (PW2). *Given a closed subgroup $G \subset_u U_N$, any of its irreducible smooth representations*

$$v : G \to U_M$$

appears inside a tensor product of the fundamental representation u and its adjoint $\bar{u}$.

Proof. In order to prove the result, we will use the following three elementary facts, regarding the spaces of coefficients introduced above:

(1) The construction $v \to C_v$ is functorial, in the sense that it maps subrepresentations into linear subspaces. This is indeed something which is routine to check.

(2) Also, it is clear that we have an inclusion of linear spaces as follows:

$$C_v \subset < g_{ij} >$$

(3) By definition of the Peter-Weyl representations, as arbitrary tensor products between the fundamental representation u and its conjugate $\bar{u}$, we have:

$$< g_{ij} > = \sum_k C_{u^{\otimes k}}$$

Now by putting together the observations (2,3) we conclude that we must have an inclusion as follows, for certain exponents $k_1,\ldots,k_p$:

$$C_v\subset C_{u^{\otimes k_1}\oplus\ldots\oplus\pi^{\otimes k_p}}$$

By using now the functoriality result from (1), we deduce from this that we have an inclusion of representations, as follows:

$$v\subset u^{\otimes k_1}\oplus\ldots\oplus u^{\otimes k_p}$$

Together with Theorem 5.10, this leads to the conclusion in the statement. □

In order to further develop now the Peter-Weyl theory, which is something very useful, we will need the following result, which is of independent interest:

PROPOSITION 5.12. *We have a Frobenius type isomorphism*

$$Hom(v,w)\simeq Fix(v\otimes\bar{w})$$

valid for any two representations v,w.

PROOF. According to the definitions, we have the following equivalences:

$$\begin{aligned}T\in Hom(v,w)&\iff Tv=wT\\&\iff \sum_jT_{aj}v_{ji}=\sum_bw_{ab}T_{bi},\forall a,i\end{aligned}$$

On the other hand, we have as well the following equivalences:

$$\begin{aligned}T\in Fix(v\otimes\bar{w})&\iff (v\otimes\bar{w})T=\xi\\&\iff \sum_{jb}v_{ij}w_{ab}^*T_{bj}=T_{ai}\forall a,i\end{aligned}$$

With these formulae in hand, both inclusions follow from the unitarity of v,w. □

We can now formulate our third Peter-Weyl theorem, as follows:

THEOREM 5.13 (PW3). *We have a direct sum decomposition as follows,*

$$C(G)=\bigoplus_{v\in Irr(G)}M_{\dim(v)}(\mathbb{C})$$

with this being an isomorphism of $*$-coalgebras, and with the summands being pairwise orthogonal with respect to the scalar product given by averaging over G.

PROOF. By combining the previous two Peter-Weyl results, we deduce that we have a linear space decomposition as follows:

$$C(G)=\sum_{v\in Irr(G)}M_{\dim(v)}(\mathbb{C})$$

Thus, in order to conclude, it is enough to prove that for any two irreducible corepresentations $v,w\in Irr(A)$, the corresponding spaces of coefficients are orthogonal:

$$v\not\sim w\implies C_v\perp C_w$$

But this follows from Proposition 5.12. Let us set indeed:

$$P_{ia,jb}=\int_Gv_{ij}w_{ab}^*$$

Then P is the orthogonal projection onto the following vector space:

$$Fix(v\otimes\bar{w})\simeq Hom(v,w)=\{0\}$$

Thus we have $P=0$, and this gives the result. □

Finally, we have the following result, completing the Peter-Weyl theory:

Theorem 5.14 (PW4). *The characters of the irreducible representations of G belong to the algebra*

$$C(G)_{central}=\left\{a\in C(G)\Big|\Sigma\Delta(a)=\Delta(a)\right\}$$

of smooth central functions on G, and form an orthonormal basis of it.

Proof. We have several things to be proved, the idea being as follows:

(1) Observe first that $C(G)_{central}$ is indeed an algebra, which contains all the characters. Conversely, consider an element $a\in C(G)$, written as follows:

$$a=\sum_{v\in Irr(G)}a_v$$

The condition $a\in C(G)_{central}$ states then that for any $v\in Irr(G)$, we must have:

$$a_v\in C(G)_{central}$$

But this means precisely that the coefficient a_v must be a scalar multiple of χ_v, and so the characters form a basis of $C(G)_{central}$, as stated.

(2) The fact that we have an orthogonal basis follows from Theorem 5.13.

(3) As for the fact that the characters have norm 1, this follows from:

$$\begin{aligned}\int_G\chi_v\chi_v^*&=\sum_{ij}\int_Gv_{ii}v_{jj}^*\\&=\sum_i\frac{1}{N}\\&=1\end{aligned}$$

Here we have used the fact, coming from Theorem 5.13, that the integrals $\int_G v_{ij}v_{kl}^*$ form the orthogonal projection onto the following vector space:

$$Fix(v\otimes\bar{v})\simeq End(v)=\mathbb{C}1$$

Thus, the proof of our theorem is now complete. □

5b. Discrete groups

Here the interesting theory regards the growth, and the notion of amenability. Of particular interest is the Kesten formulation of the notion of amenability.

5c. Compact groups

In the compact group case, the first two Peter-Weyl theorems can be established exactly as in the finite group case. For the remaining two Peter-Weyl theorems, we need to talk about integration over G. Let us begin with the following key result:

PROPOSITION 5.15. *Given a unital positive linear form $\varphi:C(G)\to\mathbb{C}$, the limit*

$$\int_\varphi f=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}(f)$$

exists, and for a coefficient of a representation $f=(\tau\otimes id)v$ we have

$$\int_\varphi f=\tau(P)$$

where P is the orthogonal projection onto the 1-eigenspace of $(id\otimes\varphi)v$.

PROOF. By linearity it is enough to prove the first assertion for elements of the following type, where v is a Peter-Weyl representation, and τ is a linear form:

$$a=(\tau\otimes id)v$$

Thus we are led into the second assertion, and more precisely we can have the whole result proved if we can establish the following formula, with $a=(\tau\otimes id)v$:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}(a)=\tau(P)$$

In order to prove this latter formula, observe that we have:

$$\begin{aligned}\varphi^{*k}(a)&=(\tau\otimes\varphi^{*k})v\\&=\tau((id\otimes\varphi^{*k})v)\end{aligned}$$

Let us set $M=(id\otimes\varphi)v$. In terms of this matrix, we have:

$$\begin{aligned}((id\otimes\varphi^{*k})v)_{i_0i_{k+1}} &= \sum_{i_1\ldots i_k} M_{i_0i_1}\ldots M_{i_ki_{k+1}}\\ &= (M^k)_{i_0i_{k+1}}\end{aligned}$$

Thus we have the following formula, for any $k\in\mathbb N$:

$$(id\otimes\varphi^{*k})v=M^k$$

It follows that our Cesàro limit is given by:

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}(a) &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\tau(M^k)\\ &= \tau\left(\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n M^k\right)\end{aligned}$$

Now since v is unitary we have $||v||=1$, and so $||M||\leq 1$. Thus the Cesàro limit on the right converges, and equals the orthogonal projection onto the 1-eigenspace of M:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n M^k=P$$

Thus our initial Cesàro limit converges as well, to $\tau(P)$, as desired. □

When the linear form $\varphi\in C(G)^*$ is chosen faithful, we obtain the following finer result:

PROPOSITION 5.16. *Given a faithful unital linear form $\varphi\in C(G)^*$, the limit*

$$\int_\varphi a=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}(a)$$

exists, and is independent of φ, given on coefficients of representations by

$$\left(id\otimes\int_\varphi\right)v=P$$

where P is the orthogonal projection onto the following space:

$$Fix(v)=\left\{\xi\in\mathbb C^n\Big|v\xi=\xi\right\}$$

PROOF. In view of Proposition 5.15, it remains to prove that when φ is faithful, the 1-eigenspace of the matrix $M=(id\otimes\varphi)v$ equals the space $Fix(v)$.

"$\supset$" This is clear, and for any φ, because we have:

$$v\xi=\xi\implies M\xi=\xi$$

"⊂" Here we must prove that, when φ is faithful, we have:

$$M\xi=\xi\implies v\xi=\xi$$

For this purpose, assume that we have $M\xi=\xi$, and consider the following element:

$$a=\sum_i\left(\sum_j v_{ij}\xi_j-\xi_i\right)\left(\sum_k v_{ik}\xi_k-\xi_i\right)^*$$

We must prove that we have $a=0$. Since v is unitary, we have:

$$\begin{aligned}
a &= \sum_i\left(\sum_j\left(v_{ij}\xi_j-\frac{1}{N}\xi_i\right)\right)\left(\sum_k\left(v_{ik}^*\bar{\xi}_k-\frac{1}{N}\bar{\xi}_i\right)\right)\\
&= \sum_{ijk} v_{ij}v_{ik}^*\xi_j\bar{\xi}_k-\frac{1}{N}v_{ij}\xi_j\bar{\xi}_i-\frac{1}{N}v_{ik}^*\xi_i\bar{\xi}_k+\frac{1}{N^2}\xi_i\bar{\xi}_i\\
&= \sum_j|\xi_j|^2-\sum_{ij}v_{ij}\xi_j\bar{\xi}_i-\sum_{ik}v_{ik}^*\xi_i\bar{\xi}_k+\sum_i|\xi_i|^2\\
&= ||\xi||^2-<v\xi,\xi>-\overline{<v\xi,\xi>}+||\xi||^2\\
&= 2(||\xi||^2-Re(<v\xi,\xi>))
\end{aligned}$$

By using now our assumption $M\xi=\xi$, we obtain from this:

$$\begin{aligned}
\varphi(a) &= 2\varphi(||\xi||^2-Re(<v\xi,\xi>))\\
&= 2(||\xi||^2-Re(<M\xi,\xi>))\\
&= 2(||\xi||^2-||\xi||^2)\\
&= 0
\end{aligned}$$

Now since φ is faithful, this gives $a=0$, and so $v\xi=\xi$, as claimed. □

We can now formulate a main result, as follows:

THEOREM 5.17. *Any compact group G has a unique Haar integration, which can be constructed by starting with any faithful positive unital state $\varphi\in C(G)^*$, and setting:*

$$\int_G=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\varphi^{*k}$$

Moreover, for any representation v we have the formula

$$\left(id\otimes\int_G\right)v=P$$

where P is the orthogonal projection onto the following linear space:

$$Fix(v)=\left\{\xi\in\mathbb{C}^n\middle|v\xi=\xi\right\}$$

PROOF. We can prove this from what we have, in several steps, as follows:

(1) Let us first go back to the general context of Proposition 5.15 above. Since convolving one more time with φ will not change the Cesàro limit appearing there, the functional $\int_\varphi \in C(G)^*$ constructed there has the following invariance property:

$$\int_\varphi *\varphi = \varphi * \int_\varphi = \int_\varphi$$

In the case where φ is assumed to be faithful, as in Proposition 5.16 above, our claim is that we have the following formula, valid this time for any $\psi \in C(G)^*$:

$$\int_\varphi *\psi = \psi * \int_\varphi = \psi(1)\int_\varphi$$

It is enough to prove this formula on a coefficient of a corepresentation:

$$a = (\tau \otimes id)v$$

In order to do so, consider the following two matrices:

$$P = \left(id \otimes \int_\varphi\right)v \quad , \quad Q = (id \otimes \psi)v$$

We have then the following computation:

$$\begin{aligned}\left(\int_\varphi *\psi\right)a &= \left(\tau \otimes \int_\varphi \otimes \psi\right)(v_{12}v_{13})\\ &= \tau(PQ)\end{aligned}$$

Similarly, we have the following computation:

$$\begin{aligned}\left(\psi * \int_\varphi\right)a &= \left(\tau \otimes \psi \otimes \int_\varphi\right)(v_{12}v_{13})\\ &= \tau(QP)\end{aligned}$$

Finally, regarding the term on the right, this is given by:

$$\psi(1)\int_\varphi a = \psi(1)\tau(P)$$

Thus, our claim is equivalent to the following equality:

$$PQ = QP = \psi(1)P$$

But this follows from the fact, coming from Proposition 5.16, that $P = (id \otimes \int_\varphi)v$ equals the orthogonal projection onto $Fix(v)$. Thus, we have proved our claim.

(2) In order to finish now, it is convenient to introduce the following abstract operation, on the continuous functions $a, b : C(G) \to \mathbb{C}$ on our group:

$$\Delta(a \otimes b)(g \otimes h) = a(g)b(h)$$

With this convention, the formula that we established above can be written as:

$$\begin{aligned}\psi\left(\int_\varphi\otimes id\right)\Delta &= \psi\left(id\otimes\int_\varphi\right)\Delta\\ &= \psi\int_\varphi(.)1\end{aligned}$$

This formula being true for any $\psi\in C(G)^*$, we can simply delete ψ. We conclude that the following invariance formula holds indeed, with $\int_G=\int_\varphi$:

$$\left(\int_G\otimes id\right)\Delta=\left(id\otimes\int_G\right)\Delta=\int_G(.)1$$

But this is exactly the left and right invariance formula we were looking for.

(3) Finally, in order to prove the uniqueness assertion, assuming that we have two invariant integrals $\int_G,\int_G'$, we have, according to the above invariance formula:

$$\begin{aligned}\left(\int_G\otimes\int_G'\right)\Delta &= \left(\int_G'\otimes\int_G\right)\Delta\\ &= \int_G(.)1\\ &= \int_G'(.)1\end{aligned}$$

Thus we have $\int_G=\int_G'$, and this finishes the proof. □

Summarizing, we can now integrate over G. As a first application, we have:

Proposition 5.18. *Given a compact group G, we have the following formula, valid for any unitary group representation $v:G\to U_M$:*

$$\int_G\chi_v=\dim(Fix(v))$$

In particular, in the unitary matrix group case, $G\subset_u U_N$, the moments of the main character $\chi=\chi_u$ are given by the following formula:

$$\int_G\chi^k=\dim(Fix(u^{\otimes k}))$$

Thus, knowing the law of the main character $\chi=\chi_u$ is the same as knowing the number of fixed points of the Peter-Weyl representations $u^{\otimes k}$.

Proof. We have three statements here, the idea being as follows:

(1) Given a unitary representation $v : G \to U_M$ as in the statement, its character χ_v is a coefficient, so we can use the integration formula for coefficients in Theorem 5.17. If we denote by P the projection onto $Fix(v)$, this formula gives, as desired:

$$\begin{aligned}\int_G \chi_v &= Tr(P)\\ &= \dim(Im(P))\\ &= dim(Fix(v))\end{aligned}$$

(2) This follows from (1), applied to the Peter-Weyl representations, as follows:

$$\begin{aligned}\int_G \chi^k &= \int_G \chi_u^k\\ &= \int_G \chi_{u^{\otimes k}}\\ &= \dim(Fix(u^{\otimes k}))\end{aligned}$$

(3) This follows from (2), and from the standard fact, which follows from definitions, that a probability measure is uniquely determined by its moments. ☐

We have in fact the following reneral integration result:

THEOREM 5.19. *The Haar integration over a closed subgroup $G \subset_u U_N$ is given on the dense subalgebra of smooth functions by the Weingarten type formula*

$$\int_G g_{i_1 j_1}^{e_1} \ldots g_{i_k j_k}^{e_k} \, dg = \sum_{\pi,\sigma \in D_k} \delta_\pi(i)\delta_\sigma(j) W_k(\pi,\sigma)$$

valid for any colored integer $k = e_1 \ldots e_k$ and any multi-indices i, j, where D_k is a linear basis of $Fix(u^{\otimes k})$, the associated generalized Kronecker symbols are given by

$$\delta_\pi(i) = <\pi, e_{i_1} \otimes \ldots \otimes e_{i_k}>$$

and $W_k = G_k^{-1}$ is the inverse of the Gram matrix, $G_k(\pi,\sigma) = <\pi,\sigma>$.

PROOF. We know from Peter-Weyl theory that the integrals in the statement form altogether the orthogonal projection P^k onto the following space:

$$Fix(u^{\otimes k}) = span(D_k)$$

Consider now the following linear map, with $D_k = \{\xi_k\}$ being as in the statement:

$$E(x) = \sum_{\pi \in D_k} < x, \xi_\pi > \xi_\pi$$

By a standard linear algebra computation, it follows that we have $P = WE$, where W is the inverse of the restriction of E to the following space:

$$K = span\left(T_\pi \Big| \pi \in D_k\right)$$

But this restriction is precisely the linear map given by the matrix G_k, and so W itself is the linear map given by the matrix W_k, and this gives the result. □

We will be back to this in later on, with some concrete applications.

Getting back to algebra, the above results, which allow us to use the Haar integration, when needed, lead to a generalization of the Peter-Weyl theory, with everything that we know about the finite groups having suitable extensions to the compact group case.

5d. Lie groups

Given a closed subgroup $G\subset U_N$, we can talk about its Lie algebra. We have in fact many examples of Lie groups, both compact and non-compact.

5e. Exercises

CHAPTER 6

Hopf algebras

6a. Definition

In this chapter we start developing quantum algebra, with inspiration from the group theory discussed in chapter 5 above. We would like to develop a theory of algebras A, which are not necessarily commutative, corresponding to "quantum groups" G.

In order to simplify the presentation, we use the following terminology:

DEFINITION 6.1. *Given an algebra A, any morphisms of type*

$$\Delta : A \to A \otimes A$$
$$\varepsilon : A \to k$$
$$S : A \to A^{opp}$$

will be called comultiplication, counit and antipode.

The terminology comes from the fact that in the basic commutative case, $A = k(X)$, the morphisms Δ, ε, S are transpose to group-type operations, as follows:

$$m : X \times X \to X$$
$$u : \{.\} \to X$$
$$i : X \to X$$

The reasons for using A^{opp} instead of A will become clear in a moment. Now with these conventions in hand, we can formulate our definition, as follows:

DEFINITION 6.2. *A Hopf algebra is an algebra A, with a comultiplication, counit and antipode, satisfying the following conditions:*

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$$
$$(\varepsilon \otimes id)\Delta = id$$
$$(id \otimes \varepsilon)\Delta = id$$
$$m(S \otimes id)\Delta = \varepsilon(.)1$$
$$m(id \otimes S)\Delta = \varepsilon(.)1$$

If the square of the antipode is the identity, $S^2 = id$, we say that A is underformed. Otherwise, in the case $S^2 \neq id$, we say that A is deformed.

Here everything is standard, except for comment in relation with the condition $S^2 = id$, which can be included or not in the definition of the Hopf algebras. This condition corresponds to the fact that, in the corresponding quantum group, we should have:

$$(g^{-1})^{-1} = g$$

We will be back to this issue, on several occasions. In fact, clarifying the relation between Hopf algebras axiomatized with $S^2 = id$, and Hopf algebras axiomatized without $S^2 = id$, will be a main theme of discussion, throughout this book.

There are several interesting examples of Hopf algebras, which are all underformed. We first have the following result, which justifies the terminology and axioms:

Proposition 6.3. *Given a finite group G, the algebra of scalar functions on it*

$$k(G) = \left\{\varphi : G \to k\right\}$$

with usual pointwise product, given by the formula

$$(\varphi\psi)(g) = \varphi(g)\psi(g)$$

is a Hopf algebra, with structural maps as follows:

$$\Delta(\varphi) = (g,h) \to \varphi(gh)$$

$$\varepsilon(\varphi) = \varphi(1)$$

$$S(\varphi) = g \to \varphi(g^{-1})$$

This Hopf algebra is finite dimensional, commutative, and underformed.

Proof. The fact that Δ, ε, S satisfy the axioms comes from the fact that these maps are the functional analytic transposes of the group operations on G, namely:

$$m : G \times G \to G$$

$$u : \{.\} \to G$$

$$i : G \to G$$

Indeed, the group axioms for G are as follows, with $\delta(g) = (g, g)$:

$$m(m \times id) = m(id \times m)$$

$$m(id \times u) = m(u \times id) = id$$

$$m(id \times i)\delta = m(i \times id)\delta = 1$$

Thus, by transposing, we obtain the axioms from Definition 6.2. Observe that we obtain in this way as well $S^2 = id$, coming by transposing the following formula:

$$i^2 = id$$

Finally, our algebra is clearly finite dimensional, and is commutative as well. □

In view of the above, we can think of any finite dimensional Hopf algebra A, not necessarily commutative, as being as follows, with G being a finite quantum group:

$$A = k(G)$$

However, all this is quite speculative, and in addition we must be careful with the axioms. One problem is whether we want to include $S^2 = id$ or not in our axioms. Another problem is that, even when assuming $S^2 = id$, nothing guarantees that a finite dimensional commutative Hopf algebra must be of the form $A = k(G)$.

Yet another problem, again axiomatic, comes from the fact that we would like the construction in Proposition 6.3 to cover other groups as well, infinite this time, such as the discrete ones, or the compact ones, or, ideally, the locally compact ones. In the framework of Definition 6.2 this is not exactly possible, due to the fact that the comultiplication Δ would have to land in the algebra $k(G \times G)$, and for an infinite group G, we have:

$$k(G) \otimes k(G) \neq k(G \times G)$$

However, there are several tricks in order to overcome this, either by allowing $\otimes$ to be a topological tensor product, or by using Lie algebras. We will be back to this.

Moving ahead now, we say that a Hopf algebra A as axiomatized above is cocommutative if, with $\Sigma(a \otimes b) = b \otimes a$ being the flip map, we have the following formula:

$$\Sigma\Delta = \Delta$$

We have the following result, which is somehow "dual" to Proposition 6.3 above, and which once again justifies the terminology and axioms:

Proposition 6.4. *Given a finite group H, the algebra of scalar functions on it*

$$k[H] = \left\{\varphi : H \to k\right\}$$

this time with usual convolution product, given by the formula

$$(\varphi * \psi)(g) = \sum_{g=hk} \varphi(h)\psi(k)$$

is a Hopf algebra, with structural maps given on group elements as follows:

$$\Delta(g) = g \otimes g$$

$$\varepsilon(g) = 1$$

$$S(g) = g^{-1}$$

This Hopf algebra is finite dimensional, cocommutative, and underformed.

Proof. As before, the fact that Δ, ε, S satisfy the axioms is clear from definitions, with the remark of course that we use the following canonical embedding:

$$H \subset k[H]$$
$$g \to \delta_g$$

Observe also that the use of the opposite multiplication $(a, b) \to a \cdot b$ in really needed in the axioms, in Definition 6.2, in order for the antipode S to be an algebra morphism:

$$\begin{aligned} S(gh) &= (gh)^{-1} \\ &= h^{-1}g^{-1} \\ &= g^{-1} \cdot h^{-1} \\ &= S(g) \cdot S(h) \end{aligned}$$

Finally, the last assertion is clear from definitions as well. □

As before with the usual function algebras, from Proposition 6.3 and from the comments afterwards, when trying to extend Proposition 6.4 to more general classes of groups, which can be compact, discrete, or more generally locally compact, but no longer finite, there are some issues to be solved. We will be back to these questions later.

For the moment, let us stay with the finite dimensional algebras. As a first theorem on the subject, which is something non-trivial, we have:

Theorem 6.5. *If G, H are finite abelian groups, dual to each other via Pontrjagin duality, then we have an identification of Hopf algebras as follows:*

$$k(G) = k[H]$$

In the case $G = H = \mathbb{Z}_N$, this identification is the usual discrete Fourier transform isomorphism. In general, we obtain a tensor product of such Fourier transforms.

Proof. All this is standard Fourier analysis, in the discrete case, and with the last result coming by writing our finite abelian group G as a product of cyclic groups:

$$G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$$

Indeed, due to the functorial properties of the Pontrjagin duality, we have as well:

$$H = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_k}$$

Thus, we are led to the conclusions in the statement. □

As before with other results, when trying to extend Theorem 6.5 to more general groups, which can be compact, discrete, or more generally locally compact, but no longer finite, there are some issues to be solved. We will be back to these questions later.

Quite remarkably, the Pontrjagin duality for the finite abelian groups can be extended to the general finite group case, in the context of the Hopf algebras, as follows:

THEOREM 6.6. *Given a finite dimensional Hopf algebra A, its dual space*

$$A^* = \Big\{\varphi : A \to k\Big\}$$

is also a finite dimensional Hopf algebra, with structural maps as follows:

$$\Delta^t : A^* \otimes A^* \to A^*$$

$$\varepsilon^t : \mathbb{C} \to A^*$$

$$m^t : A^* \to A^* \otimes A^*$$

$$u^t : A^* \to \mathbb{C}$$

$$S^t : A^* \to A^*$$

This duality makes correspond commutative algebras to cocommutative algebras. Also, this duality makes correspond $k(G)$ to $k[G]$, for any finite group G.

PROOF. Since A is, before anything, an associative algebra, its multiplication and unit maps m, u are subject to the following axioms:

$$m(m \otimes id) = m(id \otimes m)$$

$$m(id \otimes u) = id$$

$$m(u \otimes id) = id$$

We also know that A is in fact a Hopf algebra, so the following are satisfied too:

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$$

$$(\varepsilon \otimes id)\Delta = id$$

$$(id \otimes \varepsilon)\Delta = id$$

Finally, once again because A is a Hopf algebra, we have as well:

$$m(S \otimes id)\Delta = \varepsilon(.)1$$

$$m(id \otimes S)\Delta = \varepsilon(.)1$$

The point now is that the collection of these 8 formulae is "self-dual", in the sense that when transposing, we obtain exactly the same 8 formulae. Thus A^*, as constructed in the statement, is indeed a Hopf algebra. Observe that the operation $A \to A^*$ is indeed a duality, because if we dualize one more time, we obtain A itself:

$$A^{**} = A$$

Finally, the assertion about commutative and cocommutative algebras is clear from definitions, and the last assertion, regarding groups, is clear from definitions as well. □

As a conclusion to all this, we can think of any finite dimensional Hopf algebra A, not necessarily commutative or cocommutative, as being as follows, with G, H being finite quantum groups, with a generalized Pontrjagin duality between them:

$$A = k(G) = k[H]$$

However, all this is quite speculative, and in addition we must be careful with the axioms. One problem, as usual, is whether we want to include $S^2 = id$ or not in our axioms. Another problem is that, even when assuming $S^2 = id$, nothing guarantees that a finite dimensional commutative Hopf algebra must be of the form $A = k(G)$. And also, once again when assuming $S^2 = id$, nothing guarantees that a finite dimensional cocommutative Hopf algebra must be of the form $A = k[H]$. We will be back to this.

6b. Theory, examples

Some general theory can be developed for the Hopf algebras, in analogy with the basic theory of groups, by using the axioms. However, when doing group theory, you won't get very far just by playing with m, u, i, and the situation is pretty much the same with the Hopf algebras, where you won't get very far just by playing with Δ, ε, S.

There is however some interesting theory regarding the square of the antipode:

THEOREM 6.7. *Results regarding S^2.*

PROOF. This is something well-known. □

As a more exciting problematics, let us try to understand what happens beyond the finite dimensional case. As already mentioned in the above, when trying to cover various infinite groups, such as the compact, discrete, or more generally locally compact ones, the standard trick is that of using topological tensor products. We will discuss this later.

For the moment, let us point out that it is possible to cover the compact Lie groups, without changing the Hopf algebra axioms, by using a Lie algebra trick, as follows:

THEOREM 6.8. *The enveloping algebra $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is a Hopf algebra.*

PROOF. This is clear indeed from definitions. □

As a last topic here, let us discuss the duality in the arbitrary, not necessarily finite dimensional setting. We have here the following result:

THEOREM 6.9. *Duality theory, in general.*

PROOF. This is something quite technical, mixing technical algebra and technical functional analysis. Some interesting problems appear in connection with the property of reflexivity, which is automatic in the finite dimensional case. □

6c. Basic operations

We have several basic operations for the Hopf algebras, inspired from the basic operations for the groups. Passed some trivial trings, such as products and the like, the interesting operations are the intersection, generation, and the Hopf image operation.

To start with, we have the intersection and generation operations, defined at the quantum group level as follows:

Proposition 6.10. *The quantum subgroups of a given quantum group G are subject to operations as follows:*

(1) *Intersection: $H\cap K$ is the biggest quantum subgroup of H,K.*
(2) *Generation: $<H,K>$ is the smallest quantum group containing H,K.*

Proof. We must prove that the universal quantum groups in the statement exist indeed. For this purpose, let us pick writings as follows, with I,J being Hopf ideals:

$$k(H)=k(G)/I$$

$$k(K)=k(G)/J$$

We can then construct our two universal quantum groups, as follows:

$$k(H\cap K)=k(G)/<I,J>$$

$$k(<H,K>)=k(G)/(I\cap J)$$

Thus, we obtain the result. □

In practice, assuming that our quantum groups have coordinates u_{ij}, in a suitable corepresentation theory sense, the operation $\cap$ can be usually computed by using:

Proposition 6.11. *Assuming $H,K\subset G$, the intersection $H\cap K$ is given by*

$$k(H\cap K)=k(G)/\{\mathcal{R},\mathcal{P}\}$$

whenever we have writings as follows,

$$k(H)=k(G)/\mathcal{R}$$

$$k(K)=k(G)/\mathcal{P}$$

with $\mathcal{R},\mathcal{P}$ being certain sets of polynomial relations between the coordinates u_{ij}.

Proof. This follows from Proposition 6.10 above, or rather from its proof, and from the following trivial fact, regarding relations and ideals:

$$I=<\mathcal{R}>,J=<\mathcal{P}>\quad\Longrightarrow\quad <I,J>=<\mathcal{R},\mathcal{P}>$$

Thus, we obtain the result. □

In order to discuss the generation operation, let us call Hopf image of a representation $k(G) \to A$ the smallest Hopf algebra quotient $k(L)$ producing a factorization as follows:

$$k(G) \to k(L) \to A$$

The fact that this quotient exists indeed is routine, by dividing by a suitable ideal, and we will be back to this later on. This notion can be generalized as follows:

PROPOSITION 6.12. *Assuming $H, K \subset G$, the quantum group $< H, K >$ is such that*

$$k(G) \to k(H \cap K) \to k(H), k(K)$$

is the joint Hopf image of the following quotient maps:

$$k(G) \to k(H), k(K)$$

PROOF. In the particular case from the statement, the joint Hopf image appears as the smallest Hopf algebra quotient $k(L)$ producing factorizations as follows:

$$k(G) \to k(L) \to k(H), k(K)$$

We conclude from this that we have $L =< H, K >$, as desired. □

There are many other interesting things that can be said, about the above operations. We will be back to this regularly, in what follows.

6d. Further examples

There are many known interesting examples of finite dimensional Hopf algebras, in the undeformed case, and some classification results for them as well.

6e. Exercises

CHAPTER 7

Quantum groups

7a. Representations

We have seen so far some interesting theory and examples, for the Hopf algebras, mostly in the finite dimensional case. Our goal here is to develop the representation theory for the corresponding "finite quantum groups", and also to look for extensions, to a suitable class of "compact quantum groups". As results, we would like to have the existence of a Haar integration functional, and an analogue of the Peter-Weyl theory.

Inspired as usual by group theory, we have the following definition:

Definition 7.1. *A corepresentation of a Hopf algebra A is a matrix $u \in M_n(A)$ satisfying the conditions*

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$$

$$\varepsilon(u_{ij}) = \delta_{ij}$$

along with a condition regarding the values of the elements $S(u_{ij})$.

Here the last condition depends on whether A is a $*$-algebra or not, and also on the validity of $S^2 = id$. Some corepresentation theory can de developed, along these lines.

As a first remark, the corepresentations are subject to a number of operations, exactly as were the representations in the usual group case, as follows:

Proposition 7.2. *The corepresentations are subject to the following operations:*

1. *Making sums, $u + v = diag(u, v)$.*
2. *Making tensor products, $(u \otimes v)_{ia,jb} = u_{ij} v_{ab}$.*
3. *Taking conjugates, $(\bar{u})_{ij} = u_{ij}^*$.*
4. *Spinning by unitaries, $u \to VuV^*$.*

Proof. All this follows from definitions, with (3,4) in need of course for some clarifications, depending on the type of Hopf algebras that we are using. □

Next in line, we have the following key definition:

DEFINITION 7.3. *Given two corepresentations $u \in M_n(A), v \in M_m(A)$, we set*

$$Hom(u,v) = \left\{T \in M_{m\times n}(\mathbb{C})\Big| Tu = vT\right\}$$

and we use the following conventions:

(1) *We use the notations $Fix(u) = Hom(1,u)$, and $End(u) = Hom(u,u)$.*
(2) *We write $u \sim v$ when $Hom(u,v)$ contains an invertible element.*
(3) *We say that u is irreducible, and write $u \in Irr(A)$, when $End(u) = k1$.*

In the classical case $A = k(G)$ we obtain the usual notions concerning the representations. Observe also that in the group dual case, $A = k[G]$, we have:

$$g \sim h \iff g = h$$

Finally, observe that $u \sim v$ means that u, v are conjugated by an invertible matrix. Here are a few basic results, regarding the above Hom spaces:

PROPOSITION 7.4. *We have the following results:*

(1) $T \in Hom(u,v), S \in Hom(v,w) \implies ST \in Hom(u,w)$.
(2) $S \in Hom(p,q), T \in Hom(v,w) \implies S \otimes T \in Hom(p \otimes v, q \otimes w)$.
(3) $T \in Hom(v,w) \implies T^* \in Hom(w,v)$.

In other words, the Hom spaces form a tensor $*$-category.

PROOF. All this is standard, exactly as in the group case. As before, (3) is in need of some clarifications, depending on the type of Hopf algebras that we are using. □

7b. Haar integration

Under suitable assumptions, our Hopf algebras have a Haar integration functional:

THEOREM 7.5. *Suitable Hopf algebras A have a unique Haar integration, which can be constructed by starting with any faithful positive unital state $\varphi \in A^*$, and setting*

$$\int_G = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \varphi^{*k}$$

where $\phi * \psi = (\phi \otimes \psi)\Delta$. Moreover, for any corepresentation v we have

$$\left(id \otimes \int_G\right) v = P$$

where P is the orthogonal projection onto $Fix(v) = \{\xi \in k^n | v\xi = \xi\}$.

PROOF. This is something quite technical. □

7c. Peter-Weyl theory

Under suitable assumptions, our Hopf algebras are subject to a Peter-Weyl type theory. First, we have the following result:

THEOREM 7.6 (PW1). *Let $v\in M_n(A)$ be a corepresentation, consider the algebra $B=End(v)$, and write its unit as*

$$1=p_1+\ldots+p_k$$

with p_i being minimal central projections. We have then

$$v=v_1+\ldots+v_k$$

with each v_i being an irreducible corepresentation, obtained by restricting v to $Im(p_i)$.

PROOF. This is something quite technical. □

Next in line, we have the following result:

THEOREM 7.7 (PW2). *Each irreducible corepresentation of A appears as:*

$$v\subset u^{\otimes k}$$

That is, v appears inside a certain Peter-Weyl corepresentation.

PROOF. This is something quite technical as well. □

By using the Haar integration, we have as well:

THEOREM 7.8 (PW3). *The dense subalgebra $\mathcal{A}\subset A$ decomposes as a direct sum*

$$\mathcal{A}=\bigoplus_{v\in Irr(A)}M_{\dim(v)}(k)$$

with this being an isomorphism of $*$-coalgebras, and with the summands being pairwise orthogonal with respect to the scalar product given by

$$<a,b>=\int_G ab^*$$

where $\int_G$ is the Haar integration over G.

PROOF. This is, as usual, something quite technical. □

Finally, we have the folloiwing result:

THEOREM 7.9 (PW4). *The characters of the irreducible corepresentations belong to the $*$-algebra*

$$\mathcal{A}_{central}=\left\{a\in\mathcal{A}\Big|\Sigma\Delta(a)=\Delta(a)\right\}$$

of "smooth central functions" on G, and form an orthonormal basis of it.

PROOF. This is something quite technical too. □

7d. Tannakian duality

Under suitable assumptions, our Hopf algebras are subject to a Tannakian duality.

7e. Exercises

CHAPTER 8

Tensor categories

8a. Tensor categories

The Tannakian duality result found in the previous chapter raises the perspective of looking directly at tensor categories, by forgetting about the associated Hopf algebras.

8b. Basic examples

There are many interesting examples of tensor categories, as usual in the undeformed case that we are mainly interested in, for the moment.

8c. Planar algebras

Many interesting examples of tensor categories come from planar algebras in the sense of Jones. We briefly explain here this theory, coming from operator algebras.

8d. Spectral measures

This is an attempt of discussing the moment problem in semisimple tensor categories. Some things might hold only in semisimple tensor categories having substantial positivity (CQG, subfactors, tensor C^*-categories).

We do not know. In any case, to be taken with care, everything below called "Proposition" or "Theorem" being rather a conjecture.

The simplest situation is that of a semisimple tensor category which is "real", in the sense that we have $C =< X >$ with $X = X^*$. Here the result is:

THEOREM 8.1. *Assuming $C =< X >$ with $X = X^*$, the numbers*

$$M_k = \#(1 \in X^{\otimes k})$$

are the moments of a certain real probability measure μ.

PROOF. The moment problem here is the most standard one, the so-called "Hamburger moment problem". The standard criterion here is that the following determinants associated to the sequence (M_k), called Hankel determinants, must be positive:

$$\begin{vmatrix}M_0\end{vmatrix}\geq 0 \quad , \quad \begin{vmatrix}M_0&M_1\\M_1&M_2\end{vmatrix}\geq 0 \quad , \quad \begin{vmatrix}M_0&M_1&M_2\\M_1&M_2&M_3\\M_2&M_3&M_4\end{vmatrix}\geq 0 \quad , \quad \ldots$$

Equivalently, the following associated linear forms must be positive:

$$\sum_{i,j=1}^n c_i\bar{c}_jM_{i+j}\geq 0$$

This is something very classical, in one sense coming from the following formula:

$$\int_{\mathbb R}\left|\sum_{i=1}^nc_ix^i\right|^2d\mu(x)=\sum_{i,j=1}^nc_i\bar{c}_jM_{i+j}$$

In the other sense, this comes from some standard functional analysis. Now in connection with our tensor category problem, we have:

$$\begin{aligned}\sum_{i,j=1}^nc_i\bar{c}_jM_{i+j}&=\sum_{i,j=1}^nc_i\bar{c}_j\#(1\in X^{\otimes i+j})\\&=\dim\left(End\left(\sum_{i=1}^nc_iX^{\otimes i}\right)\right)\end{aligned}$$

Here on the right we have a "virtual object", with such things being defined in the general semisimple category setting, and with the dimension being ≥ 0, as desired. □

The next problem is that of a semisimple tensor category which is "classical", or perhaps "braided", in the sense that we have $C=<X>$ with:

$$X\otimes X^*=X^*\otimes X$$

Here we are looking for a complex probability measure having as moments the following numbers:

$$M_{kl}=\#\big(1\in X^{\otimes k}\otimes X^{*\otimes l}\big)$$

The problem, however, is that the moment problem in this setting, called "full complex moment problem" (or "truncated complex moment problem", under finiteness assumptions) is something very complicated, with non-trivial operator theory involved, and with, at the end of the day, not a very clear answer, that we can exploit in our setting.

Thus, we must find something else.

For this purpose, let us go back to the real setting, $C=<X>$ with $X=X^*$. We have the following alternative to Theorem 8.1 and its proof, which is something more concrete, provided that it works indeed:

THEOREM 8.2. *Assuming $C =< X >$ with $X = X^*$, the numbers*

$$M_k = \#(1 \in X^{\otimes k})$$

are the moments of a certain real probability measure μ, which is the law of the matrix

$$\Delta_{YZ} = dim(Hom(Y, Z \otimes X))$$

with indices $Y, Z \in Irr(C)$, with respect to the following functional:

$$\int T = T_{11}$$

Moreover, this matrix is symmetric, $\Delta = \Delta^t$.

PROOF. The fact that the matrix Δ in the statement is indeed symmetric follows from Frobenius reciprocity, and from $X = X^*$, as follows:

$$\begin{aligned}
\Delta_{ZY} &= dim(Hom(Z, Y \otimes X)) \\
&= \#(1 \in Z^* \otimes Y \otimes X) \\
&= \#(1 \in X^* \otimes Y^* \otimes Z) \\
&= \#(1 \in Y^* \otimes Z \otimes X^*) \\
&= \#(1 \in Y^* \otimes Z \otimes X) \\
&= dim(Hom(Y, Z \otimes X)) \\
&= \Delta_{YZ}
\end{aligned}$$

Our claim now is that for any $k \in \mathbb{N}$ we have the following formula:

$$(\Delta^k)_{11} = M_k$$

But this is something which seems to hold indeed, by decomposing everything into irreducibles, and we obtain the result. □

Before getting further with the investigation of the general case, $X \neq X^*$, let us record as well the following useful version of Theorem 8.2:

THEOREM 8.3. *Assuming $C =< X >$ with $X = X^*$, the numbers*

$$M_k = \#(1 \in X^{\otimes k})$$

are the moments of a real probability measure μ, which is the spectral measure of the pointed graph $(\Gamma, 1)$ having as vertices the elements of $Irr(C)$, with multiple edges

$$\#(Y - Z) = dim(Hom(Y, Z \otimes X))$$

with this meaning that we have the moment formula

$$M_k = \#\Big(\underbrace{1 - \ldots - 1}_{k-loop\ at\ 1}\Big)$$

or equivalently, meaning that μ is the law of the Laplacian of Γ with respect to $\int T = T_{11}$.

PROOF. This is just a fancy reformulation of Theorem 8.2 above, with the k-loops in the statement being the numbers $(\Delta^k)_{11}$ from Theorem 8.2 and its proof. □

Getting now into the general case, $X \neq X^*$, we should probably have something as follows, generalizing at the same time Theorem 8.2 and Theorem 1.3:

THEOREM 8.4. *Assuming $C =< X >$, the following numbers, with $k = \circ\bullet\circ\bullet\ldots$*

$$M_k = \#(1 \in X^{\otimes k})$$

are the moments of a certain abstract distribution μ, which is the law of the matrix

$$\Delta_{YZ} = dim(Hom(Y, Z \otimes X))$$

with indices $Y, Z \in Irr(C)$, with respect to the following functional:

$$\int T = T_{11}$$

Moreover, μ is as well the spectral measure, in the sense of abstract distributions, of the pointed graph $(\Gamma, 1)$ having as vertices the elements of $Irr(C)$, with multiple edges

$$\#(Y - Z) = dim(Hom(Y, Z \otimes X))$$

with this meaning that we have the moment formula

$$M_k = \#\Big(\underbrace{1 - \ldots - 1}_{k-loop\ at\ 1}\Big)$$

or equivalently, meaning that μ is the law of the Laplacian of Γ with respect to $\int T = T_{11}$.

PROOF. This looks like a straightforward extension of Theorem 8.2 and Theorem 8.3 taken altogether, with the only difference coming from the fact that the matrix Δ is no longer symmetric, and so its distribution, which is not a measure in the usual sense, must be defined in terms of "colored moments", indexed by colored integers $k = \circ\bullet\circ\bullet\ldots$ □

Now getting back to the case that we are really interested in, namely $C =< X >$ with $X \otimes X^* = X^* \otimes X$, we have here the following statement:

THEOREM 8.5. *Assuming $C =< X >$ with $X \otimes X^* = X^* \otimes X$, the numbers*

$$M_{kl} = \#\big(1 \in X^{\otimes k} \otimes X^{*\otimes l}\big)$$

are the moments of a certain complex probability measure μ.

PROOF. As already mentioned above, before the statement of Theorem 8.2, a quick and direct proof, in the spirit of the proof of Theorem 8.1 above, seems impossible, due to the fact that the complex moment problem is far more complicated than the real moment problem. However, we can probably trick by using Theorem 8.4, if that statement holds indeed, by arguing that in the case $X \otimes X^* = X^* \otimes X$, the abstract distribution that we construct there is in fact a complex probability measure. □

Summarizing, what we have is Theorem 8.1, which is definitely known to hold, and then we have its extension Theorem 8.5, which is for the moment conjectural.

The issues to be checked are multiple, due on one hand to the fact that the tricks in Theorem 8.2 and 8.3 and 8.4 are rather things known under strong positivity assumptions, and also with $X = X^*$, so many checks to be done here, and on the other hand due to infinite dimensionality issues, when talking about infinite matrices Δ and infinite graphs Γ, with to be done here first being probably a complete check in the "finite" case.

We are interested in computing the spectral measures μ of tensor category objects $X \in C$. The general phenomenon, which is something very beautiful, and 100% verified in the compact quantum group case, is that "the more interesting is the pair (C, X), the more interesting is the measure μ, and vice versa". Beyond compact quantum groups, this phenomenon seems to be verified too, at least for the only computation that we have so far, leading to the arcsine law, which is something fundamental in probability, random matrices and statistical mechanics. Below are some technical details, all well-known, but worth including, on one hand as a concrete illustration for the Stieltjes inversion formula, and on the other hand due to the potential importance of this arcsine law example, which might be well the "central example" in all that we want to do.

The Stieltjes inversion formula is as follows:

THEOREM 8.6. *The density of a real probability measure μ can be recaptured from the sequence of moments (M_k) via the Stieltjes inversion formula*

$$d\mu(x) = \lim_{t \searrow 0} -\frac{1}{\pi} Im\left(G(x+it)\right) \cdot dx$$

where $G(\xi) = \xi^{-1} + M_1\xi^{-2} + M_2\xi^{-3} + \ldots$ is the Cauchy transform.

PROOF. This is something very standard, and we refer here to the literature. □

In relation now with the arcsine law, the result is as follows:

PROPOSITION 8.7. *The real probability measure having as moments the central binomial coefficients, $D_k = \binom{2k}{k}$, is the arcsine law on $[0, 4]$, given by:*

$$d\mu(x) = \frac{1}{\pi\sqrt{x(4-x)}}\, dx$$

There are as well variations of this result, regarding measures having modifications of the sequence (D_k) as moments, which equally lead to arcsine laws, on other intervals $[a, b]$.

Proof. In order to apply the Stieltjes inversion formula, we need a simple formula for the Cauchy transform. The trick here is to use the following formula, which comes either from Taylor, or from the generalized binomial formula, with exponent $-1/2$:

$$\frac{1}{\sqrt{1+t}}=\sum_{k=0}^\infty D_k\left(-\frac{t}{4}\right)^k$$

By using this formula with $t=-4/\xi$ we obtain the following formula:

$$\begin{aligned}G(\xi)&=\xi^{-1}\sum_{k=0}^\infty D_k\xi^{-k}\\&=\frac{1}{\xi}\sum_{k=0}^\infty D_k\left(-\frac{t}{4}\right)^k\\&=\frac{1}{\xi}\cdot\frac{1}{\sqrt{1-4/\xi}}\\&=\frac{1}{\sqrt{\xi(\xi-4)}}\end{aligned}$$

Thus by Stieltjes inversion we obtain the density in the statement, namely:

$$d\mu(x)=\frac{1}{\pi\sqrt{x(4-x)}}\,dx$$

Here is as well a proof by "cheating", i.e. knowing the measure in advance. We just have to compute the moments of μ, and with $x=4\cos^2t$, we have indeed:

$$\begin{aligned}M_k&=\frac{1}{\pi}\int_0^4\frac{x^k}{\sqrt{x(4-x)}}\,dx\\&=\frac{1}{\pi}\int_0^{\pi/2}\frac{(4\cos^2t)^k}{2\cos t\cdot 2\sin t}\,8\sin t\cos t\,dt\\&=\frac{2}{\pi}\cdot 4^k\int_0^{\pi/2}\cos^{2k}t\,dt\\&=\frac{2}{\pi}\cdot 4^k\cdot\frac{\pi}{2}\cdot\frac{(2k)!!}{(2k+1)!!}\\&=4^k\cdot\frac{1\cdot 3\cdot 5\ldots(2k-1)}{2\cdot 4\cdot 6\ldots(2k)}\\&=4^k\cdot\frac{(2k)!/(2^kk!)}{2^kk!}\\&=\binom{2k}{k}\end{aligned}$$

Here we have used a standard double factorial formula for integrating $\cos^{2k} t$. Finally, as mentioned in the statement, by suitably modifying either of the above proofs, we can obtain variations of the result, regarding measures having modifications of the sequence (D_k) as moments, which are equally arcsine laws, on other intervals $[a, b]$. □

We will need in what follows the "square root" of the arcsine law too. Consider the middle binomial coefficients, which are given by:

$$E_k = \binom{k}{[k/2]}$$

In terms of the central binomial coefficients D_k used above, we have:

$$E_{2k} = \binom{2k}{k} = \frac{(2k)!}{k!k!} = D_k$$

$$E_{2k-1} = \binom{2k-1}{k} = \frac{(2k-1)!}{k!(k-1)!} = \frac{D_k}{2}$$

Numerically, the sequence of the middle binomial coeffients is as follows:

$$1, 1, 2, 3, 6, 10, 20, 35, \ldots$$

In connection now with our questions, we have the following result:

Theorem 8.8. *The real probability measure having as moments the middle binomial coefficients, $E_k = \binom{k}{[k/2]}$, is the following law on $[-2, 2]$,*

$$d\mu(x) = \frac{1}{2\pi}\sqrt{\frac{2+x}{2-x}}\,dx$$

which can be regarded as being a "square root" of the arcsine law on $[0, 4]$.

Proof. Standard calculus based on the Taylor formula for $(1+t)^{-1/2}$ from the proof of Proposition 8.7 gives, via the above conversion formulae between D_k, E_k numbers, the following formula for the generating function of the numbers E_k, which is well-known:

$$\frac{1}{2x}\left(\sqrt{\frac{1+2x}{1-2x}} - 1\right) = \sum_{k=0}^{\infty} E_k x^k$$

With $x=\xi^{-1}$ we obtain the following formula for the Cauchy transform of the real probability measure that we are looking for:

$$\begin{aligned}G(\xi) &= \xi^{-1}\sum_{k=0}^\infty E_k\xi^{-k}\\ &= \frac{1}{\xi}\left(\sqrt{\frac{1+2/\xi}{1-2/\xi}}-1\right)\\ &= \frac{1}{\xi}\left(\sqrt{\frac{\xi+2}{\xi-2}}-1\right)\end{aligned}$$

By Stieltjes inversion we obtain the density in the statement, on $[-2,2]$, namely:

$$d\mu(x)=\frac{1}{2\pi}\sqrt{\frac{2+x}{2-x}}\,dx$$

Here is as well a proof by "cheating", i.e. knowing the measure in advance. We just have to compute the moments of μ, and with $x=2\cos 2t$, then $t=s/2$, we have:

$$\begin{aligned}M_k &= \frac{1}{2\pi}\int_{-2}^2\sqrt{\frac{2+x}{2-x}}\,x^k\,dx\\ &= \frac{1}{2\pi}\int_0^{\pi/2}\sqrt{\frac{1+\cos 2t}{1-\cos 2t}}\,(2\cos 2t)^k 4\sin 2t\,dt\\ &= \frac{2^{k+2}}{\pi}\int_0^{\pi/2}\frac{\cos t}{\sin t}\,(\cos 2t)^k\sin t\cos t\,dt\\ &= \frac{2^{k+2}}{\pi}\int_0^{\pi/2}(\cos 2t)^k\cos^2 t\,dt\\ &= \frac{2^{k+2}}{\pi}\int_0^{\pi}(\cos s)^k\,\frac{1+\cos s}{2}\cdot\frac{ds}{2}\\ &= \frac{2^k}{\pi}\int_0^\pi \cos^k s+\cos^{k+1}s\,ds\end{aligned}$$

By using the formulae in the proof of Proposition 8.7, the even moments are:

$$\begin{aligned}M_{2l} &= \frac{4^l}{\pi}\int_0^\pi\cos^{2l}s\,ds\\ &= \frac{4^l}{\pi}\times 2\int_0^{\pi/2}\cos^{2l}s\,ds\\ &= \binom{2l}{l}\end{aligned}$$

As for the even moments, by using the same formulae, these are as follows:

$$\begin{aligned} M_{2l-1} &= \frac{2^{2l-1}}{\pi}\int_0^\pi \cos^{2l} s\, ds \\ &= \frac{4^l}{\pi}\int_0^{\pi/2} \cos^{2l} s\, ds \\ &= \frac{1}{2}\binom{2l}{l} \end{aligned}$$

Thus, we have $M_k = E_k$, and are led to the conclusion in the statement. □

We are interested in the following questions:

(1) Given a tensor category $C =< X >$ when do we have "positivity", in the sense that X has a spectral measure, in the sense of probability theory,

(2) When do we have "freeness" of $C =< X >$, once again in a sense beyond pure algebra, with some probability involved,

(3) Can we axiomatize the "asymptotic tensor categories" $C = (C_N)$ with $N \in \mathbb{N}$, perhaps even with $N \in [1, \infty)$, Deligne-style, and do more advanced probability theory on them, with $N \to \infty$.

Given a compact group $G \subset U_N$, with fundamental representation denoted X, the category $C = Rep(G)$ is generated by X. Assuming $G \subset O_N$, by Peter-Weyl we have:

$$\int_G Tr(g)^k dg = \#(1 \in X^{\otimes k})$$

Thus the spectral measure of the main character $\chi = Tr$, which is by definition the real probability measure induced on the spectrum $\sigma(\chi) \subset [-N, N]$ by the Haar measure on G, has the numbers $M_k = \#(1 \in X^{\otimes k})$ as moments.

In the general complex case, $G \subset U_N$, the formula that we need, once again coming from Peter-Weyl, and which is a bit more general, is:

$$\int_G Tr(g)^k\, \overline{Tr(g)}^l dg = \#(1 \in X^{\otimes k} \otimes X^{*\otimes l})$$

Thus the spectral measure of the main character $\chi = Tr$, which is now by definition the complex probability measure induced on the spectrum $\sigma(\chi) \subset D_0(N)$ by the Haar measure on G, has the numbers $M_{kl} = \#(1 \in X^{\otimes k} \otimes X^{*\otimes l})$ as moments.

Exactly the same formulae hold for compact quantum groups in the sense of Woronowicz, due to his analogue of the Peter-Weyl theory. With the subtlety, however, that in the general, non-orthogonal case, $X \neq X^*$, and non-braided case either, $X \otimes X^* \neq X^* \otimes X$, the main character χ and its adjoint χ^* do not longer commute, and so the spectral measure of the main character χ is something rather abstract, having as moments the numbers of the following type, indexed by words on two letters $\circ$ and $\bullet$:

$$M_{\circ\bullet\bullet\circ\ldots} = \#(1 \in X \otimes X^* \otimes X^* \otimes X \otimes \ldots)$$

As main examples, the spectral measures for the groups S_N, O_N are with $N \to \infty$ Poisson and Gaussian, and the spectral measures for the free quantum groups S_N^+, O_N^+ are with $N \to \infty$ free Poisson and free Gaussian, a.k.a. Marchenko-Pastur and Wigner. There are many explanations for this phenomenon, and more on this later.

All this raises the question of understanding what happens in the general tensor category case. Assuming indeed that we have a tensor category generated by an object, $C =< X >$, we would like to understand if X has a spectral measure or not, and then what exactly can we do with this measure.

Let us first discuss the case $X = X^*$. Here we can say that X has a spectral mesure μ if the numbers $M_k = \#(1 \in X^{\otimes k})$ are the moments of a certain real probability measure μ. In this case, we also say that $C =< X >$ has "positivity".

As a first remark, such a real probability measure μ, if it exists, is unique. Indeed, knowing the moments is the same as knowing how to integrate each power function x^k, and by linearity we know how to integrate any polynomial $P(x)$, and then by continuity we know how to integrate any continuous function $f(x)$, and once we have this, the Riesz theorem tells us that the formal integration $f \to \int f$ that we constructed in this way must be indeed the integration with respect to a real probability measure μ.

Regarding now the existence of μ, we are led here into using abstract "moment problem" techniques. A sequence of numbers M_k is known to be the sequence of moments of a real probability measure μ if the corresponding Hankel determinants are all positive, and this is the same as saying that some associated quadratic forms are all positive definite.

In the semisimple case, the computation, using the quadratic form approach, leads to a certain multiplicity, which is positive, and so we have indeed "positivity". In the non-semisimple case, apparently examples where we have "positivity" abound, and this actually raises the question on whether counterexamples do actually exist.

In what regards the explicit computation of μ, out of the moments, this is usually done via the "Stieltjes inversion formula", which is more advanced "moment theory", and which amounts in performing certain real and complex analysis manipulations on the series $f(z) = \sum_k M_k z^k$, as to end up with the density of μ. This is usually non-trivial calculus, and even for the most basic measures (Gaussian, Poisson, Wigner, Marchenko-Pastur) this is something which takes some time. As another basic example, which is well-known, and can be probably found in the random matrix literature, where such things appear, for the central binomial coefficients we obtain the arcsine law.

As a last topic regarding the case $X = X^*$, we have some interesting questions in relation with the support of μ. In the group and quantum group case, discussed above, this is contained in $[-N, N]$, and this due to the fact that we have $||\chi|| \leq N$, and so the spectrum of χ, which by Riesz is the support of the spectral measure, satisfies $\sigma(\chi) \subset [-N, N]$. Less trivial is the fact that for compact groups G the support of μ must contain either N or $-N$, that is, our estimate is sharp, and that for general compact quantum groups G, this happens precisely when the discrete quantum group dual $\Gamma = \widehat{G}$ is amenable. This is in fact the "quantum Kesten theorem", whose proof is quite tricky, using advanced functional analysis techniques, and with the name coming from the fact that when $G = \widehat{\Gamma}$ is by definition the abstract dual of a discrete group Γ, in the sense of the Woronowicz theory, what we get is the Kesten theorem for Γ.

In the general tensor category setting now, we can probably expect something similar to happen, provided that μ exists, and with $N = \dim(X)$ being the dimension taken in the appropriate sense, and with all this being probably known and worked out in the tensor C^*-category setting, in various follow-up papers to Doplicher-Roberts, at least at the combinatorial level, with perhaps no explicit measure μ involved.

Next in line, we have the case where X commutes with X^*. Here we can say that X has a spectral mesure μ if the numbers $M_{kl} = \#(1 \in X^{\otimes k} \otimes X^{*\otimes l})$ are the moments of a certain complex probability measure μ, in these sense that these are the integrals of $z^k \bar{z}^l$ with respect to μ. In this case, we also say that $C =< X >$ has "positivity".

As before, μ is unique, if it exists, thanks to the Riesz theorem. Regarding the existence, this looks like advanced abstract "moment method" theory, and we must find here the relevant method, replacing the Hankel determinants/quadratic form positivity criteria, and then apply it to the semisimple case, and see if things work.

Regarding the explicit computation of μ, out of its moments, once again this looks like advanced "moment method" theory, and we must find here the relevant method, replacing the Stieltjes inversion formula, and apply it to the examples that we have in mind.

Finally, regarding the support of μ, and amenability questions, in the quantum group case the trick is to use the spectral measure of $\chi + \chi^*$, the result being that, with the notations above, $\Gamma = \widehat{G}$ is amenable precisely when $N \in supp(law(Re(\chi)))$, with basically no new computations needed. For tensor category extensions, in the C^*-category setting the follow-ups to Doplicher-Roberts are probably to be examined first.

In the general case now, where X does not commute with X^*, things are quite unclear, and this even in the compact quantum group case. We do not have a probability measure in the usual sense, but rather something abstract, described by its moments, which are numbers $M_{\circ\bullet\bullet\circ\ldots}$ indexed by words on two letters, $\circ$ and $\bullet$. To be studied later.

As a second topic to be investigated, we have the question of understanding when a singly generated tensor category $C =< X >$ can be called "free".

In order to discuss this, let us first recall that $C = Rep(G)$ with $G \subset U_N$, with fundamental representation denoted X, is "classical", in the sense that $X \otimes X^* = X^* \otimes X$. More generally, assuming that we have a braided quantum group G, the associated tensor category $Rep(G)$ is "classical" in this somewhat naive sense.

Now recall from the above that basic groups like S_N, O_N have free counterparts S_N^+, O_N^+. This list is actually substantially longer, related to the classification of the so-called "easy quantum groups". The problem is that of axiomatizing the freeness property of the associated tensor categories $C = Rep(G)$, so that we can have a notion of "free tensor category", coming as a complement to the above notion of "classical tensor category".

There are actually several proposals here, already, in the compact quantum group literature, but not necessarily formulated in terms of tensor categories, and not necessarily agreeing either with each other either. All this material is to be studied and processed, from our present the tensor category viewpoint, in order to reach to a clear and nice definition, that has a chance to extend to the general tensor category setting.

As mentioned before, the spectral measures for the groups S_N, O_N are with $N \to \infty$ Poisson and Gaussian, and the spectral measures for the free quantum groups S_N^+, O_N^+ are with $N \to \infty$ free Poisson and free Gaussian, a.k.a. Marchenko-Pastur and Wigner. The reason behind this is the fact that $S = (S_N)$ and $O = (O_N)$ are "easy", in the sense that, thanks to the Schur-Weyl or Brauer theorems, the associated categories are spanned by certain linear maps canonically associated to partitions, the partitions in question being respectively P and P_2, all partitions, and all pairings. The quantum groups $S^+ = (S_N^+)$ and $O^+ = (O_N^+)$ are easy as well, with the partitions in question being this time NC and NC_2, all noncrossing partitions, and all noncrossing pairings. And this gives the above-mentioned probability results with $N \to \infty$, via standard combinatorics.

The problem, in relation to this, is that of axiomatizing the "asymptotic tensor categories" $C = (C_N)$ with $N \in \mathbb{N}$, and do more advanced probability theory on them, with $N \to \infty$. It is probably possible to talk about $N \in [1, \infty)$, on one hand due to various examples coming from subfactor theory (TL, FC..) where the index can vary continuously, and on the other hand from the Deligne construction of $Rep(S_t)$ with $t > 1$. As main probability questions, we must understand whether $\mu = \lim_{N \to \infty} \mu_N$ is infinitely divisible, with respect to the classical or free convolution. Technically speaking, this amounts in computing Fourier and Voiculescu R-transforms, in the $N \to \infty$ limit.

8e. Exercises

Part III

Quantum spaces

Maybe one day we'll be united
And our love won't be divided
Maybe one day we'll be united
And our love won't be divided

CHAPTER 9

Liberation theory

9a. Liberation theory

We go back here to the Hopf algebras, and later to tensor categories and planar algebras, with the aim of looking for quantum spaces X which do not depend on the ground field k, and also if possible, which do not depend on the value of $N \in \mathbb{N}$.

In order to do this, it is convenient to adopt the "liberation" philosophy, which consists in regarding a quantum space X as the liberation of the classical space X_{class}, obtained by dividing the corresponding Hopf algebra by its commutator ideal.

9b. Brauer theorems

In order to formulate our Brauer type theorems, providing us with examples of "absolute" spaces as above, let us begin with a general definition, from [20], as follows:

Definition 9.1. *Let $P(k,l)$ be the set of partitions between an upper colored integer k, and a lower colored integer l. A collection of subsets*

$$D = \bigsqcup_{k,l} D(k,l)$$

with $D(k,l) \subset P(k,l)$ is called a category of partitions when it has the following properties:

1. *Stability under the horizontal concatenation, $(\pi,\sigma) \to [\pi\sigma]$.*
2. *Stability under vertical concatenation $(\pi,\sigma) \to [^\sigma_\pi]$, with matching middle symbols.*
3. *Stability under the upside-down turning $*$, with switching of colors, $\circ \leftrightarrow \bullet$.*
4. *Each set $P(k,k)$ contains the identity partition $|| \ldots ||$.*
5. *The sets $P(\emptyset, \circ\bullet)$ and $P(\emptyset, \bullet\circ)$ both contain the semicircle $\cap$.*

As basic examples of such categories, we have various categories of pairings, matching or not, and crossing or not, with inclusions between them as follows:

$$\begin{array}{ccc} \mathcal{P}_2 & \longleftarrow & \mathcal{NC}_2 \\ \downarrow & & \downarrow \\ P_2 & \longleftarrow & NC_2 \end{array}$$

There are many other examples, as for instance P itself, or the category $NC \subset P$ of all noncrossing partitions. We have as well various categories of partitions formed by the partitions having even blocks. These form a diagram as follows:

$$\begin{array}{ccc} \mathcal{P}_{even} & \leftarrow & \mathcal{NC}_{even} \\ \downarrow & & \downarrow \\ P_{even} & \leftarrow & NC_{even} \end{array}$$

The relation with the Tannakian categories of linear maps comes from the fact that we can associate linear maps to the pairings, as in [20], as follows:

Definition 9.2. *Associated to any partition $\pi \in P(k,l)$ and any integer $N \in \mathbb{N}$ is the linear map*

$$T_\pi : (k^N)^{\otimes k} \to (k^N)^{\otimes l}$$

given by the following formula, with $\{e_1, \ldots, e_N\}$ being the standard basis of k^N,

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

and with the Kronecker symbols $\delta_\pi \in \{0,1\}$ depending on whether the indices fit or not.

To be more precise here, in the definition of the Kronecker symbols, we agree to put the two multi-indices on the two rows of points of the pairing, in the obvious way. The Kronecker symbols are then defined by $\delta_\pi = 1$ when all the strings of π join equal indices, and by $\delta_\pi = 0$ otherwise. Observe that all this is independent of the coloring.

Here are a few basic examples of such linear maps:

Proposition 9.3. *The correspondence $\pi \to T_\pi$ has the following properties, where R is the operator mapping $1 \to \sum_i e_i \otimes e_i$, and $\Sigma(a \otimes b) = b \otimes a$ is the flip operator:*

1. *$T_\cap = R$.*
2. *$T_\cup = R^*$.*
3. *$T_{||\ldots||} = id$.*
4. *$T_{/\!\!\!\backslash} = \Sigma$.*

Proof. We can assume if we want that all the upper and lower legs of π are colored $\circ$. With this assumption made, the proof goes as follows:

(1) We have $\cap\in P_2(\emptyset,\circ\circ)$, and so the corresponding operator is a certain linear map $T_\cap:k\to k^N\otimes k^N$. The formula of this map is as follows:

$$\begin{aligned}T_\cap(1)&=\sum_{ij}\delta_\cap(i\ j)e_i\otimes e_j\\&=\sum_{ij}\delta_{ij}e_i\otimes e_j\\&=\sum_i e_i\otimes e_i\end{aligned}$$

We recognize here the formula of $R(1)$, and so we have $T_\cap=R$, as claimed.

(2) Here we have $\cup\in P_2(\circ\circ,\emptyset)$, and so the corresponding operator is a certain linear form $T_\cap:k^N\otimes k^N\to k$. The formula of this linear form is as follows:

$$\begin{aligned}T_\cap(e_i\otimes e_j)&=\delta_\cap(i\ j)\\&=\delta_{ij}\end{aligned}$$

Since this is the same as $R^*(e_i\otimes e_j)$, we have $T_\cup=R^*$, as claimed.

(3) Consider indeed the "identity" pairing $||\ldots||\in P_2(k,k)$, with $k=\circ\circ\ldots\circ\circ$. The corresponding linear map is then the identity, because we have:

$$\begin{aligned}T_{||\ldots||}(e_{i_1}\otimes\ldots\otimes e_{i_k})&=\sum_{j_1\ldots j_k}\delta_{||\ldots||}\begin{pmatrix}i_1&\ldots&i_k\\j_1&\ldots&j_k\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_k}\\&=\sum_{j_1\ldots j_k}\delta_{i_1j_1}\ldots\delta_{i_kj_k}e_{j_1}\otimes\ldots\otimes e_{j_k}\\&=e_{i_1}\otimes\ldots\otimes e_{i_k}\end{aligned}$$

(4) In the case of the basic crossing $\slash\!\!\!\backslash\in P_2(\circ\circ,\circ\circ)$, the corresponding linear map $T_{\slash\!\!\!\backslash}:k^N\otimes k^N\to k^N\otimes k^N$ can be computed as follows:

$$\begin{aligned}T_{\slash\!\!\!\backslash}(e_i\otimes e_j)&=\sum_{kl}\delta_{\slash\!\!\!\backslash}\begin{pmatrix}i&j\\k&l\end{pmatrix}e_k\otimes e_l\\&=\sum_{kl}\delta_{il}\delta_{jk}e_k\otimes e_l\\&=e_j\otimes e_i\end{aligned}$$

Thus we obtain the flip operator $\Sigma(a\otimes b)=b\otimes a$, as claimed. □

Summarizing, the correspondence $\pi\to T_\pi$ provides us with some simple formulae for the operators R,R^* that we are interested in, and for other important operators, such as the flip $\Sigma(a\otimes b)=b\otimes a$, and has as well some interesting categorical properties.

Let us further explore these properties, and make the link with the Tannakian categories. We have the following result, from [20]:

Proposition 9.4. *The assignement $\pi\to T_\pi$ is categorical, in the sense that we have*

$$T_\pi\otimes T_\sigma=T_{[\pi\sigma]}$$

$$T_\pi T_\sigma=N^{c(\pi,\sigma)}T_{[^\sigma_\pi]}$$

$$T_\pi^*=T_{\pi^*}$$

where $c(\pi,\sigma)$ are certain integers, coming from the erased components in the middle.

Proof. The formulae in the statement are all elementary, as follows:

(1) The concatenation axiom follows from the following computation:

$$\begin{aligned}
&(T_\pi\otimes T_\sigma)(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})\\
=&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}\delta_\sigma\begin{pmatrix}k_1&\ldots&k_r\\ l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
=&\sum_{j_1\ldots j_q}\sum_{l_1\ldots l_s}\delta_{[\pi\sigma]}\begin{pmatrix}i_1&\ldots&i_p&k_1&\ldots&k_r\\ j_1&\ldots&j_q&l_1&\ldots&l_s\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_q}\otimes e_{l_1}\otimes\ldots\otimes e_{l_s}\\
=&T_{[\pi\sigma]}(e_{i_1}\otimes\ldots\otimes e_{i_p}\otimes e_{k_1}\otimes\ldots\otimes e_{k_r})
\end{aligned}$$

(2) The composition axiom follows from the following computation:

$$\begin{aligned}
&T_\pi T_\sigma(e_{i_1}\otimes\ldots\otimes e_{i_p})\\
=&\sum_{j_1\ldots j_q}\delta_\sigma\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}\sum_{k_1\ldots k_r}\delta_\pi\begin{pmatrix}j_1&\ldots&j_q\\ k_1&\ldots&k_r\end{pmatrix}e_{k_1}\otimes\ldots\otimes e_{k_r}\\
=&\sum_{k_1\ldots k_r}N^{c(\pi,\sigma)}\delta_{[^\sigma_\pi]}\begin{pmatrix}i_1&\ldots&i_p\\ k_1&\ldots&k_r\end{pmatrix}e_{k_1}\otimes\ldots\otimes e_{k_r}\\
=&N^{c(\pi,\sigma)}T_{[^\sigma_\pi]}(e_{i_1}\otimes\ldots\otimes e_{i_p})
\end{aligned}$$

(3) Finally, the involution axiom follows from the following computation:

$$\begin{aligned}
&T_\pi^*(e_{j_1}\otimes\ldots\otimes e_{j_q})\\
=&\sum_{i_1\ldots i_p}<T_\pi^*(e_{j_1}\otimes\ldots\otimes e_{j_q}),e_{i_1}\otimes\ldots\otimes e_{i_p}>e_{i_1}\otimes\ldots\otimes e_{i_p}\\
=&\sum_{i_1\ldots i_p}\delta_\pi\begin{pmatrix}i_1&\ldots&i_p\\ j_1&\ldots&j_q\end{pmatrix}e_{i_1}\otimes\ldots\otimes e_{i_p}\\
=&T_{\pi^*}(e_{j_1}\otimes\ldots\otimes e_{j_q})
\end{aligned}$$

Summarizing, our correspondence is indeed categorical. □

In relation with the quantum groups, we have the following result, from [20]:

THEOREM 9.5. *Each category of partitions $D = (D(k,l))$ produces a family of compact quantum groups $G = (G_N)$, one for each $N \in \mathbb{N}$, via the formula*

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

which produces a Tannakian category, and the Tannakian duality correspondence.

PROOF. This follows indeed from Tannakian duality. Indeed, let us set:

$$C(k,l) = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

By using the axioms in Definition 9.1, and the categorical properties of the operation $\pi \to T_\pi$, from Proposition 9.4 above, we deduce that $C = (C(k,l))$ is a Tannakian category. Thus the Tannakian duality applies, and gives the result. □

As an application, we have the following result:

THEOREM 9.6. *Brauer type theorems for unitary quantum groups.*

PROOF. This is something a bit technical, done in full generality by Bichon. □

We have as well the following result:

THEOREM 9.7. *Brauer type theorems for orthogonal quantum groups.*

PROOF. This is something even more technical than the unitary result above, due to some well-known issues regarding the correct orthogonal groups to be used, depending on the ground field k, once again done in full generality by Bichon. □

Moving ahead, we have as well the following fundamental result:

THEOREM 9.8. *Brauer type theorems for quantum permutation groups.*

PROOF. As before, this is something a bit technical, depending on the ground field k, done in full generality by Bichon. □

Finally, as a generalization of Theorem 9.8, we have:

THEOREM 9.9. *Brauer type theorems for quantum reflection groups.*

PROOF. This basically follows by adpapting the above-mentioned techniques of Bichon, from the quantum permutation group case. □

There are of course many other Brauer type theorems, for instance involving the bishochastic groups. Also, there is an extension covering the symplectic groups as well. However, a clear axiomatization covering the ABCD cases is not known yet. Not known either, even over $k = \mathbb{C}$, is the liberation theory for the exceptional Lie groups.

9c. Half-liberation

A technically less difficult question, leading to very concrete results, going beyond the combinatorial setting and the Brauer type theorems, concerns the study of the half-liberations. We have indeed here the results of Bichon and Dubois-Violette.

9d. Intermediate liberations

Things here are quite technical, even in the $k=\mathbb{C}$ case.

9e. Exercises

CHAPTER 10

Absolute spaces

10a. Quotient spaces

It is possible to talk about various homogeneous spaces $X = G/H$.

10b. Compact objects

It is possible to talk about more general algebraic manifolds, of the same type.

10c. Axiomatization

The axiomatization problems are in fact open, at least so far.

10d. Local compactness

We are interested in liberating and twisting the basic examples of locally compact Lie groups $G \subset GL_N(\mathbb{C})$, in analogy with the known liberation and twisting of the basic compact Lie groups $G \subset U_N$. Before starting, let us mention that the general "liberation and twisting" scheme from the compact case comprises in fact 5 main objects:

(1) First, we have the group itself, $G \subset U_N$.

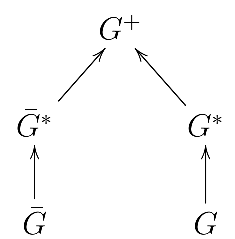
(2) We have then a liberation, $G^+ \subset U_N^+$.

(3) We have as well a half-liberation, $G^* \subset U_N^*$.

(4) Independently of this, we have a $q = -1$ twist, $\bar{G} \subset \bar{U}_N$.

(5) And for completing the scheme, we have a twisted half-liberation, $\bar{G}^* \subset \bar{U}_N^*$.

That is, the general idea is that the basic compact Lie groups $G \subset U_N$ do not appear alone, but rather as part of 5-item families, as follows:



In this scheme, best thought as lying above the complex plane, the right part corresponds to $q = 1$, and the left part corresponds to $q = -1$. Going upwards means liberating, and the biggest liberation, G^+, is at the same time "twisted and untwisted".

In practice now, the above scheme needs some strong assumptions on $G \subset U_N$, in order to fully exist. The simplest case is when $H_N \subset G \subset U_N$ is easy, with this situation notably covering $G = H_N, K_N, O_N, U_N$, and here the above 5-scheme can be constructed by using standard techniques, that we will remind in what follows.

In more general cases, twists might exist but not liberations, or vice versa, and so on, with all this depending a lot on our precise group $G \subset U_N$. Once again, we will remind these methods, in what follows.

Let us mention as well, in addition to this, that more complicated constructions are possible, such as technical intermediate liberations $G^\times \neq G^*$, using various methods from the easy quantum group literature. We have as well matrix model truncations, $G^{(K)} \subset G$, with these being in general not quantum groups. It is also possible to intersect twisted and untwisted objects, as to extend the above 5-diagram into a 9-diagram.

Finally, as a last comment, all these "liberation and twisting" methods can be applied, more generally, to certain special classes of compact homogeneous spaces, or more generally, to certain special classes of algebraic submanifolds of the complex sphere.

Our purpose in what follows will be that of discussing the possible "liberation and twisting" methods in the locally compact group case. That is, our goal will be that of liberating and twisting the basic locally compact Lie groups $G \subset GL_N(\mathbb{C})$.

Before even starting, we should mention that the problem here is of very different nature from the one in the compact case. Indeed, in the compact case the problem is basically of algebraic nature, with all the functional analytic aspects being taken care of by Woronowicz's papers, which contain all the needed functional analysis tools.

In the locally compact case, the situation is opposite, with the problem basically belonging to functional analysis. There is of course some algebra to be done, in order to get started, but the main problem, once a candidate for a half-liberation or twist of $G \subset GL_N(\mathbb{C})$ is found, is that of establishing a Haar measure existence result.

Long story short, we are here, modulo some algebra, for construcing Haar measures, by using all kinds of advanced functional analysis tools.

Half-liberation. In what regards the half-liberation, things look much easier. Following Bichon-Dubois-Violette and others, we can use 3 equivalent known techniques, namely: (1) 2×2 matrix models, (2) crossed products by $\mathbb{Z}_2$, (3) lifting projective versions.

Let us comment on the 2×2 matrix model method, which is probably the simplest. The general idea here would be that of defining $GL_N^*(\mathbb{R})$ as being the quantum group having coordinates as follows, with a_{ij} being the standard coordinates on $GL_N(\mathbb{C})$:

$$u_{ij} = \begin{pmatrix} 0 & a_{ij} \\ a_{ij}^* & 0 \end{pmatrix}$$

In what regards $GL_N^*(\mathbb{C})$, here the coordinates would be as follows, with a_{ij}, b_{ij} being the standard coordinates on two copies of $GL_N(\mathbb{C})$:

$$v_{ij} = \begin{pmatrix} 0 & a_{ij} \\ b_{ij} & 0 \end{pmatrix}$$

The problem however is that these coordinates are not bounded, and not L^∞ either. Thus, if we want to use operator algebras, we must rather use polynomial functions in these coordinates, or rather limits of such polynomials in the coordinates, which are bounded. Algebrically speaking this would cause no problems, because the monomials in the coordinates are still modelizable by 2×2 matrices, either diagonal or antidiagonal, depeding on the length of the monomial. Thus, there is some abstract algebra to be done here, basically by "splitting everything in 2 parts", and then some functional analysis work, consisting in defining the algebras, along with the associated Haar functionals.

In what regards $SL_N(\mathbb{R})$, $SL_N(\mathbb{C})$, these do not half-liberate, the reasons for this being the non-existence of a "half-classical determinant", and the non-existence of a "free determinant" either. Here the things are quite folklore, with a precise statement in this sense being the fact that SO_N^* should collapse to SO_N. Another reason for this comes from noncommutative geometry and orientation. There are some diagrammatic explanations for this phenomenon as well, the idea being that "det produces the crossing".

In short, many interesting questions here. The problem is that of choosing the right approach to half-liberation, then getting into Haar measure existence questions. From the perspective of the known locally compact quantum group literature, the most reasonable option would be probably that of using (2) above, crossed products by $\mathbb{Z}_2$. Indeed, this would make our problem quite close to the known things about twisting.

Twisting. Generally speaking, the problems here are quite similar to those that we met above, in connection with the half-liberation, needing a $\mathbb{Z}_2$ study. Indeed, in the context of the twisting we have relations of the following type:

$$\prod_i x_{\sigma(i)} = \varepsilon(\sigma) \prod_i x_i$$

Thus, we have a potential approach using algebra. However, all this is probably best investigated by using cocycle twisting methods.

Half-liberation and twisting. This comes after, by combining half-liberation and twisting. In fact, nothing much is known here, even in the compact case.

As a conclusion, the main questions which are waiting to be investigated are the half-liberation of $GL_N(\mathbb{R})$, $GL_N(\mathbb{C})$, the twisting of $GL_N(\mathbb{R})$, $GL_N(\mathbb{C})$, $SL_N(\mathbb{R})$, $SL_N(\mathbb{C})$, and the half-liberation and twisting of $GL_N(\mathbb{R})$, $GL_N(\mathbb{C})$.

The problem is that of coming with candidates for these half-liberations and twists, and then getting into Haar measure existence results, by using known twisting methods, from the locally compact quantum group literature.

As in what regards the liberation, things here look more delicate, with some questions however in the "discrete" case, in connection with tori and their known liberations.

Liberation. No one ever managed to liberate any non-compact locally Lie compact group $G\subset GL_N(\mathbb{C})$. Even worse, no one ever managed to come up with a candidate, for such a liberation. So this is probably a question which is very difficult, if not undoable. The folklore in fact is that the non-compact locally compact Lie groups $G\subset GL_N(\mathbb{C})$ cannot be liberated, and that more general non-compact locally compact manifolds cannot be liberated either. But who really knows. To be investigated a bit, in any case.

This being said, in the non-Lie group case, or rather in non-continuous Lie group case, we have plenty of infinite discrete groups which "liberate", in some reasonable sense. Basic examples here include the standard torus $\mathbb{T}_N=\mathbb{Z}^N$, which liberates into the free torus $\mathbb{T}_N^+=\widehat{F_N}$, and the real torus, or cube, $T_N=\mathbb{Z}_2^N$, which liberates into the free real torus, or free cube, $T_N^+=\widehat{\mathbb{Z}_2^{*N}}$. These are both very natural examples of "liberations". Perhaps some work to be done here, in order to "take off" in this discrete direction, with liberations of things which are very close to infinite groups.

10e. Exercises

CHAPTER 11

Differential geometry

11a. K-theory

Generally speaking, K-theory does not work well, because it is sensitive to the choice of the algebra A, with this being well-known even in the group dual case. However, K-theory does work well as long as we are far enough from freeness, and from non-amenability in general, as for instance in the case of various crossed products, or half-liberations.

11b. Smoothness issues

Things here are again quite tricky, with once again things working well as long as we are far enough from freeness, and from non-amenability in general, as for instance in the case of various crossed products, or half-liberations.

11c. Riemannian geometry

Let us begin with a discussion regarding quantum isometry groups. According to a standard differential geometry result, whose proof is elementary, the classical isometry group $\mathcal{G}(X)$ of a Riemannian manifold X is then the group of diffeomorphisms $\varphi : X \to X$ whose induced action on $C^\infty(X)$ commutes with Δ.

In view of this, following now Goswami, we can formulate:

Definition 11.1. *Associated to a compact Riemannian manifold X are:*
(1) $\mathcal{D}^+(X)$*: the category of compact quantum groups acting on X.*
(2) $\mathcal{G}^+(X) \in \mathcal{D}^+(X)$*: the universal object with a coaction commuting with Δ.*

Here the coactions maps $\Phi : C(X) \to C(X) \otimes C(G)$ that we consider in (1) must satisfy by definition the usual axioms for the algebraic coactions, namely:

$$(\Phi \otimes id)\Phi = (id \otimes \Delta)\Phi$$

$$(id \otimes \varepsilon)\Phi = id$$

In addition, these are subject as well to the following smoothness assumption:

$$\Phi(C^\infty(X)) \subset C^\infty(X) \otimes C(G)$$

As for the commutation condition with Δ in (2) above, this regards the canonical extension of the action to the space $L^2(X)$.

We have the following well-known weak version of Goswami's main rigidity result, which is something rather elementary:

PROPOSITION 11.2. *A compact connected Riemannian manifold X cannot have genuine group dual isometries.*

PROOF. We recall that for a connected Riemannian manifold X, the eigenfunctions of the Laplacian have the domain property, namely:

$$f,g\neq 0\implies fg\neq 0$$

This is for instance because the set of zeros of each nonzero eigenfunction of the Laplacian is known to have Hausdorff dimension $\dim X-1$, and hence measure zero. Now assume that we have a group dual coaction, as follows:

$$\Phi:C(X)\to C(X)\otimes C^*(\Gamma)$$

Let $E=E_1\oplus E_2$ be the direct sum of two eigenspaces of the Laplacian Δ. Pick a basis $\{x_i\}$ such that the corresponding corepresentation on E becomes diagonal, in the sense that we have, for certain group elements $g_i\in\Gamma$:

$$\Phi(x_i)=x_i\otimes g_i$$

The formula $\Phi(x_ix_j)=\Phi(x_jx_i)$ reads then:

$$x_ix_j\otimes g_ig_j=x_ix_j\otimes g_jg_i$$

Thus, by using the domain property of the eigenfunctions of Δ, we obtain:

$$g_ig_j=g_jg_i$$

Also, the formula $\Phi(x_i\bar{x}_j)=\Phi(\bar{x}_jx_i)$ reads:

$$x_i\bar{x}_j\otimes g_ig_j^{-1}=x_i\bar{x}_j\otimes g_j^{-1}g_i$$

Thus by using the domain property again, we obtain:

$$g_ig_j^{-1}=g_j^{-1}g_i$$

Thus the elements $\{g_i,g_i^{-1}\}$ pairwise commute, and with the eigenspace E varying, this shows that our group Γ must be abelian, as claimed. □

Let us discuss now the case of the noncommutative Riemannian manifolds. We will use in what follows some very light axioms, inspired from Connes' ones from [36]:

DEFINITION 11.3. *A baby spectral triple $X=(A,H,D)$ consists of the following:*

1. *A is a unital C^*-algebra.*
2. *H is a Hilbert space, on which A acts.*
3. *D is an unbounded self-adjoint operator on H, with compact resolvents, such that $[D,\phi]$ has a bounded extension, for any ϕ in a dense $*$-subalgebra $\mathcal{A}\subset A$.*

The guiding examples come from the compact Riemannian manifolds X. Indeed, associated to any such manifold X are several triples (A, H, D), with the dense $*$-subalgebra $\mathcal{A} \subset A$ being the algebra $C^\infty(X) \subset C(X)$, as follows:

(1) H is the space of square-integrable spinors, and D is the Dirac operator.

(2) H is the space of forms on X, and D is the Hodge-Dirac operator $d + d^*$.

(3) $H = L^2(X, dv)$, dv being the Riemannian volume, and $D = d^*d$.

In view of Definition 11.1 (2), however, the last example in the above list will be in fact the one that we will be interested in. Once again following Goswami, we have:

DEFINITION 11.4. *Associated to a baby spectral triple $X = (A, H, D)$ are:*
(1) *$\mathcal{D}^+(X)$: the category of compact quantum groups acting on (A, H).*
(2) *$\mathcal{G}^+(X) \in \mathcal{D}^+(X)$: the universal object with a coaction commuting with D.*

In other words, $\mathcal{G}^+(X)$ must have a unitary representation U on H which commutes with D, satisfies $U1_A = 1 \otimes 1_A$, and is such that ad_U maps A'' into itself.

In relation with the real quantum spheres, we have the following result:

PROPOSITION 11.5. *Associated to any real sphere $S^{N-1}_{\mathbb{R},\times}$ is the baby triple (A, H, D), where $A = C(S^{N-1}_{\mathbb{R},\times})$, and where D acting on $H = L^2(A, tr)$ is defined as follows:*
(1) *Consider the following linear space:*

$$H_k = span\left(x_{i_1} \ldots x_{i_r}\Big| r \leq k\right)$$

(2) *Define $E_k = H_k \cap H_{k-1}^\perp$, so that we have:*

$$H = \bigoplus_{k=0}^\infty E_k$$

(3) *Finally, set $Dx = \lambda_k x$, for any $x \in E_k$, where λ_k are distinct numbers.*

PROOF. We have to show that the operator $[D, T_i]$ is bounded, where T_i is the left multiplication by x_i. Since $x_i \in A$ is self-adjoint, so is the corresponding operator T_i. Now since we have $T_i(H_k) \subset H_{k+1}$, by self-adjointness we get:

$$T_i(H_k^\perp) \subset H_{k-1}^\perp$$

Thus we have inclusions as follows:

$$T_i(E_k) \subset E_{k-1} \oplus E_k \oplus E_{k+1}$$

This gives a decomposition of the following type:

$$T_i = T_i^{-1} + T_i^0 + T_i^1$$

We have then $[D, T_i^\alpha] = \alpha T_i^\alpha$ for any $\alpha \in \{-1, 0, 1\}$, and this gives the result. □

Summarizing, what we constructed above is some kind of algebraic structure on our spheres, coming from the eigenspaces of the Laplacian. This structure misses however the fine geometric aspects, coming from the eigenvalues, at least in the above formulation.

However, with our formalism, we can now prove:

THEOREM 11.6. *We have the quantum isometry group formula*

$$\mathcal{G}^+(S^{N-1}_{\mathbb{R},\times}) = O_N^\times$$

with respect to the baby spectral triple structure constructed above.

PROOF. Consider the universal affine coaction map on our sphere:

$$\Phi : C(S^{N-1}_{\mathbb{R},\times}) \to C(S^{N-1}_{\mathbb{R},\times}) \otimes C(O_N^\times)$$

This coaction map extends to a unitary representation on the GNS space H, that we denote by U. We have then an inclusion, as follows:

$$\Phi(H_k) \subset H_k \otimes C(O_N^\times)$$

Now observe that this formula reads:

$$U(H_k) \subset H_k$$

By unitarity we obtain as well:

$$U(H_k^\perp) \subset H_k^\perp$$

Thus each space E_k is U-invariant, and U, D must commute. We conclude that Φ is isometric with respect to D. Finally, the universality of $O_N^\times$ is clear. □

There are several interesting questions in relation with the above. First, we have the question of understanding what happens for the complex spheres, and also for the tori, real or complex. Also, we have the question of understanding what the eigenvalues of the Laplacian are, and whether this resulting Laplacian can be used in order to formulate basic PDE over our spheres. We refer here to the literature, for a discussion.

11d. Integration theory

There is a lot of theory which has been developed here.

11e. Exercises

CHAPTER 12

Algebraic geometry

12a. Basic manifolds

We are first interested in the compact hypersurfaces $X \subset \mathbb{R}^N_+$. First of all, we can talk about free spheres, in the following way:

Definition 12.1. *We have compact quantum spaces, constructed as follows,*

$$C(S^{N-1}_{\mathbb{R},+}) = C^*\left(x_1,\ldots,x_N\Big|x_i = x_i^*,\sum_i x_i^2 = 1\right)$$

$$C(S^{N-1}_{\mathbb{C},+}) = C^*\left(x_1,\ldots,x_N\Big|\sum_i x_ix_i^* = \sum_i x_i^*x_i = 1\right)$$

called respectively the free real sphere, and the free complex sphere.

Observe that the C^*-norm exists indeed, because the quadratic conditions give:

$$||x_i||^2 = ||x_ix_i^*|| \leq ||\sum_i x_ix_i^*|| = 1$$

Given a compact quantum space X, its classical version is the space X_{class} obtained by dividing $C(X)$ by its commutator ideal, and then applying the Gelfand theorem:

$$C(X_{class}) = C(X)/I \quad , \quad I =< [a,b] >$$

Observe that we have an embedding of compact quantum spaces $X_{class} \subset X$. In this situation, we also say that X appears as a "liberation" of X. We have:

Proposition 12.2. *We have embeddings of compact quantum spaces*

$$\begin{array}{ccc}
S^{N-1}_{\mathbb{C}} & \longrightarrow & S^{N-1}_{\mathbb{C},+}\\
\uparrow & & \uparrow\\
S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{R},+}
\end{array}$$

and the spaces on the right appear as liberations of the spaces of the left.

Proof. The first assertion is clear. For the second one, we must establish the following isomorphisms, where C^*_{comm} stands for "universal commutative C^*-algebra":

$$C(S^{N-1}_\mathbb{R}) = C^*_{comm}\left(x_1,\ldots,x_N\Big|x_i = x_i^*, \sum_i x_i^2 = 1\right)$$

$$C(S^{N-1}_\mathbb{C}) = C^*_{comm}\left(x_1,\ldots,x_N\Big|\sum_i x_ix_i^* = \sum_i x_i^*x_i = 1\right)$$

But these isomorphisms are both clear, by using the Gelfand theorem. □

We can talk as well about tori, in the following way:

Definition 12.3. *Given a closed subspace $S \subset S^{N-1}_{\mathbb{C},+}$, the subspace $T \subset S$ given by*

$$C(T) = C(S)\Big/\left\langle x_ix_i^* = x_i^*x_i = \frac{1}{N}\right\rangle$$

is called associated torus. In the real case, $S \subset S^{N-1}_{\mathbb{R},+}$, we also call T cube.

As a basic example here, for $S = S^{N-1}_\mathbb{C}$ the corresponding subspace $T \subset S$ appears by imposing the relations $|x_i| = \frac{1}{\sqrt{N}}$ to the coordinates, so we obtain a torus:

$$S = S^{N-1}_\mathbb{C} \implies T = \left\{x \in \mathbb{C}^N\Big||x_i| = \frac{1}{\sqrt{N}}\right\}$$

As for the case of the real sphere, $S = S^{N-1}_\mathbb{R}$, here the subspace $T \subset S$ appears by imposing the relations $x_i = \pm\frac{1}{\sqrt{N}}$ to the coordinates, so we obtain a cube:

$$S = S^{N-1}_\mathbb{R} \implies T = \left\{x \in \mathbb{R}^N\Big|x_i = \pm\frac{1}{\sqrt{N}}\right\}$$

Now by getting back to the spheres, we have the following result:

Theorem 12.4. *The tori of the basic spheres are all group duals, as follows,*

$$\begin{array}{ccc}
\mathbb{T}^N & \longrightarrow & \widehat{F_N}\\
\uparrow & & \uparrow\\
\mathbb{Z}_2^N & \longrightarrow & \widehat{\mathbb{Z}_2^{*N}}
\end{array}$$

where F_N is the free group on N generators, and $*$ is a group-theoretical free product.

Proof. This follows from the definitions of the various objects involved. □

Let us discuss now the compact hypersurfaces $X \subset \mathbb{R}^N_+$. These hypersurfaces fit into the C^*-algebra formalism, their definition being as follows:

Definition 12.5. *A real compact hypersurface in N variables, denoted $X_f\subset\mathbb R^N_+$, is the abstract spectrum of a universal C^*-algebra of the following type,*

$$C(X_f)=C^*\left(x_1,\ldots,x_N\Big|x_i=x_i^*,f(x_1,\ldots,x_N)=0\right)$$

with the noncommutative polynomial $f\in\mathbb R<x_1,\ldots,x_N>$ being such the maximal C^*-norm on the complex $*$-algebra $\mathbb C<x_1,\ldots,x_N>/(f)$ is bounded.

As a first result here, coming from the Gelfand theorem, we have:

Theorem 12.6. *In order for X_f to exist, the real algebraic manifold*

$$X_f^\times=\left\{x\in\mathbb R^N\Big|f(x_1,\ldots,x_N)=0\right\}$$

must be compact. In addition, in this case we have $||x_i||_\times\leq||x_i||$, for any i.

Proof. Assuming that X_f exists, the Gelfand theorem shows that the algebra of continuous functions on the manifold $X_f^\times$ in the statement appears as:

$$C(X_f^\times)=C(X_f)\Big/\Big\langle[x_i,x_j]=0\Big\rangle$$

Thus we have an embedding of compact quantum spaces $X_f^\times\subset X_f$, and the norm estimate is clear as well, because such embeddings increase the norms. □

Let us first discuss the quadratic case. We have here:

Proposition 12.7. *Given a quadratic polynomial $f\in\mathbb R<x_1,\ldots,x_N>$, written as*

$$f=\sum_{ij}A_{ij}x_ix_j+\sum_iB_ix_i+C$$

the following conditions are equivalent:

1. *X_f exists.*
2. *$X_f^\times$ is compact.*
3. *The symmetric matrix $Q=\frac{A+A^t}{2}$ is positive or negative.*

Proof. The implication (1) $\implies$ (2) being known from Theorem 12.6 above, and (2) $\iff$ (3) being well-known, we are left with proving (3) $\implies$ (1). As a first remark here, by applying the adjoint, our manifold X_f is defined by:

$$\begin{cases}\sum_{ij}A_{ij}x_ix_j+\sum_iB_ix_i+C=0\\ \sum_{ij}A_{ij}x_jx_i+\sum_iB_ix_i+C=0\end{cases}$$

In terms of $P=\frac{A-A^t}{2}$ and $Q=\frac{A+A^t}{2}$, these equations can be written as:

$$\begin{cases}\sum_{ij}P_{ij}x_ix_j=0\\ \sum_{ij}Q_{ij}x_ix_j+\sum_iB_ix_i+C=0\end{cases}$$

Let us first examine the second equation. When regarding x as a column vector, and B as a row vector, this equation becomes an equality of 1×1 matrices, as follows:

$$x^t Qx + Bx + C = 0$$

Now let us assume that Q is positive or negative. Up to a sign change, we can assume $Q > 0$. We can write $Q = UDU^t$, with $D = diag(d_i)$ and $d_i > 0$, and with $U \in O_N$. In terms of the vector $y = U^t x$, and with $E = BU$, our equation becomes:

$$y^t Dy + Ey + C = 0$$

By reverting back to sums and indices, this equation reads:

$$\sum_i d_i y_i^2 + \sum_i e_i y_i + C = 0$$

Now by making squares, this equation takes the following form:

$$\sum_i d_i \left(y_i + \frac{e_i}{2d_i} \right)^2 = c$$

By positivity, we deduce that we have the following estimate:

$$\left\| y_i + \frac{e_i}{2d_i} \right\|^2 \leq \frac{|c|}{d_i}$$

Thus our hypersurface X_f is well-defined, and we are done. □

We have in fact the following result:

THEOREM 12.8. *Up to linear changes of coordinates, the free compact quadrics in* $\mathbb{R}^N_+$ *are the empty set, the point, the standard free sphere* $S^{N-1}_{\mathbb{R},+}$*, defined by*

$$\sum_i x_i^2 = 1$$

and some intermediate spheres $S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{R},+}$*, which can be explicitly characterized. Moreover, for all these free quadrics, we have* $||x_i|| = ||x_i||_\times$*, for any* i*.*

PROOF. We use the computations from the proof of Proposition 12.7. The first equation there, making appear the matrix $P = \frac{A-A^t}{2}$, is as follows:

$$\sum_{ij} P_{ij} x_i x_j = 0$$

As for the second equation, up to a linear change of the coordinates, this reads:

$$\sum_i z_i^2 = c$$

At $c<0$ we obtain the empty set. At $c=0$ we must have $z=0$, and depending on whether the first equation is satisfied or not, we obtain either a point, or the empty set. At $c>0$ now, we can assume by rescaling $c=1$, and our second equation reads:

$$X_f\subset S^{N-1}_{\mathbb R,+}$$

As a conclusion, the solutions here are certain subspaces $S\subset S^{N-1}_{\mathbb R,+}$ which appear via equations of type $\sum_{ij}P_{ij}x_ix_j=0$, with $P\in M_N(\mathbb R)$ being antisymmetric, and with $x_1,\ldots,x_N$ appearing via $z_1,\ldots,z_N$ via a linear change of variables. Note that this latter change of variables is a priori related to P, so all this needs to be clarified.

As a first remark, when redoing the above computation with $X_f^\times$ at the place of X_f, we obtain $X_f=S^{N-1}_{\mathbb R}$, and this, because the equations $\sum_{ij}P_{ij}x_ix_j=0$ are trivial for commuting variables. We conclude that our subspaces $S\subset S^{N-1}_{\mathbb R,+}$ must satisfy:

$$S^{N-1}_{\mathbb R}\subset S\subset S^{N-1}_{\mathbb R,+}$$

Thus, we are left with investigating which such subspaces can indeed be solutions. Observe that both the extreme cases can appear as solutions, as shown by:

$$\begin{aligned}X_{2x^2+y^2+\frac{3}{2}xy+\frac{1}{2}yx}&=S^1_{\mathbb R}\\ X_{2x^2+y^2+xy+yx}&=S^1_{\mathbb R,+}\end{aligned}$$

Finally, the last assertion is clear for the empty set and for the point, and for the remaining hypersurfaces, this follows from $S^{N-1}_{\mathbb R}\subset S\subset S^{N-1}_{\mathbb R,+}$. □

Here is now yet another version of Proposition 12.7:

PROPOSITION 12.9. *Given M real linear functions $L_1,\ldots,L_M$ in N noncommuting variables $x_1,\ldots,x_N$, the following are equivalent:*

1. *$\sum_k L_k(x_1,\ldots,x_N)^2=1$ defines a compact hypersurface in $\mathbb R^N$.*
2. *$\sum_k L_k(x_1,\ldots,x_N)^2=1$ defines a compact quantum hypersurface.*
3. *The matrix formed by the coefficients of $L_1,\ldots,L_M$ has rank N.*

PROOF. The equivalence (1) $\iff$ (2) follows from the equivalence (1) $\iff$ (2) in Proposition 12.7, because the surfaces under investigation are quadrics. As for the equivalence (2) $\iff$ (3), this is well-known. More precisely, our equation read:

$$\begin{aligned}1&=\sum_k L_k(x_1,\ldots,x_N)^2\\ &=\sum_k\sum_i L_{ki}x_i\sum_j L_{kj}x_j\\ &=\sum_{ij}(L^tL)_{ij}x_ix_j\end{aligned}$$

Thus, in the context of Proposition 12.7, the underlying square matrix $A \in M_N(\mathbb{R})$ is given by $A = L^tL$. It follows that we have $Q = A = L^tL$, and so the condition $Q > 0$ is equivalent to L^tL being invertible, and so to L to have rank N, as claimed. □

In order to construct more examples, we will need the following basic fact:

Proposition 12.10. *In a C^*-algebra we have $x = x^*, ||x||^p \leq 1 \implies ||x|| \leq 1$.*

Proof. With $n \in \mathbb{N}$ being such that $2^n \geq p$, we have:

$$||x||^{2^n} = ||x^2||^{2^{n-1}} = \ldots = ||x^{2^n}|| \leq ||x^p|| \cdot ||x^{2^n-p}|| \leq 1 \cdot ||x||^{2^n-p}$$

Thus, we obtain $||x||^p \leq 1$, and so $||x|| \leq 1$, as desired. □

As an application, we have the following construction:

Proposition 12.11. *Given integers $p_i \in \mathbb{N}$, the equation*

$$\sum_i x_i^{2p_i} = 1$$

defines a noncommutative hypersurface.

Proof. This follows indeed from Proposition 12.10 above, by positivity. □

More generally, we have the following result, covering our various examples, so far:

Proposition 12.12. *Given M real linear functions $L_1, \ldots, L_M$ in N noncommuting variables $x_1, \ldots, x_N$, and exponents $p_1, \ldots, p_M \in \mathbb{N}$, the equation*

$$\sum_k L_k(x_1, \ldots, x_N)^{2p_i} = 1$$

defines a quantum hypersurface, provided that the $M \times N$ matrix formed by the coefficients of $L_1, \ldots, L_M$ has rank N.

Proof. By positivity, imposing the above equation leads to:

$$||L_k(x_1, \ldots, x_N)|| \leq 1 \quad , \quad \forall k$$

We are therefore left with the problem of uniformly bounding the norms $||x_i||$, and normally we can proceed here exactly as in the classical case. □

More generally now, we have the following result:

Theorem 12.13. *General construction of hypersurfaces, via equations of type*

$$\sum_k L_k L_k^* = 1$$

with $L_k \in \mathbb{R} < x_1, \ldots, x_N >$, improving the construction from Proposition 12.12.

Proof. This does not look obvious at all. As usual, there are some norm estimates to be worked out too, in relation with the basic inequality $||x_i||_\times \leq ||x_i||$. □

Going beyond the above looks like a non-trivial question.

12b. Advanced theory

What is $\mathbb{R}^N_+$? There are several approaches to this problem, and in each case we are looking for a triple (A,Δ,h) consisting of an operator algebra A, typically a non-unital C^*-algebra, then a comultiplication Δ, understood to come accompanied by maps ε,S too, and then a Haar integration functional h. As a starting point, we have:

1. Products. Using $\mathbb{R}^N=(\mathbb{R})^N$. At the algebra level we have $C_0(\mathbb{R}^N)=C_0(\mathbb{R})^{\otimes N}$, and this suggests setting $C_0(\mathbb{R}^N_+)=C_0(\mathbb{R})^{*N}$. Thus we have a well-defined algebra A, and we have a comultiplication Δ too. The problem is with the Haar integration h. Our belief is that this problem can be solved by using suitable $N\times N$ matrix models, with our algebra A appearing on the diagonal. This looks quite tricky.

2. Polar coordinates. Using $[0,\infty)\times S^{N-1}_\mathbb{R}\to\mathbb{R}^N$. At the algebra level we have $C_0(\mathbb{R}^N)\subset C_0[0,\infty)\otimes C(S^{N-1}_\mathbb{R})$, and the very first question is that of understanding what the subalgebra $C_0(\mathbb{R}^N)$ exactly is. Since the quotient map $[0,\infty)\times S^{N-1}_\mathbb{R}\to\mathbb{R}^N$, given by $(r,x)\to rx$, has the property $0x=0y$ for any x,y, this suggests that $C_0(\mathbb{R}^N)\subset C_0[0,\infty)\otimes C(S^{N-1}_\mathbb{R})$ consists of functions such that $f(0,x)$ does not depend on x. It is not very clear what this means, algebrically. Once this difficulty solved, we can probably go ahead and construct something similar in the free case, $C_0(\mathbb{R}^N_+)\subset C_0[0,\infty)*C(S^{N-1}_{\mathbb{R},+})$, then look for a comultiplication Δ, and a Haar functional h.

2b. An alternative approach here would be by using $\mathbb{R}^N-\{0\}=(0,\infty)\times S^{N-1}_\mathbb{R}$. Here we have at the algebra level $C_0(\mathbb{R}^N-\{0\})=C_0(0,\infty)\otimes C(S^{N-1}_\mathbb{R})$, so at least we have a clearly defined algebra, that we can generalize right away to the free settting, in the form of something of type $C_0(\mathbb{R}^N_+-\{0\})=C_0(0,\infty)*C(S^{N-1}_{\mathbb{R},+})$. However, we cannot really investigate the Δ problem in this setting, so we run once again into a difficulty, namely constructing the correct lifts $C_0(\mathbb{R}^N)$ and $C_0(\mathbb{R}^N_+)$. This being said, the question of investigating the Haar functional h seems to make sense, even in this "$-\{0\}$" setting, meaning without solving the lifting problem. This is actually quite unclear.

2c. Yet another alternative approach would be by using $P^{N-1}_\mathbb{R}$ instead of $S^{N-1}_\mathbb{R}$. The first question here is that of understanding the precise relation between the spaces $\mathbb{R}\times P^{N-1}_\mathbb{R}$ and $\mathbb{R}^N$, which is probably something well-known, but looks quite geometric and tricky. Assuming this geometric problem solved, we can probably have $C_0(\mathbb{R}^N)$ constructed afterwards in terms of $C_0(\mathbb{R})\otimes C(P^{N-1}_\mathbb{R})$, and then at the free level, we can have $C_0(\mathbb{R}^N_+)$ constructed in terms of $C_0(\mathbb{R})*C(P^{N-1}_{\mathbb{R},+})$, and then look for Δ, and for h.

2d. In fact, in modern terms, we are looking for a "free suspension of the free sphere".

3. Compactification. Using $\mathbb{R}^N=S^N_\mathbb{R}-\{\infty\}$. To be more precise, we want to use the fact that $S^N_\mathbb{R}$ appears as the 1-point compactification of $\mathbb{R}^N$, with the isomorphism being

the standard stereographic projection map. This might look like a weird idea, because it is not group-theoretical at all, the main feature of the stereographic projection being the fact that it is conformal, preserving angles, and so useful in geometry, but not in group theory. This being said, this is an idea to be explored too, especially since the formula for h should be not that complicated, and here are some preliminary computations:

Let us start with some abstract considerations. The 1-point compactification of $\mathbb{R}^N$ is indeed the sphere $S^N_\mathbb{R}$, and for precise formulae and everything, to be given later, the best is to say that the 1-point compactification of $\mathbb{R}^N = \mathbb{R}^N \times \{0\} \subset \mathbb{R}^{N+1}$ is the unit sphere $S^N_\mathbb{R} \subset \mathbb{R}^{N+1}$, with the convention that the point which is added is $\infty = (1, 0, \ldots, 0)$. Also, we make the convention that the coordinates on $\mathbb{R}^{N+1}$ are denoted $x_0, \ldots, x_N$.

In functional analysis terms, we have a diagram as follows, with all horizontal maps being inclusions, with the bar on $C_0(\mathbb{R}^N)$ standing for unitization, and with the 0 subscript to $C(S^N_\mathbb{R})$ standing for taking the ideal generated by the first coordinate x_0:

$$\begin{array}{ccccc}
C_0(\mathbb{R}^N) & \longrightarrow & \bar{C}_0(\mathbb{R}^N) & \longrightarrow & C_b(\mathbb{R}^N) \\
\Vert & & \Vert & & \\
C(S^N_\mathbb{R})_0 & \longrightarrow & C(S^N_\mathbb{R}) & &
\end{array}$$

In view of our motivations, this is not bad, because in the free case we can normally talk as well about the ideal $C(S^N_{\mathbb{R},+})_0 \subset C(S^N_{\mathbb{R},+})$ generated by the first coordinate x_0. The problem is whether we can declare this ideal to be $C_0(\mathbb{R}^N_+)$, with a Δ and h.

In order to comment on this, let us do some computations, in the classical case. We first need the precise formulae of the isomorphism $\mathbb{R}^N \simeq S^N_\mathbb{R} - \{\infty\}$, obtained in practice by identifying $\mathbb{R}^N = \mathbb{R}^N \times \{0\} \subset \mathbb{R}^{N+1}$ with the unit sphere $S^N_\mathbb{R} \subset \mathbb{R}^{N+1}$, with the convention that the point which is added is $\infty = (1, 0, \ldots, 0)$, via the stereographic projection. That is, we need the precise formulae of two inverse maps, as follows:

$$\Phi : \mathbb{R}^N \to S^N_\mathbb{R} - \{\infty\}$$

$$\Psi : S^N_\mathbb{R} - \{\infty\} \to \mathbb{R}^N$$

In one sense we must have $\Phi(v) = t(0, v) + (1-t)(1, 0)$, with $t \in (0, 1)$ being such that $||\Phi(v)|| = 1$. The equation here is $(1-t)^2 + t^2||v||^2 = 1$, which simplifies to $t^2(1 + ||v||^2) = 2t$, with solution $t = \frac{2}{1+||v||^2}$, and so the formula of Φ is as follows:

$$\Phi(v) = (1, 0) + \frac{2}{1 + ||v||^2}(-1, v)$$

In the other sense we must have $(0, \Psi(c, x)) = \alpha(c, x) + (1 - \alpha)(1, 0)$ for a certain $\alpha \in \mathbb{R}$, and from $\alpha c + 1 - \alpha = 0$ we get $\alpha = \frac{1}{1-c}$, so the formula of Ψ is as follows:

$$\Psi(c, x) = \frac{x}{1 - c}$$

Here, as before, and in what follows too, we use $\mathbb{R}^{N+1} = \mathbb{R} \times \mathbb{R}^N$, with the coordinate of $\mathbb{R}$ denoted x_0, and with the coordinates of $\mathbb{R}^N$ denoted $x_1, \ldots, x_N$.

Let us discuss now the Δ problematics. We can transport the group structure of $\mathbb{R}^N$ to a group structure on $S^N_\mathbb{R} - \{\infty\}$, as follows:

$$p \cdot q = \Phi(\Psi(p) + \Psi(q))$$

In view of the above formulae of Φ, Ψ, the multiplication on $S^N_\mathbb{R} - \{\infty\}$ that we obtain is given by the following formula:

$$\begin{aligned}
(c, x) \cdot (d, y) &= \Phi(\Psi(c, x) + \Psi(d, y)) \\
&= \Phi\left(\frac{x}{1-c} + \frac{y}{1-d}\right) \\
&= (1, 0) + \frac{2}{1+t}\left(-1, \frac{x}{1-c} + \frac{y}{1-d}\right)
\end{aligned}$$

Here the parameter t is given by the following formula:

$$t = \left|\left|\frac{x}{1-c} + \frac{y}{1-d}\right|\right|^2$$

Now by transposing, we obtain a comultiplication map as follows, with $C(S^N_\mathbb{R})_0 \subset C(S^N_\mathbb{R})$ being the ideal generated by the first coordinate x_0:

$$\Delta : C(S^N_\mathbb{R})_0 \to C(S^N_\mathbb{R})_0 \otimes C(S^N_\mathbb{R})_0$$

$$f \to \Big[(c, x), (d, y) \to f((c, x) \cdot (d, y))\Big]$$

The problem is that of slowly working out the details of this map Δ, on various products of coordinates and so on, and see if we can get a decent formula for Δ out of this, and then if this formula has a free generalization or not.

Let us discuss now the Haar problematics, which is the point where we wanted to get, where things might get simpler. As before with Δ, we can transport the Haar integration over $\mathbb{R}^N$ into an integration over $S^N_\mathbb{R} - \{\infty\}$, according to the following formula:

$$\int_{S^N_\mathbb{R} - \{\infty\}} f(x) = \int_{\mathbb{R}^N} f(\Phi(v))dv$$

In practice, according to the above formula of Φ, the precise formula is:

$$\int_{S^N_\mathbb{R}-\{\infty\}}f(x)=\int_{\mathbb{R}^N}f\left((1,0)+\frac{2}{1+||v||^2}(-1,v)\right)dv$$

Passed the details of this formula, which might look quite complicated, the transport of the Haar integration over $\mathbb{R}^N$ into an integration over $S^N_\mathbb{R}-\{\infty\}$ looks like something quite simple. Indeed, the measure on $S^N_\mathbb{R}-\{\infty\}$ should not be that far from the usual Haar measure of $S^N_\mathbb{R}$, with just a density added on the x_0 direction, and this because both measures, the transported one on $S^N_\mathbb{R}-\{\infty\}$, and the Haar one on $S^N_\mathbb{R}$, are invariant under the action of O_N, acting on the coordinates $x_1,\ldots,x_N$.

In short, we should have a formula as follows, with on the right the integration being the usual Haar one on $S^N_\mathbb{R}$, and with $\varphi:[-1,1]\to(0,\infty)$ being a certain density:

$$\int_{S^N_\mathbb{R}-\{\infty\}}f(x)=\int_{S^N_\mathbb{R}}f(x)\varphi(x_0)dx$$

Assuming all this understood, and φ explicitely computed, the extension to the free case would be probably quite routine, our conjecture being that the integration on $\mathbb{R}^N_+$, in a "free stereographic picture", should be just a modification of the usual Weingarten formula for $S^N_{\mathbb{R},+}$, via a horizontal density $\psi:[-1,1]\to(0,\infty)$, appearing as the free version of $\varphi:[-1,1]\to(0,\infty)$, in the sense of the Bercovici-Pata bijection.

12c. Matrix models

We denote by $\mathbb{C}\{x_1,\ldots,x_N\}$ the algebra of the complex noncommutative polynomials in N variables, subject to the half-commutation conditions $abc=cba$. This algebra is a $*$-algebra, with involution given on the generators by $x_i^*=x_i$. We have:

PROPOSITION 12.14. *We have an embedding of $*$-algebras*

$$\mathbb{C}\{x_1,\ldots,x_N\}\subset M_2\left(\bar{\mathbb{C}}[z_1,\ldots,z_N]\right)\quad,\quad x_i=\begin{pmatrix}0&z_i\\ \bar{z}_i&0\end{pmatrix}$$

where $\bar{\mathbb{C}}[z_1,\ldots,z_N]$ are the polynomials in N commuting variables, and their conjugates.

PROOF. We have indeed a morphism from left to right, because the following matrices are self-adjoint, and half-commute as well:

$$X_i=\begin{pmatrix}0&z_i\\ \bar{z}_i&0\end{pmatrix}$$

Indeed, the products of such matrices are given by:

$$X_iX_j=\begin{pmatrix}0&z_i\\ \bar{z}_i&0\end{pmatrix}\begin{pmatrix}0&z_j\\ \bar{z}_j&0\end{pmatrix}=\begin{pmatrix}z_i\bar{z}_j&0\\ 0&\bar{z}_iz_j\end{pmatrix}$$

Thus the 3-fold products are given by:

$$X_iX_jX_k=\begin{pmatrix}z_i\bar{z}_j&0\\0&\bar{z}_iz_j\end{pmatrix}\begin{pmatrix}0&z_k\\\bar{z}_k&0\end{pmatrix}=\begin{pmatrix}0&z_i\bar{z}_jz_k\\\bar{z}_iz_j\bar{z}_k&0\end{pmatrix}$$

Now since this quantity is symmetric in i,k, we have $X_iX_kX_k=X_kX_jX_i$, as desired. In order to prove now the injectivity, we use the following vector space decomposition, coming by considering the spans of the monomials of even and odd length:

$$\mathbb{C}\{x_1,\ldots,x_N\}=\mathbb{C}_{even}\{x_1,\ldots,x_N\}\oplus\mathbb{C}_{odd}\{x_1,\ldots,x_N\}$$

On the even part, which is a subalgebra, our morphism is given by:

$$x_{i_1}x_{j_1}\ldots x_{i_k}x_{j_k}\to\begin{pmatrix}z_{i_1}\bar{z}_{j_1}\ldots z_k\bar{z}_{j_k}&0\\0&\bar{z}_{i_1}z_{j_1}\ldots\bar{z}_{i_k}z_{j_k}\end{pmatrix}$$

Since both the monomials at left, and the upper left entries at right, at a given length $k\in\mathbb{N}$, are only subject to the double action of S_k on the i,j indices, we have the injectivity. On the odd part the situation is similar, our morphism being given by:

$$x_{i_1}x_{j_1}\ldots x_{i_k}x_{j_k}x_{i_{k+1}}\to\begin{pmatrix}0&z_{i_1}\bar{z}_{j_1}\ldots z_k\bar{z}_{j_k}z_{i_{k+1}}\\\bar{z}_{i_1}z_{j_1}\ldots\bar{z}_{i_k}z_{j_k}\bar{z}_{i_{k+1}}&0\end{pmatrix}$$

At length $k\in\mathbb{N}$, both the monomials at left, and those at the upper right corner at right, being only subject to the actions of S_{k+1},S_k on the i,j indices. □

We have the following statement:

THEOREM 12.15. *The half-classical hypersurfaces X_f which are symmetric, in the sense that $f(x_1,\ldots,x_N)=f(-x_1,\ldots,-x_N)$, have matrix models of type*

$$C(X_f)\subset M_2\left(C(Z_{\tilde{f}})\right)\quad,\quad x_i=\begin{pmatrix}0&z_i\\\bar{z}_i&0\end{pmatrix}$$

where $\tilde{f}\in\bar{\mathbb{C}}[z_1,\ldots,z_N]$ is a certain polynomial associated to $f\in\mathbb{C}\{x_1,\ldots,x_N\}$. In addition, the construction $f\to\tilde{f}$ is such that X_f exists precisely when $Z_{\tilde{f}}$ is bounded. There should be as well an extension of all this, to the general (non-symmetric) case.

PROOF. The first piece of evidence comes from the following embedding:

$$C(S^{N-1}_{\mathbb{R},*})\subset M_2(C(S^{N-1}_{\mathbb{C}}))$$

In our present language, this embedding reformulates as:

$$f(x_1,\ldots,x_N)=\sum_i x_i^2-1\implies\tilde{f}(z_1,\ldots,z_N)=\sum_i|z_i|^2-1$$

This suggests to construct $f\to\tilde{f}$ in general by suitably replacing $x_i\to z_i,\bar{z}_i$, with the exact choice here depending on the position of each x_i inside the monomial.

As for a potential proof, the idea here would be to say that $Z_{\tilde{f}}$ is the classical lift of the projective version PX_f, and to conclude by using a grading trick. □

Let us recall that we have:

PROPOSITION 12.16. *Given any real compact quantum manifold, appearing via a formula of type $C(X)=C^*(x_1,\ldots,x_N|x_i=x_i^*,f_i(x)=0)$, we can construct a filtration*

$$X^\times=X^{(1)}\subset X^{(2)}\subset\ldots\subset X^{(\infty)}\subset X$$

with $X^{(K)}$ with $K<\infty$ being constructed by restricting the attention to the $K\times K$ matrix realizations of the coordinates, and with $X^{(\infty)}=\cup_{K<\infty}X^{(K)}$, provided that $X^\times\neq\emptyset$.

PROOF. All this is known, without the real manifold assumption $x_i=x_i^*$, and under the supplementary assumption $X\subset S^{N-1}_{\mathbb{C},+}$, which becomes in our case $X\subset S^{N-1}_{\mathbb{R},+}$, and which is in fact not needed. To be more precise, the situation is as follows:

(1) The construction of each $X^{(K)}$ with $K<\infty$ is straightforward, by dividing $C(X)$ by a suitable ideal, and the fact that we have $X^{(1)}=X^\times$ is clear as well.

(2) The fact that each $X^{(K)}$ is algebraic is known under the assumption $X^\times\neq\emptyset$, but is it a bit unclear whether this assumption is really needed, or not. The inclusions $X^{(K)}\subset X^{(K+1)}$ are constructed by using a point $p\in X^\times$, and here once again there is some work to be done, with the exact dependence on this point in need to be clarified. Note that all this is not entirely trivial, as it uses advanced material.

(3) Finally, once we have the inclusions $X^{(K)}\subset X^{(K+1)}$ we can talk about the inductive limit $X^{(\infty)}=\cup_{K<\infty}X^{(K)}$ as well, and besides clarifying the functoriality aspects, one problem here is that of fully understanding when $X^{(\infty)}\subset X$ is an equality. □

The above result applies of course to our quantum hypersurfaces X_f, and so in particular by looking at the various norms of the coordinates we have:

$$||x_i||_\times=||x_i||_1\leq||x_i||_2\leq\ldots\leq||x_i||_\infty\leq||x_i||$$

This refines our original observation $||x_i||_\times\leq||x_i||$. There is probably some interesting work to be done here, once we have a big class of examples $\{X_f\}$ to play with:

THEOREM 12.17. *Matrix truncations of the real compact quantum hypersurfaces: generalities, and some norm estimates.*

PROOF. For the moment, it is not very clear what we can do here. We would need, as a minimal input, a complete theory of quantum quadrics. □

12d. Projective manifolds

There are several simplifications and subtleties in the projective setting.

12e. Exercises

Part IV

Deformation theory

Not bothered about religion
Not bothered about belief
In the house of love
Everybody's free

CHAPTER 13

Standard twists

13a. Ad-hoc twisting

In order to get started, the best is to deform first the simplest objects that we have, namely the quantum spheres. This can be done as follows:

THEOREM 13.1. *We have quantum spheres as follows, obtained via the twisted commutation relations $ab = \pm ba$, and twisted half-commutation relations $abc = \pm cba$,*

$$\begin{array}{ccccc}
S^{N-1}_{\mathbb{R},+} & \longrightarrow & \mathbb{T}S^{N-1}_{\mathbb{R},+} & \longrightarrow & S^{N-1}_{\mathbb{C},+}\\
\uparrow & & \uparrow & & \uparrow\\
\bar{S}^{N-1}_{\mathbb{R},*} & \longrightarrow & \mathbb{T}\bar{S}^{N-1}_{\mathbb{R},*} & \longrightarrow & \bar{S}^{N-1}_{\mathbb{C},*}\\
\uparrow & & \uparrow & & \uparrow\\
\bar{S}^{N-1}_{\mathbb{R}} & \longrightarrow & \mathbb{T}\bar{S}^{N-1}_{\mathbb{R}} & \longrightarrow & \bar{S}^{N-1}_{\mathbb{C}}
\end{array}$$

with the precise signs being as follows:

(1) *The signs on the bottom correspond to the anticommutation of distinct coordinates, and their adjoints. That is, with $z_i = x_i, x_i^*$ and $\varepsilon_{ij} = 1 - \delta_{ij}$:*

$$z_i z_j = (-1)^{\varepsilon_{ij}} z_j z_i$$

(2) *The signs in the middle come from functoriality, as for the spheres in the middle to contain those on the bottom. That is, the formula is:*

$$z_i z_j z_k = (-1)^{\varepsilon_{ij}+\varepsilon_{jk}+\varepsilon_{ik}} z_k z_j z_i$$

PROOF. All this is elementary, as follows:

(1) Here there is nothing to prove, because we can define the spheres on the bottom by the following formulae, with $z_i = x_i, x_i^*$ and $\varepsilon_{ij} = 1 - \delta_{ij}$ being as above:

$$C(\bar{S}^{N-1}_{\mathbb{R}}) = C(S^{N-1}_{\mathbb{R},+}) \Big/ \Big\langle x_i x_j = (-1)^{\varepsilon_{ij}} x_j x_i \Big\rangle$$

$$C(\bar{S}^{N-1}_{\mathbb{C}}) = C(S^{N-1}_{\mathbb{C},+}) \Big/ \Big\langle z_i z_j = (-1)^{\varepsilon_{ij}} z_j z_i \Big\rangle$$

(2) Here our claim is that, if we want to construct half-classical twisted spheres, via relations of type $abc = \pm cba$ between the coordinates x_i and their adjoints x_i^*, as for these spheres to contain the twisted spheres constructed in (1), the only possible choice for these relations is as follows, with $z_i = x_i, x_i^*$ and $\varepsilon_{ij} = 1 - \delta_{ij}$ being as above:

$$z_i z_j z_k = (-1)^{\varepsilon_{ij}+\varepsilon_{jk}+\varepsilon_{ik}} z_k z_j z_i$$

But this is something clear, coming from the following computation, inside of the quotient algebras corresponding to the twisted spheres constructed in (1) above:

$$\begin{aligned} z_i z_j z_k &= (-1)^{\varepsilon_{ij}} z_j z_i z_k \\ &= (-1)^{\varepsilon_{ij}+\varepsilon_{ik}} z_j z_k z_i \\ &= (-1)^{\varepsilon_{ij}+\varepsilon_{jk}+\varepsilon_{ik}} z_k z_j z_i \end{aligned}$$

Thus, we are led to the conclusion in the statement, the spheres being given by:

$$C(\bar{S}^{N-1}_{\mathbb{R},*}) = C(S^{N-1}_{\mathbb{R},+}) \Big/ \Big\langle x_i x_j x_k = (-1)^{\varepsilon_{ij}+\varepsilon_{jk}+\varepsilon_{ik}} x_k x_j x_i \Big\rangle$$

$$C(\bar{S}^{N-1}_{\mathbb{C},*}) = C(S^{N-1}_{\mathbb{C},+}) \Big/ \Big\langle z_i z_j z_k = (-1)^{\varepsilon_{ij}+\varepsilon_{jk}+\varepsilon_{ik}} z_k z_j z_i \Big\rangle$$

Thus, we have constructed our spheres, and embeddings, as desired. □

Let us first discuss the twisting of O_N, U_N. Following [13] in the orthogonal case, and the known literaure in the unitary case, the result here is as follows:

THEOREM 13.2. *We have twisted orthogonal and unitary groups, as follows,*

$$\begin{array}{ccc} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ \bar{O}_N & \longrightarrow & \bar{U}_N \end{array}$$

defined via the following relations, with the convention $\alpha = a, a^*$ and $\beta = b, b^*$:

$$\alpha\beta = \begin{cases} -\beta\alpha & \text{for } a, b \in \{u_{ij}\} \text{ distinct, on the same row or column of } u \\ \beta\alpha & \text{otherwise} \end{cases}$$

These quantum groups act on the corresponding twisted real and complex spheres.

PROOF. Let us first discuss the construction of the quantum group $\bar{O}_N$. We must prove that the algebra $C(\bar{O}_N)$ obtained from $C(O_N^+)$ via the relations in the statement has a comultiplication Δ, a counit ε, and an antipode S. Regarding Δ, let us set:

$$U_{ij} = \sum_k u_{ik} \otimes u_{kj}$$

For $j\neq k$ we have the following computation:

$$\begin{aligned}U_{ij}U_{ik} &= \sum_{s\neq t}u_{is}u_{it}\otimes u_{sj}u_{tk}+\sum_s u_{is}u_{is}\otimes u_{sj}u_{sk}\\ &= \sum_{s\neq t}-u_{it}u_{is}\otimes u_{tk}u_{sj}+\sum_s u_{is}u_{is}\otimes(-u_{sk}u_{sj})\\ &= -U_{ik}U_{ij}\end{aligned}$$

Also, for $i\neq k,j\neq l$ we have the following computation:

$$\begin{aligned}U_{ij}U_{kl} &= \sum_{s\neq t}u_{is}u_{kt}\otimes u_{sj}u_{tl}+\sum_s u_{is}u_{ks}\otimes u_{sj}u_{sl}\\ &= \sum_{s\neq t}u_{kt}u_{is}\otimes u_{tl}u_{sj}+\sum_s(-u_{ks}u_{is})\otimes(-u_{sl}u_{sj})\\ &= U_{kl}U_{ij}\end{aligned}$$

Thus, we can define a comultiplication map for $C(\bar{O}_N)$, by setting:

$$\Delta(u_{ij})=U_{ij}$$

Regarding now the counit ε and the antipode S, things are clear here, by using the same method, and with no computations needed, the formulae to be satisfied being trivially satisfied. We conclude that $\bar{O}_N$ is a compact quantum group, and the proof for $\bar{U}_N$ is similar, by adding $*$ exponents everywhere in the above computations.

Finally, the last assertion is clear too, by doing some elementary computations, of the same type as above, and with the remark that the converse holds too, in the sense that if we want a quantum group $U\subset U_N^+$ to be defined by relations of type $ab=\pm ba$, and to have an action $U\curvearrowright S$ on the corresponding twisted sphere, we are led to the relations in the statement. We refer to the literature for further details on all this. □

In order to discuss now the half-classical case, given three coordinates $a,b,c\in\{u_{ij}\}$, let us set $span(a,b,c)=(r,c)$, where $r,c\in\{1,2,3\}$ are the number of rows and columns spanned by a,b,c. In other words, if we write $a=u_{ij},b=u_{kl},c=u_{pq}$ then $r=\#\{i,k,p\}$ and $l=\#\{j,l,q\}$. With this convention, we have the following result:

THEOREM 13.3. *We have intermediate quantum groups as follows,*

$$\begin{array}{ccccc}
O_N^+ & \longrightarrow & \mathbb{T}O_N^+ & \longrightarrow & U_N^+\\
\uparrow & & \uparrow & & \uparrow\\
\bar{O}_N^* & \longrightarrow & \mathbb{T}\bar{O}_N^* & \longrightarrow & \bar{U}_N^*\\
\uparrow & & \uparrow & & \uparrow\\
\bar{O}_N & \longrightarrow & \mathbb{T}\bar{O}_N & \longrightarrow & \bar{U}_N
\end{array}$$

defined via the following relations, with $\alpha = a, a^*$, $\beta = b, b^*$ and $\gamma = c, c^*$,

$$\alpha\beta\gamma = \begin{cases} -\gamma\beta\alpha & \text{for } a,b,c \in \{u_{ij}\} \text{ with } span(a,b,c) = (\leq 2, 3) \text{ or } (3, \leq 2)\\ \gamma\beta\alpha & \text{otherwise}\end{cases}$$

which act on the corresponding twisted half-classical real and complex spheres.

PROOF. Observe first that the rules in the statement can be summarized as follows:

$r\backslash c$	1	2	3
1	+	+	−
2	+	+	−
3	−	−	+

Let us first prove the result for $\bar{O}_N^*$. The proof goes as follows:

(1) We first construct Δ. For this purpose, we must prove that $U_{ij} = \sum_k u_{ik} \otimes u_{kj}$ satisfy the relations in the statement. We have the following computation:

$$\begin{aligned}
U_{ia}U_{jb}U_{kc} &= \sum_{xyz} u_{ix}u_{jy}u_{kz} \otimes u_{xa}u_{yb}u_{zc}\\
&= \sum_{xyz} \pm u_{kz}u_{jy}u_{ix} \otimes \pm u_{zc}u_{yb}u_{xa}\\
&= \pm U_{kc}U_{jb}U_{ia}
\end{aligned}$$

We must show that, when examining the precise two $\pm$ signs in the middle formula, their product produces the correct $\pm$ sign at the end. But the point is that both these signs depend only on $s = span(x, y, z)$, and for $s = 1, 2, 3$ respectively, we have:

– For a $(3, 1)$ span we obtain $+-$, $+-$, $-+$, so a product $-$ as needed.

– For a $(2, 1)$ span we obtain $++$, $++$, $--$, so a product $+$ as needed.

– For a $(3, 3)$ span we obtain $--$, $--$, $++$, so a product $+$ as needed.

– For a $(3, 2)$ span we obtain $+-$, $+-$, $-+$, so a product $-$ as needed.

– For a $(2, 2)$ span we obtain $++$, $++$, $--$, so a product $+$ as needed.

Together with the fact that our problem is invariant under $(r,c)\to(c,r)$, and with the fact that for a $(1,1)$ span there is nothing to prove, this finishes the proof for Δ.

(2) The construction of the counit, via the formula $\varepsilon(u_{ij})=\delta_{ij}$, requires the Kronecker symbols δ_{ij} to commute/anticommute according to the above table. Equivalently, we must prove that the situation $\delta_{ij}\delta_{kl}\delta_{pq}=1$ can appear only in a case where the above table indicates "+". But this is clear, because $\delta_{ij}\delta_{kl}\delta_{pq}=1$ implies $r=c$.

(3) Finally, the construction of the antipode, via the formula $S(u_{ij})=u_{ji}$, is clear too, because this requires the choice of our $\pm$ signs to be invariant under transposition, and this is true, the above table being symmetric.

We conclude that $\bar{O}_N^*$ is indeed a compact quantum group, and the proof for $\bar{U}_N^*$ is similar, by adding $*$ exponents everywhere in the above computations.

Finally, the last assertion is clear too, by doing some elementary computations, of the same type as above, and with the remark that the converse holds too, in the sense that if we want a quantum group $U\subset U_N^+$ to be defined by relations of type $abc=\pm cba$, and to have an action $U\curvearrowright S$ on the corresponding half-classical twisted sphere, we are led to the relations in the statement. We refer to the literature for further details on all this. $\square$

The above results can be summarized as follows:

THEOREM 13.4. *We have quantum groups as follows, obtained via the twisted commutation relations $ab=\pm ba$, and twisted half-commutation relations $abc=\pm cba$,*

$$\begin{array}{ccccc}
O_N^+ & \longrightarrow & \mathbb{T}O_N^+ & \longrightarrow & U_N^+\\
\uparrow & & \uparrow & & \uparrow\\
\bar{O}_N^* & \longrightarrow & \mathbb{T}\bar{O}_N^* & \longrightarrow & \bar{U}_N^*\\
\uparrow & & \uparrow & & \uparrow\\
\bar{O}_N & \longrightarrow & \mathbb{T}\bar{O}_N & \longrightarrow & \bar{U}_N
\end{array}$$

with the various signs coming as follows:

1. *The signs for $\bar{O}_N$ correspond to anticommutation of distinct entries on rows and columns, and commutation otherwise, with this coming from $\bar{O}_N\curvearrowright\bar{S}^{N-1}_{\mathbb{R}}$.*
2. *The signs for $\bar{O}_N^*,\bar{U}_N,\bar{U}_N^*$ come as well from the signs for $\bar{S}^{N-1}_{\mathbb{R}}$, either via the requirement $\bar{O}_N\subset U$, or via the requirement $U\curvearrowright S$.*

PROOF. This is a summary of Theorem 13.2 and Theorem 13.3, and their proofs. $\square$

13b. Schur-Weyl twists

Let us start with something that we already know, from the liberation theory developed in the above, using Brauer type algebras, namely:

PROPOSITION 13.5. *The intermediate easy quantum groups*

$$H_N \subset G \subset U_N^+$$

come via Tannakian duality from the intermediate categories of partitions

$$P_{even} \supset D \supset \mathcal{NC}_2$$

with $P_{even}(k,l) \subset P(k,l)$ being the category of partitions whose blocks have even size.

PROOF. Indeed, the easy quantum groups appear as certain intermediate compact quantum groups, as follows:

$$S_N \subset G \subset U_N^+$$

To be more precise, such a quantum group is easy when the corresponding Tannakian category comes from an intermediate category of partitions, as follows:

$$P \supset D \supset \mathcal{NC}_2$$

Now since this correspondence makes correspond $H_N \leftrightarrow P_{even}$, once again as explained in section 2 above, we are led to the conclusion in the statement. □

Summarizing, we must do some combinatorics, for the partitions having even blocks. Given a partition $\tau \in P(k,l)$, let us call "switch" the operation which consists in switching two neighbors, belonging to different blocks, in the upper row, or in the lower row. Also, we use the standard embedding $S_k \subset P_2(k,k)$, via the pairings having only up-to-down strings. With these conventions, we have the following result:

THEOREM 13.6. *There is a signature map $\varepsilon : P_{even} \to \{-1,1\}$, given by*

$$\varepsilon(\tau) = (-1)^c$$

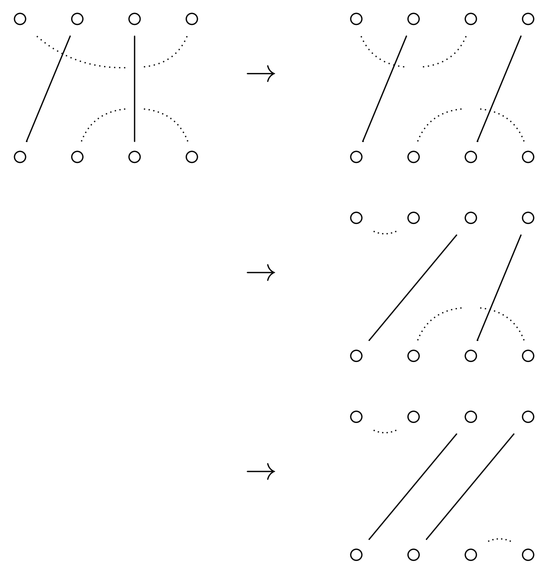
where c is the number of switches needed to make τ noncrossing. In addition:

(1) *For $\tau \in S_k$, this is the usual signature.*
(2) *For $\tau \in P_2$ we have $(-1)^c$, where c is the number of crossings.*
(3) *For $\tau \leq \pi \in NC_{even}$, the signature is 1.*

PROOF. In order to show that the signature map $\varepsilon : P_{even} \to \{-1,1\}$ in the statement, given by $\varepsilon(\tau) = (-1)^c$, is well-defined, we must prove that the number c in the statement is well-defined modulo 2. It is enough to perform the verification for the noncrossing partitions. More precisely, given $\tau, \tau' \in NC_{even}$ having the same block structure, we must prove that the number of switches c required for the passage $\tau \to \tau'$ is even.

In order to do so, observe that any partition $\tau \in P(k,l)$ can be put in "standard form", by ordering its blocks according to the appearence of the first leg in each block, counting clockwise from top left, and then by performing the switches as for block 1 to be at left, then for block 2 to be at left, and so on. Here the required switches are also uniquely determined, by the order coming from counting clockwise from top left.

Here is an example of such an algorithmic switching operation, with block 1 being first put at left, by using two switches, then with block 2 left unchanged, and then with block 3 being put at left as well, but at right of blocks 1 and 2, with one switch:



The point now is that, under the assumption $\tau \in NC_{even}(k,l)$, each of the moves required for putting a leg at left, and hence for putting a whole block at left, requires an even number of switches. Thus, putting τ is standard form requires an even number of switches. Now given $\tau,\tau' \in NC_{even}$ having the same block structure, the standard form coincides, so the number of switches c required for the passage $\tau \to \tau'$ is indeed even.

Regarding now the remaining assertions, these are all elementary:

(1) For $\tau \in S_k$ the standard form is $\tau' = id$, and the passage $\tau \to id$ comes by composing with a number of transpositions, which gives the signature.

(2) For a general $\tau \in P_2$, the standard form is of type $\tau' = |\ldots|^{\cup\ldots\cup}_{\cap\ldots\cap}$, and the passage $\tau \to \tau'$ requires c mod 2 switches, where c is the number of crossings.

(3) Assuming that $\tau \in P_{even}$ comes from $\pi \in NC_{even}$ by merging a certain number of blocks, we can prove that the signature is 1 by proceeding by recurrence. $\square$

We define the kernel of a multi-index $\binom{i}{j}$ to be the partition obtained by joining the equal indices. Also, we write $\pi \leq \sigma$ if each block of π is contained in a block of σ.

With these conventions, and the above result in hand, we can now formulate:

Definition 13.7. *Associated to any partition $\pi\in P_{even}(k,l)$ is the linear map*

$$\bar{T}_\pi:(\mathbb{C}^N)^{\otimes k}\to(\mathbb{C}^N)^{\otimes l}$$

given by the following formula, with $e_1,\ldots,e_N$ being the standard basis of $\mathbb{C}^N$,

$$\bar{T}_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j_1\ldots j_l}\bar{\delta}_\pi\begin{pmatrix}i_1&\ldots&i_k\\ j_1&\ldots&j_l\end{pmatrix}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

and where $\bar{\delta}_\pi\in\{-1,0,1\}$ is $\bar{\delta}_\pi=\varepsilon(\tau)$ if $\tau\geq\pi$, and $\bar{\delta}_\pi=0$ otherwise, with:

$$\tau=\ker\begin{pmatrix}i\\ j\end{pmatrix}$$

In other words, what we are doing here is to add signatures to the usual formula of T_π. Indeed, observe that the usual formula for T_π can be written as folllows:

$$T_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{j:\ker(^i_j)\geq\pi}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

Now by inserting signs, coming from the signature map $\varepsilon:P_{even}\to\{\pm1\}$, we are led to the following formula, which coincides with the one given above:

$$\bar{T}_\pi(e_{i_1}\otimes\ldots\otimes e_{i_k})=\sum_{\tau\geq\pi}\varepsilon(\tau)\sum_{j:\ker(^i_j)=\tau}e_{j_1}\otimes\ldots\otimes e_{j_l}$$

We will be back later to this analogy, with more details on what can be done with it. For the moment, we must first prove a key categorical result, as follows:

Proposition 13.8. *The assignement $\pi\to\bar{T}_\pi$ is categorical, in the sense that*

$$\bar{T}_\pi\otimes\bar{T}_\sigma=\bar{T}_{[\pi\sigma]}\quad,\quad\bar{T}_\pi\bar{T}_\sigma=N^{c(\pi,\sigma)}\bar{T}_{[^\sigma_\pi]}\quad,\quad\bar{T}_\pi^*=\bar{T}_{\pi^*}$$

where $c(\pi,\sigma)$ are certain positive integers.

Proof. We have to go back to the proof from the untwisted case, and insert signs. We have to check three conditions, as follows:

1. Concatenation. In the untwisted case, this was based on the following formula:

$$\delta_\pi\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}\delta_\sigma\begin{pmatrix}k_1\ldots k_r\\ l_1\ldots l_s\end{pmatrix}=\delta_{[\pi\sigma]}\begin{pmatrix}i_1\ldots i_p&k_1\ldots k_r\\ j_1\ldots j_q&l_1\ldots l_s\end{pmatrix}$$

In the twisted case, it is enough to check the following formula:

$$\varepsilon\left(\ker\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}\right)\varepsilon\left(\ker\begin{pmatrix}k_1\ldots k_r\\ l_1\ldots l_s\end{pmatrix}\right)=\varepsilon\left(\ker\begin{pmatrix}i_1\ldots i_p&k_1\ldots k_r\\ j_1\ldots j_q&l_1\ldots l_s\end{pmatrix}\right)$$

Let us denote by τ,ν the partitions on the left, so that the partition on the right is of the form $\rho\leq[\tau\nu]$. Now by switching to the noncrossing form, $\tau\to\tau'$ and $\nu\to\nu'$, the partition on the right transforms into:

$$\rho\to\rho'\leq[\tau'\nu']$$

Now since the partition $[\tau'\nu']$ is noncrossing, we can use Theorem 13.6 (3), and we obtain the result.

2. Composition. In the untwisted case, this was based on the following formula:

$$\sum_{j_1\ldots j_q}\delta_\pi\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}\delta_\sigma\begin{pmatrix}j_1\ldots j_q\\ k_1\ldots k_r\end{pmatrix}=N^{c(\pi,\sigma)}\delta_{[^\pi_\sigma]}\begin{pmatrix}i_1\ldots i_p\\ k_1\ldots k_r\end{pmatrix}$$

In order to prove now the result in the twisted case, it is enough to check that the signs match. More precisely, we must establish the following formula:

$$\varepsilon\left(\ker\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}\right)\varepsilon\left(\ker\begin{pmatrix}j_1\ldots j_q\\ k_1\ldots k_r\end{pmatrix}\right)=\varepsilon\left(\ker\begin{pmatrix}i_1\ldots i_p\\ k_1\ldots k_r\end{pmatrix}\right)$$

Let τ,ν be the partitions on the left, so that the partition on the right is of the form $\rho\leq[^\tau_\nu]$. Our claim is that we can jointly switch τ,ν to the noncrossing form. Indeed, we can first switch as for $\ker(j_1\ldots j_q)$ to become noncrossing, and then switch the upper legs of τ, and the lower legs of ν, as for both these partitions to become noncrossing. Now observe that when switching in this way to the noncrossing form, $\tau\to\tau'$ and $\nu\to\nu'$, the partition on the right transforms into:

$$\rho\to\rho'\leq[^{\tau'}_{\nu'}]$$

Now since the partition $[^{\tau'}_{\nu'}]$ is noncrossing, we can apply Theorem 13.6 (3), and we obtain the result.

3. Involution. Here we must prove the following formula:

$$\bar{\delta}_\pi\begin{pmatrix}i_1\ldots i_p\\ j_1\ldots j_q\end{pmatrix}=\bar{\delta}_{\pi^*}\begin{pmatrix}j_1\ldots j_q\\ i_1\ldots i_p\end{pmatrix}$$

But this is clear from the definition of $\bar{\delta}_\pi$, and we are done. □

As a conclusion, our twisted construction $\pi\to\bar{T}_\pi$ has all the needed properties for producing quantum groups, via Tannakian duality, and we can now formulate:

THEOREM 13.9. *Given a category of partitions $D\subset P_{even}$, the construction*

$$Hom(u^{\otimes k},u^{\otimes l})=span\left(\bar{T}_\pi\Big|\pi\in D(k,l)\right)$$

produces via Tannakian duality a quantum group $\bar{G}_N\subset U_N^+$, for any $N\in\mathbb{N}$.

PROOF. This follows indeed from Tannakian duality. □

We can unify the easy quantum groups, or at least the examples coming from categories $D \subset P_{even}$, with the quantum groups constructed above, as follows:

Definition 13.10. *A closed subgroup $G \subset U_N^+$ is called q-easy, or quizzy, with deformation parameter $q = \pm 1$, when its tensor category appears as follows,*

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(\dot{T}_\pi \Big| \pi \in D(k,l)\right)$$

for a certain category of partitions $D \subset P_{even}$, where, for $q = -1, 1$:

$$\dot{T} = \bar{T}, T$$

The Schur-Weyl twist of G is the quizzy quantum group $\bar{G} \subset U_N^+$ obtained via $q \to -q$.

We will see later on that the easy quantum group associated to P_{even} itself is the hyperochahedral group H_N, and so that our assumption $D \subset P_{even}$, replacing $D \subset P$, simply corresponds to $H_N \subset G$, replacing the usual condition $S_N \subset G$.

For the moment, our most pressing task is that of checking that, when applying the Schur-Weyl twisting to the basic unitary quantum groups, we obtain the ad-hoc twists that we previously constructed. This is indeed the case:

Theorem 13.11. *The twisted unitary quantum groups introduced before,*

$$\begin{matrix} O_N^+ & \longrightarrow & \mathbb{T}O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow & & \uparrow \\ \bar{O}_N^* & \longrightarrow & \mathbb{T}\bar{O}_N^* & \longrightarrow & \bar{U}_N^* \\ \uparrow & & \uparrow & & \uparrow \\ \bar{O}_N & \longrightarrow & \mathbb{T}\bar{O}_N & \longrightarrow & \bar{U}_N \end{matrix}$$

appear as Schur-Weyl twists of the basic unitary quantum groups.

Proof. This is something routine, in several steps, as follows:

(1) The basic crossing, $\ker\binom{ij}{ji}$ with $i \neq j$, comes from the transposition $\tau \in S_2$, so its signature is -1. As for its degenerated version $\ker\binom{ii}{ii}$, this is noncrossing, so here the signature is 1. We conclude that the linear map associated to the basic crossing is:

$$\bar{T}_{\slash\!\!\!\backslash}(e_i \otimes e_j) = \begin{cases} -e_j \otimes e_i & \text{for } i \neq j \\ e_j \otimes e_i & \text{otherwise} \end{cases}$$

For the half-classical crossing, namely $\ker\binom{ijk}{kji}$ with i, j, k distinct, the signature is once again -1, and by examining the signatures of the various degenerations of this half-classical crossing, we are led to the following formula:

$$\bar{T}_{\slash\!\!\!|\!\!\!\backslash}(e_i \otimes e_j \otimes e_k) = \begin{cases} -e_k \otimes e_j \otimes e_i & \text{for } i,j,k \text{ distinct} \\ e_k \otimes e_j \otimes e_i & \text{otherwise} \end{cases}$$

(2) Our claim now if that for an orthogonal quantum group G, the following holds, with the quantum group $\bar{O}_N$ being the one in Theorem 13.2:

$$\bar{T}_{\slash\!\!\!\backslash} \in End(u^{\otimes 2}) \iff G \subset \bar{O}_N$$

Indeed, by using the formula of $\bar{T}_{\slash\!\!\!\backslash}$ found in (1) above, we obtain:

$$\begin{aligned}(\bar{T}_{\slash\!\!\!\backslash} \otimes 1)u^{\otimes 2}(e_i \otimes e_j \otimes 1) &= \sum_k e_k \otimes e_k \otimes u_{ki}u_{kj} \\ &- \sum_{k \neq l} e_l \otimes e_k \otimes u_{ki}u_{lj}\end{aligned}$$

On the other hand, we have as well the following formula:

$$u^{\otimes 2}(\bar{T}_{\slash\!\!\!\backslash} \otimes 1)(e_i \otimes e_j \otimes 1) = \begin{cases} \sum_{kl} e_l \otimes e_k \otimes u_{li}u_{ki} & \text{if } i = j \\ -\sum_{kl} e_l \otimes e_k \otimes u_{lj}u_{ki} & \text{if } i \neq j \end{cases}$$

For $i = j$ the conditions are $u_{ki}^2 = u_{ki}^2$ for any k, and $u_{ki}u_{li} = -u_{li}u_{ki}$ for any $k \neq l$. For $i \neq j$ the conditions are $u_{ki}u_{kj} = -u_{kj}u_{ki}$ for any k, and $u_{ki}u_{lj} = u_{lj}u_{ki}$ for any $k \neq l$. Thus we have exactly the relations between the coordinates of $\bar{O}_N$, and we are done.

(3) Our claim now if that for an orthogonal quantum group G, the following holds, with the quantum group $\bar{O}_N^*$ being the one in Theorem 13.3:

$$\bar{T}_{\slash\!\!\!|\!\!\!\backslash} \in End(u^{\otimes 3}) \iff G \subset \bar{O}_N^*$$

Indeed, by using the formula of $\bar{T}_{\slash\!\!\!|\!\!\!\backslash}$ found in (1) above, we obtain:

$$\begin{aligned}(\bar{T}_{\slash\!\!\!|\!\!\!\backslash} \otimes 1)u^{\otimes 2}(e_i \otimes e_j \otimes e_k \otimes 1) &= \sum_{abc\ not\ distinct} e_c \otimes e_b \otimes e_a \otimes u_{ai}u_{bj}u_{ck} \\ &- \sum_{a,b,c\ distinct} e_c \otimes e_b \otimes e_a \otimes u_{ai}u_{bj}u_{ck}\end{aligned}$$

On the other hand, we have as well the following formula:

$$\begin{aligned}&u^{\otimes 2}(\bar{T}_{\slash\!\!\!|\!\!\!\backslash} \otimes 1)(e_i \otimes e_j \otimes e_k \otimes 1) \\ &= \begin{cases} \sum_{abc} e_c \otimes e_b \otimes e_a \otimes u_{ck}u_{bj}u_{ai} & \text{for } i,j,k \text{ not distinct} \\ -\sum_{abc} e_c \otimes e_b \otimes e_a \otimes u_{ck}u_{bj}u_{ai} & \text{for } i,j,k \text{ distinct} \end{cases}\end{aligned}$$

For i,j,k not distinct the conditions are $u_{ai}u_{bj}u_{ck} = u_{ck}u_{bj}u_{ai}$ for a,b,c not distinct, and $u_{ai}u_{bj}u_{ck} = -u_{ck}u_{bj}u_{ai}$ for a,b,c distinct. For i,j,k distinct the conditions are $u_{ai}u_{bj}u_{ck} = -u_{ck}u_{bj}u_{ai}$ for a,b,c not distinct, and $u_{ai}u_{bj}u_{ck} = u_{ck}u_{bj}u_{ai}$ for a,b,c

distinct. Thus we have exactly the relations between the coordinates of $\bar{O}_N^*$, and we are done.

(4) Now with the above results in hand, we obtain that the Schur-Weyl twists of O_N, O_N^* are indeed the quantum groups $\bar{O}_N, \bar{O}_N^*$ from Theorem 13.2 and Theorem 13.3.

(4) The proof in the unitary case is similar, by adding signs in the above computations (2,3), the conclusion being that the Schur-Weyl twists of U_N, U_N^* are indeed $\bar{U}_N, \bar{U}_N^*$. □

Let us clarify now the relation between the maps $T_\pi, \bar{T}_\pi$. We recall that the Möbius function of any lattice, and in particular of P_{even}, is given by:

$$\mu(\sigma,\pi)=\begin{cases}1 & \text{if } \sigma=\pi\\ -\sum_{\sigma\leq\tau<\pi}\mu(\sigma,\tau) & \text{if } \sigma<\pi\\ 0 & \text{if } \sigma\not\leq\pi\end{cases}$$

With this notation, we have the following result:

Proposition 13.12. *For any partition $\pi\in P_{even}$ we have the formula*

$$\bar{T}_\pi=\sum_{\tau\leq\pi}\alpha_\tau T_\tau$$

where $\alpha_\sigma=\sum_{\sigma\leq\tau\leq\pi}\varepsilon(\tau)\mu(\sigma,\tau)$, with μ being the Möbius function of P_{even}.

Proof. The linear combinations $T=\sum_{\tau\leq\pi}\alpha_\tau T_\tau$ acts on tensors as follows:

$$\begin{aligned}T(e_{i_1}\otimes\ldots\otimes e_{i_k}) &= \sum_{\tau\leq\pi}\alpha_\tau T_\tau(e_{i_1}\otimes\ldots\otimes e_{i_k})\\ &= \sum_{\tau\leq\pi}\alpha_\tau\sum_{\sigma\leq\tau}\sum_{j:\ker\binom{i}{j}=\sigma}e_{j_1}\otimes\ldots\otimes e_{j_l}\\ &= \sum_{\sigma\leq\pi}\left(\sum_{\sigma\leq\tau\leq\pi}\alpha_\tau\right)\sum_{j:\ker\binom{i}{j}=\sigma}e_{j_1}\otimes\ldots\otimes e_{j_l}\end{aligned}$$

Thus, in order to have $\bar{T}_\pi=\sum_{\tau\leq\pi}\alpha_\tau T_\tau$, we must have $\varepsilon(\sigma)=\sum_{\sigma\leq\tau\leq\pi}\alpha_\tau$, for any $\sigma\leq\pi$. But this problem can be solved by using the Möbius inversion formula, and we obtain the numbers $\alpha_\sigma=\sum_{\sigma\leq\tau\leq\pi}\varepsilon(\tau)\mu(\sigma,\tau)$ in the statement. □

With the above results in hand, let us go back now to the question of twisting the quantum reflection groups. We can now prove:

THEOREM 13.13. *The basic quantum reflection groups, namely*

$$\begin{matrix}
H_N^+ & \longrightarrow & \mathbb{T}H_N^+ & \longrightarrow & K_N^+\\
\uparrow & & \uparrow & & \uparrow\\
H_N^* & \longrightarrow & \mathbb{T}H_N^* & \longrightarrow & K_N^*\\
\uparrow & & \uparrow & & \uparrow\\
H_N & \longrightarrow & \mathbb{T}H_N & \longrightarrow & K_N
\end{matrix}$$

equal their own Schur-Weyl twists.

PROOF. This result basically comes from the results that we have:

(1) In the real case, the verifications are as follows:

– H_N^+. We know that for $\pi \in NC_{even}$ we have $\bar{T}_\pi = T_\pi$, and since we are in the situation $D \subset NC_{even}$, the definitions of $G, \bar{G}$ coincide.

– $H_N^{[\infty]}$. Here we can use the same argument as in (1), based this time on the well-known description of $P_{even}^{[\infty]}$ involving the signatures.

– H_N^*. We have $H_N^* = H_N^{[\infty]} \cap O_N^*$, so $\bar{H}_N^* \subset H_N^{[\infty]}$ is the subgroup obtained via the defining relations for $\bar{O}_N^*$. But all the $abc = -cba$ relations defining $\bar{H}_N^*$ are automatic, of type $0 = 0$, and it follows that $\bar{H}_N^* \subset H_N^{[\infty]}$ is the subgroup obtained via the relations $abc = cba$, for any $a, b, c \in \{u_{ij}\}$. Thus we have $\bar{H}_N^* = H_N^{[\infty]} \cap O_N^* = H_N^*$, as claimed.

– H_N. We have $H_N = H_N^* \cap O_N$, and by functoriality, $\bar{H}_N = \bar{H}_N^* \cap \bar{O}_N = H_N^* \cap \bar{O}_N$. But this latter intersection is easily seen to be equal to H_N, as claimed.

(2) In the complex case the proof is similar. □

13c. Cocycle twisting

More general twists come by cocycle twisting.

13d. Twisted geometry

In relation with the twisted quantum groups, we first have:

THEOREM 13.14. *We have the Weingarten type formula*

$$\int_{\dot{G}} u_{i_1j_1}^{e_1} \ldots u_{i_kj_k}^{e_k} = \sum_{\pi,\sigma \in P_\times(\alpha)} \dot{\delta}_\pi(i_1 \ldots i_k) \dot{\delta}_\sigma(j_1 \ldots j_k) W_{kN}(\pi, \sigma)$$

where $W_{kN} = G_{kN}^{-1}$, *with* $G_{kN}(\pi, \sigma) = N^{|\pi \vee \sigma|}$, *for* $\pi, \sigma \in D(k)$.

PROOF. This follows exactly as in the untwisted case, the idea being that the signs will cancel. Let us recall indeed that the twisted vectors $\bar{\xi}_\pi$ are as follows:

$$\bar{\xi}_\pi = \sum_{\tau\geq\pi}\varepsilon(\tau)\sum_{i:\ker(i)=\tau} e_{i_1}\otimes\ldots\otimes e_{i_k}$$

Thus, the Gram matrix of these vectors is given by:

$$\begin{aligned} <\xi_\pi,\xi_\sigma> &= \sum_{\tau\geq\pi\vee\sigma}\varepsilon(\tau)^2\left|\left\{(i_1,\ldots,i_k)\Big|\ker i=\tau\right\}\right| \\ &= \sum_{\tau\geq\pi\vee\sigma}\left|\left\{(i_1,\ldots,i_k)\Big|\ker i=\tau\right\}\right| \\ &= N^{|\pi\vee\sigma|} \end{aligned}$$

Thus the Gram matrix is the same as in the untwisted case, and so the Weingarten matrix is the same as well as in the untwisted case, and this gives the result. □

In relation now with the spheres, we have the following result:

THEOREM 13.15. *The twisted spheres have the following properties:*

(1) *They have affine actions of the twisted unitary quantum groups.*
(2) *They have unique invariant Haar functionals, which are ergodic.*
(3) *Their Haar functionals are given by Weingarten type formulae.*
(4) *They appear, via the GNS construction, as first row spaces.*

PROOF. The proofs here are similar to those from the untwisted case, via a routine computation, by adding signs where needed, and with the main technical ingredient, namely the Weingarten formula, being available from Theorem 13.14 above. □

13e. Exercises

CHAPTER 14

Drinfeld-Jimbo

14a. Formal twists

We discuss here the construction of the Drinfeld-Jimbo algebras $U_q\mathfrak{g}$, at generic values of the parameter $q \in k$.

These can be thought of as corresponding to quantum groups G_q, where G is the corresponding compact form of $\mathfrak{g}$.

14b. Specializations

We discuss here the properties of the Drinfeld-Jimbo algebras $U_q\mathfrak{g}$, at special values of the parameter $q \in k$.

The basic example is $G = SU_2$, in the case $k = \mathbb{C}$ and $q > 0$.

Let us first explain the relation between O_N^+ and SU_2^q. To any matrix $F \in GL_N(\mathbb{R})$ satisfying $F^2 = 1$ we associate the following universal algebra:

$$C(O_F^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u = F\bar{u}F = \text{unitary}\right)$$

Observe that $O_{I_N}^+ = O_N^+$. In general, the above algebra satisfies Woronowicz's generalized axioms in [99], which do not include the strong antipode axiom $S^2 = id$.

At $N = 2$, up to a trivial equivalence relation on the matrices F, and on the quantum groups O_F^+, we can assume that F is as follows, with $q \in [-1, 0)$:

$$F = \begin{pmatrix}0 & \sqrt{-q}\\ 1/\sqrt{-q} & 0\end{pmatrix}$$

Our claim is that for this matrix we have:

$$O_F^+ = SU_2^q$$

Indeed, the relations $u = F\bar{u}F$ tell us that u must be of the following special form:

$$u = \begin{pmatrix}\alpha & -q\gamma^*\\ \gamma & \alpha^*\end{pmatrix}$$

Thus $C(O_F^+)$ is the universal algebra generated by two elements α,γ, with the relations making the above matrix u unitary. But these unitarity conditions are:

$$\alpha\gamma = q\gamma\alpha$$

$$\alpha\gamma^* = q\gamma^*\alpha$$

$$\gamma\gamma^* = \gamma^*\gamma$$

$$\alpha^*\alpha + \gamma^*\gamma = 1$$

$$\alpha\alpha^* + q^2\gamma\gamma^* = 1$$

We recognize here the relations in [99] defining the algebra $C(SU_2^q)$, and it follows that we have an isomorphism of Hopf C^*-algebras:

$$C(O_F^+) \simeq C(SU_2^q)$$

Now back to the general case, let us try to understand the integration over O_F^+. Given $\pi \in NC_2(2k)$ and $i = (i_1, \ldots, i_{2k})$, we set:

$$\delta_\pi^F(i) = \prod_{s\in\pi} F_{i_{s_l} i_{s_r}}$$

Here the product is over all strings of π, denoted as follows:

$$s = \{s_l \curvearrowright s_r\}$$

Our claim now is that the following family of vectors, with $\pi \in NC_2(2k)$, spans the space of fixed vectors of $u^{\otimes 2k}$:

$$\xi_\pi = \sum_i \delta_\pi^F(i) e_{i_1} \otimes \ldots \otimes e_{i_{2k}}$$

Indeed, having $\xi_\cap$ fixed by $u^{\otimes 2}$ is equivalent to assuming that $u = F\bar{u}F$ is unitary.

We have a matrix model due to Woronowicz [99], where the standard generators α,γ are mapped as follows:

$$\begin{aligned} \pi_u(\alpha)e_k &= \sqrt{1-q^{2k}}e_{k-1} \\ \pi_u(\gamma)e_k &= uq^k e_k \end{aligned}$$

Here $u \in \mathbb{T}$ is a parameter, and (e_k) is the standard basis of $l^2(\mathbb{N})$.

Going ahead now, there is a relation here with the symplectic groups. We first have the following result:

PROPOSITION 14.1. *Given a closed subgroup $G\subset U_N^+$, with irreducible fundamental corepresentation $u=(u_{ij})$, this corepresentation is self-adjoint, $u\sim\bar{u}$, precisely when*

$$u=F\bar{u}F^{-1}$$

for some unitary matrix $F\in U_N$, satisfying the following condition:

$$F\bar{F}=\pm 1$$

Moreover, when N is odd we must have $F\bar{F}=1$.

PROOF. Since u is self-adjoint, $u\sim\bar{u}$, we must have $u=F\bar{u}F^{-1}$, for a certain matrix $F\in GL_N(\mathbb{C})$. We obtain from this, by using our assumption that u is irreducible:

$$\begin{aligned}
u=F\bar{u}F^{-1}\quad&\implies\quad \bar{u}=\bar{F}u\bar{F}^{-1}\\
&\implies\quad u=(F\bar{F})u(F\bar{F})^{-1}\\
&\implies\quad F\bar{F}=c1\\
&\implies\quad \bar{F}F=\bar{c}1\\
&\implies\quad c\in\mathbb{R}
\end{aligned}$$

Now by rescaling we can assume $c=\pm 1$, so we have proved so far that:

$$F\bar{F}=\pm 1$$

In order to establish now the formula $FF^*=1$, we can proceed as follows:

$$\begin{aligned}
(id\otimes S)u=u^*\quad&\implies\quad (id\otimes S)\bar{u}=u^t\\
&\implies\quad (id\otimes S)(F\bar{u}F^{-1})=Fu^tF^{-1}\\
&\implies\quad u^*=Fu^tF^{-1}\\
&\implies\quad u=(F^*)^{-1}\bar{u}F^*\\
&\implies\quad \bar{u}=F^*u(F^*)^{-1}\\
&\implies\quad \bar{u}=F^*F\bar{u}F^{-1}(F^*)^{-1}\\
&\implies\quad FF^*=d1
\end{aligned}$$

We have $FF^*>0$, so $d>0$. On the other hand, from $F\bar{F}=\pm 1$, $FF^*=d1$ we get:

$$|\det F|^2=\det(F\bar{F})=(\pm 1)^N$$

$$|\det F|^2=\det(FF^*)=d^N$$

Since $d>0$ we obtain from this $d=1$, and so $FF^*=1$ as claimed. We obtain as well that when N is odd the sign must be 1, and so $F\bar{F}=1$, as claimed. $\square$

It is convenient to diagonalize the matrices F that we found. Up to an orthogonal base change, we can assume that our matrix is as follows, where $N=2p+q$ and $\varepsilon=\pm 1$, with the 1_q block at right disappearing if $\varepsilon = -1$:

$$F = \begin{pmatrix} 0 & 1 \\ \varepsilon 1 & 0_{(0)} \\ & & \ddots \\ & & & 0 & 1 \\ & & & \varepsilon 1 & 0_{(p)} \\ & & & & & 1_{(1)} \\ & & & & & & \ddots \\ & & & & & & & 1_{(q)} \end{pmatrix}$$

We are therefore led into the following definition:

Definition 14.2. *The "super-space" $\mathbb{C}^N_F$ is the usual space $\mathbb{C}^N$, with its standard basis $\{e_1, \ldots, e_N\}$, with a chosen sign $\varepsilon = \pm 1$, and a chosen involution on the set of indices,*

$$i \to \bar{i}$$

with F being the "super-identity" matrix, $F_{ij} = \delta_{i\bar{j}}$ for $i \leq j$ and $F_{ij} = \varepsilon\delta_{i\bar{j}}$ for $i \geq j$.

In what follows we will usually assume that F is the explicit matrix appearing above. Indeed, up to a permutation of the indices, we have a decomposition $n = 2p + q$ such that the involution is, in standard permutation notation:

$$(12)\ldots(2p-1, 2p)(2p+1)\ldots(q)$$

Let us construct now some basic compact quantum groups, in our "super" setting. Let us formulate:

Definition 14.3. *Associated to the super-space $\mathbb{C}^N_F$ are the following objects:*

(1) *The super-orthogonal group, given by:*

$$O_F = \left\{ U \in U_N \middle| U = F\bar{U}F^{-1} \right\}$$

(2) *The super-orthogonal quantum group, given by:*

$$C(O_F^+) = C^*\left((u_{ij})_{i,j=1,\ldots,n} \middle| u = F\bar{u}F^{-1} = \text{unitary} \right)$$

We have the following result, making the connection with O_N^+ and SU_2:

Theorem 14.4. *The basic orthogonal groups and quantum groups are as follows:*

(1) *At $\varepsilon = -1$ we have $O_F = Sp_N$ and $O_F^+ = Sp_N^+$.*
(2) *At $\varepsilon = -1$ and $N = 2$ we have $O_F = O_F^+ = SU_2$.*
(3) *At $\varepsilon = 1$ we have $O_F = O_N$ and $O_F^+ = O_N^+$.*

PROOF. These results are all elementary, as follows:

(1) At $\varepsilon=-1$ this follows from definitions, because the symplectic group $Sp_N\subset U_N$ is by definition the following group:

$$Sp_N=\left\{U\in U_N\Big|U=F\bar{U}F^{-1}\right\}$$

(2) Still at $\varepsilon=-1$, the equation $U=F\bar{U}F^{-1}$ tells us that the symplectic matrices $U\in Sp_N$ are exactly the unitaries $U\in U_N$ which are patterned as follows:

$$U=\begin{pmatrix}a&b&\ldots\\-\bar{b}&\bar{a}\\ \vdots&&\ddots\end{pmatrix}$$

In particular, the symplectic matrices at $N=2$ are as follows:

$$U=\begin{pmatrix}a&b\\-\bar{b}&\bar{a}\end{pmatrix}$$

Thus we have $Sp_2=U_2$, and the formula $Sp_2^+=Sp_2$ is elementary as well.

(3) At $\varepsilon=1$ now, consider the root of unity $\rho=e^{\pi i/4}$, and set:

$$J=\frac{1}{\sqrt{2}}\begin{pmatrix}\rho&\rho^7\\ \rho^3&\rho^5\end{pmatrix}$$

This matrix J is then unitary, and we have:

$$J\begin{pmatrix}0&1\\1&0\end{pmatrix}J^t=1$$

Thus the following matrix is unitary as well, and satisfies $KFK^t=1$:

$$K=\begin{pmatrix}J^{(1)}\\&\ddots\\&&J^{(p)}\\&&&1_q\end{pmatrix}$$

Thus in terms of the matrix $V=KUK^*$ we have:

$$U=F\bar{U}F^{-1}=\text{unitary}\quad\Longleftrightarrow\quad V=\bar{V}=\text{unitary}$$

We obtain in this way an isomorphism $O_F^+=O_N^+$ as in the statement, and by passing to classical versions, we obtain as well $O_F=O_N$, as desired. □

With the above formalism and results in hand, we can now formulate the unification result for O_N^+ and SU_2, which in complete form is as follows:

Theorem 14.5. *For the quantum group $O_F^+ \in \{O_N^+, Sp_N^+\}$ with $N \geq 2$, the main character follows the standard Wigner semicircle law,*

$$\chi \sim \frac{1}{2\pi}\sqrt{4-x^2}dx$$

the irreducible representations are all self-adjoint, and can be labelled by positive integers, with their fusion rules being the Clebsch-Gordan ones,

$$r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+2} + \ldots + r_{k+l}$$

and the dimensions of these representations are given by

$$\dim r_k = \frac{q^{k+1} - q^{-k-1}}{q - q^{-1}}$$

where q, q^{-1} are the solutions of $X^2 - NX + 1 = 0$. Also, we have $Sp_2^+ = SU_2$.

Proof. This is a straightforward unification of the results that we already have for O_N^+ and SU_2, the technical details being all standard. □

14c. Roots of unity

Here the situation is more complicated, especially at $q = -1$.

14d. Problems and fixes

We discuss here the fix at $q = -1$.

14e. Exercises

CHAPTER 15

Compact forms

15a. General theory

The general theory, from the underformed case, extends well. Under suitable assumptions, our Hopf algebras have a Haar integration functional:

THEOREM 15.1. *Suitable Hopf algebras A have a unique Haar integration, which can be constructed by starting with any faithful positive unital state $\varphi \in A^*$, and setting*

$$\int_G = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi^{*k}$$

where $\phi * \psi = (\phi \otimes \psi)\Delta$. Moreover, for any corepresentation v we have

$$\left(id \otimes \int_G\right) v = P$$

where P is the orthogonal projection onto $Fix(v) = \{\xi \in k^n | v\xi = \xi\}$.

PROOF. This is something quite technical. □

Under suitable assumptions, our Hopf algebras are subject to a Peter-Weyl type theory. First, we have the following result:

THEOREM 15.2 (PW1). *Let $v \in M_n(A)$ be a corepresentation, consider the algebra $B = End(v)$, and write its unit as*

$$1 = p_1 + \ldots + p_k$$

with p_i being minimal central projections. We have then

$$v = v_1 + \ldots + v_k$$

with each v_i being an irreducible corepresentation, obtained by restricting v to $Im(p_i)$.

PROOF. This is something quite technical. □

Next in line, we have the following result:

THEOREM 15.3 (PW2). *Each irreducible corepresentation of A appears as:*

$$v \subset u^{\otimes k}$$

That is, v appears inside a certain Peter-Weyl corepresentation.

PROOF. This is something quite technical as well. □

By using the Haar integration, we have as well:

THEOREM 15.4 (PW3). *The dense subalgebra $\mathcal{A}\subset A$ decomposes as a direct sum*

$$\mathcal{A}=\bigoplus_{v\in Irr(A)}M_{\dim(v)}(k)$$

with this being an isomorphism of $*$-coalgebras, and with the summands being pairwise orthogonal with respect to the scalar product given by

$$<a,b>=\int_Gab^*$$

where $\int_G$ is the Haar integration over G.

PROOF. This is, as usual, something quite technical. □

Finally, we have the folloiwing result:

THEOREM 15.5 (PW4). *The characters of the irreducible corepresentations belong to the $*$-algebra*

$$\mathcal{A}_{central}=\left\{a\in\mathcal{A}\Big|\Sigma\Delta(a)=\Delta(a)\right\}$$

of "smooth central functions" on G, and form an orthonormal basis of it.

PROOF. This is something quite technical too. □

Finally, under suitable assumptions, our Hopf algebras are subject to a Tannakian duality. Thus, we have a complete theory for them.

15b. Real parameters

Let us first explain the relation between O_N^+ and SU_2^q. To any matrix $F\in GL_N(\mathbb{R})$ satisfying $F^2=1$ we associate the following universal algebra:

$$C(O_F^+)=C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big|u=F\bar{u}F=\text{unitary}\right)$$

Observe that $O_{I_N}^+=O_N^+$. In general, the above algebra satisfies Woronowicz's generalized axioms in [99], which do not include the strong antipode axiom $S^2=id$.

At $N=2$, up to a trivial equivalence relation on the matrices F, and on the quantum groups O_F^+, we can assume that F is as follows, with $q\in[-1,0)$:

$$F=\begin{pmatrix}0&\sqrt{-q}\\1/\sqrt{-q}&0\end{pmatrix}$$

Our claim is that for this matrix we have:

$$O_F^+=SU_2^q$$

Indeed, the relations $u=F\bar{u}F$ tell us that u must be of the following special form:

$$u=\begin{pmatrix}\alpha&-q\gamma^*\\ \gamma&\alpha^*\end{pmatrix}$$

Thus $C(O_F^+)$ is the universal algebra generated by two elements α,γ, with the relations making the above matrix u unitary. But these unitarity conditions are:

$$\alpha\gamma=q\gamma\alpha$$
$$\alpha\gamma^*=q\gamma^*\alpha$$
$$\gamma\gamma^*=\gamma^*\gamma$$
$$\alpha^*\alpha+\gamma^*\gamma=1$$
$$\alpha\alpha^*+q^2\gamma\gamma^*=1$$

We recognize here the relations in [99] defining the algebra $C(SU_2^q)$, and it follows that we have an isomorphism of Hopf C^*-algebras:

$$C(O_F^+)\simeq C(SU_2^q)$$

Now back to the general case, let us try to understand the integration over O_F^+. Given $\pi\in NC_2(2k)$ and $i=(i_1,\ldots,i_{2k})$, we set:

$$\delta_\pi^F(i)=\prod_{s\in\pi}F_{i_{s_l}i_{s_r}}$$

Here the product is over all strings of π, denoted as follows:

$$s=\{s_l\curvearrowright s_r\}$$

Our claim now is that the following family of vectors, with $\pi\in NC_2(2k)$, spans the space of fixed vectors of $u^{\otimes 2k}$:

$$\xi_\pi=\sum_i\delta_\pi^F(i)e_{i_1}\otimes\ldots\otimes e_{i_{2k}}$$

Indeed, having $\xi_\cap$ fixed by $u^{\otimes 2}$ is equivalent to assuming that $u=F\bar{u}F$ is unitary.

15c. Toral parameters

This is something that we basically discussed in the above.

15d. Beyond compactness

Here the problems are quite technical.

15e. Exercises

CHAPTER 16

Geometry and physics

16a. Lattice models

We will need some basic theory, regarding the matrix models for the quantum groups, or for more general manifolds. Let us start with:

Definition 16.1. *A matrix model for $G\subset U_N^+$ is a morphism of C^*-algebras*

$$\pi:C(G)\to M_K(C(T))$$

where T is a compact space, and $K\geq 1$ is an integer.

More generally, we can try to model in this way the standard coordinates $x_i\in C(X)$ of the various algebraic manifolds $X\subset S^{N-1}_{\mathbb{C},+}$. Indeed, these manifolds generalize the compact matrix quantum groups, which appear as:

$$G\subset U_N^+\subset S^{N^2-1}_{\mathbb{C},+}$$

Thus we have many other interesting examples of such manifolds, such as the homogeneous spaces discussed before. However, at this level of generality, not much general theory is available. It is elementary to show that, under the technical assumption $X^c\neq\emptyset$, there exists a universal $K\times K$ model for the algebra $C(X)$, which factorizes as follows, with $X^{(K)}\subset X$ being a certain algebraic submanifold:

$$\pi_K:C(X)\to C(X^{(K)})\subset M_K(C(T_K))$$

To be more precise, the universal $K\times K$ model space T_K appears by imposing to the complex $K\times K$ matrices the relations defining X, and the algebra $C(X^{(K)})$ is then by definition the image of π_K. In relation with this, we can set as well:

$$X^{(\infty)}=\bigcup_{K\in\mathbb{N}}X^{(K)}$$

We are led in this way to a filtration of X, as follows:

$$X^c=X^{(1)}\subset X^{(2)}\subset X^{(3)}\subset\ldots\ldots\subset X^{(\infty)}\subset X$$

It is possible to say a few non-trivial things about these manifolds $X^{(K)}$, by using algebraic and functional analytic techniques, and we refer here to the literature.

In the compact quantum group case, however, that we are mainly interested in here, the matrix truncations $G^{(K)}\subset G$ are generically not subgroups at $K\geq 2$, and so this theory is a priori not very useful, at least in its basic form presented here.

In order to reach, however, to some results, let us introduce as well:

Definition 16.2. *A matrix model $\pi:C(G)\to M_K(C(T))$ is called stationary when*

$$\int_G=\left(tr\otimes\int_T\right)\pi$$

where $\int_T$ is the integration with respect to a given probability measure on T.

Observe that this definition can be extended as well to the algebraic manifold case, $X\subset S^{N-1}_{\mathbb{C},+}$, provided that our manifolds have certain integration functionals $\int_X$. This is the case for instance with the homogeneous spaces discussed before, where $\int_X$ appears as the unique G-invariant trace, with respect to the underlying quantum group G. However, the axiomatization of such manifolds being not available yet, we will keep this as a remark, and get back in what follows, until the end, to the quantum groups.

So, back to Definition 16.2, as it is, our first comment concerns the terminology. The term "stationary" comes from a functional analytic interpretation of all this, with a certain Cesàro limit being needed to be stationary, and this will be explained later on. Yet another explanation comes from a certain relation with the lattice models, but this relation is rather something folklore, not axiomatized yet. We will be back to this later.

As a first result now, the stationarity property implies the faithfulness:

Theorem 16.3. *Assuming that $G\subset U_N^+$ has a stationary model,*

$$\pi:C(G)\to M_K(C(T))$$

$$\int_G=\left(tr\otimes\int_T\right)\pi$$

it follows that G must be coamenable, and that the model is faithful.

Proof. We have two assertions to be proved, the idea being as follows:

(1) Assume that we have a stationary model, as in the statement. By performing the GNS construction with respect to $\int_G$, we obtain a factorization as follows, which commutes with the respective canonical integration functionals:

$$\pi:C(G)\to C(G)_{red}\subset M_K(C(T))$$

Thus, in what regards the coamenability question, we can assume that π is faithful. With this assumption made, observe that we have embeddings as follows:

$$C^\infty(G)\subset C(G)\subset M_K(C(T))$$

The point now is that the GNS construction gives a better embedding, as follows:

$$L^\infty(G)\subset M_K(L^\infty(T))$$

Now since the von Neumann algebra on the right is of type I, so must be its subalgebra $A=L^\infty(G)$. This means that, when writing the center of this latter algebra as $Z(A)=L^\infty(X)$, the whole algebra decomposes over X, as an integral of type I factors:

$$L^\infty(G)=\int_X M_{K_x}(\mathbb C)\,dx$$

In particular, we can see from this that $C^\infty(G)\subset L^\infty(G)$ has a unique C^*-norm, and so G is coamenable. Thus we have proved our first assertion.

(2) The second assertion follows as well from the above, because our factorization of π consists of the identity, and of an inclusion. □

Regarding now the examples of stationary models, we first have:

Proposition 16.4. *The following have stationary models:*

(1) *The compact Lie groups.*
(2) *The finite quantum groups.*

Proof. Both these assertions are elementary, with the proofs being as follows:

(1) This is clear, because we can use the identity $id:C(G)\to M_1(C(G))$.

(2) Here we can use the regular representation $\lambda:C(G)\to M_{|G|}(\mathbb C)$. Indeed, let us endow the linear space $H=C(G)$ with the scalar product $<a,b>=\int_G ab^*$. We have then a representation, as follows:

$$\lambda:C(G)\to B(H)$$

$$a\to[b\to ab]$$

Now since we have $H\simeq\mathbb C^{|G|}$ with $|G|=\dim A$, we can view λ as a matrix model map, as above, and the stationarity axiom $\int_G=tr\circ\lambda$ is satisfied, as desired. □

In order to discuss now the group duals, consider a model as follows:

$$\pi:C^*(\Gamma)\to M_K(C(T))$$

According to the general theory of group algebras, such a matrix model must come from a group representation, as follows:

$$\rho:\Gamma\to C(T,U_K)$$

With this identification made, we have:

Proposition 16.5. *An matrix model $\rho:\Gamma\subset C(T,U_K)$ is stationary when:*

$$\int_T tr(g^x)dx=0,\forall g\neq 1$$

Moreover, the examples include all the abelian groups, and all finite groups.

Proof. Consider indeed a group embedding $\rho:\Gamma\subset C(T,U_K)$, which produces by linearity a matrix model, as follows:

$$\pi:C^*(\Gamma)\to M_K(C(T))$$

It is enough to formulate the stationarity condition on the group elements $g\in C^*(\Gamma)$. Let us set $\rho(g)=(x\to g^x)$. With this notation, the stationarity condition reads:

$$\int_T tr(g^x)dx=\delta_{g,1}$$

Since this equality is trivially satisfied at $g=1$, where by unitality of our representation we must have $g^x=1$ for any $x\in T$, we are led to the condition in the statement. Regarding now the examples, these are both clear. More precisely:

(1) When Γ is abelian we can use the following trivial embedding:

$$\Gamma\subset C(\widehat{\Gamma},U_1)$$
$$g\to[\chi\to\chi(g)]$$

(2) When Γ is finite we can use the left regular representation:

$$\Gamma\subset\mathcal{L}(\mathbb{C}\Gamma)$$
$$g\to[h\to gh]$$

Indeed, in both cases, the stationarity condition is trivially satisfied. □

In order to further advance, and to come up with some tools for discussing the non-stationary case as well, let us keep looking at the group duals $G=\widehat{\Gamma}$.

We know that a matrix model $\pi:C^*(\Gamma)\to M_K(C(T))$ must come from a unitary group representation $\rho:\Gamma\to C(T,U_K)$.

Now observe that when ρ is faithful, the representation π is in general not faithful, for instance because when $T=\{.\}$ its target algebra is finite dimensional. On the other hand, this representation obviously "reminds" Γ, and so can be used in order to fully understand Γ.

Summarizing, we have a new idea here, basically saying that, for practical purposes, the faithfuless property can be replaced with something much weaker.

This weaker notion is called "inner faithfulness", something that we have already met in the above, and the theory here is as follows:

DEFINITION 16.6. *Let $\pi : C(G) \to M_K(C(T))$ be a matrix model.*

(1) *The Hopf image of π is the smallest quotient Hopf C^*-algebra $C(G) \to C(H)$ producing a factorization of type $\pi : C(G) \to C(H) \to M_K(C(T))$.*
(2) *When the inclusion $H \subset G$ is an isomorphism, i.e. when there is no non-trivial factorization as above, we say that π is inner faithful.*

As a basic illustration for these notions, in the case where $G = \widehat{\Gamma}$ is a group dual, π must come from a group representation, as follows:

$$\rho : \Gamma \to C(T, U_K)$$

We conclude that in this case, the minimal factorization constructed in Definition 16.6 is simply the one obtained by taking the image:

$$\rho : \Gamma \to \Lambda \subset C(T, U_K)$$

Thus π is inner faithful when our group satisfies:

$$\Gamma \subset C(T, U_K)$$

As a second illustration now, given a compact group G, and elements $g_1, \ldots, g_K \in G$, we have a representation $\pi : C(G) \to \mathbb{C}^K$, given by:

$$f \to (f(g_1), \ldots, f(g_K))$$

The minimal factorization of π is then via $C(H)$, with:

$$H = \overline{< g_1, \ldots, g_K >}$$

Thus π is inner faithful precisely when our group satisfies:

$$G = H$$

In general, the existence and uniqueness of the Hopf image comes from dividing $C(G)$ by a suitable ideal, as explained before. In Tannakian terms, we have:

THEOREM 16.7. *Consider a closed subgroup $G \subset U_N^+$, with fundamental corepresentation denoted $u = (u_{ij})$. The Hopf image of a matrix model*

$$\pi : C(G) \to M_K(C(T))$$

comes then from the Tannakian category

$$C_{kl} = Hom(U^{\otimes k}, U^{\otimes l})$$

where $U_{ij} = \pi(u_{ij})$, and where the spaces on the right are taken in a formal sense.

PROOF. Since the morphisms increase the intertwining spaces, when defined either in a representation theory sense, or just formally, we have inclusions as follows:

$$Hom(u^{\otimes k}, u^{\otimes l}) \subset Hom(U^{\otimes k}, U^{\otimes l})$$

More generally, we have such inclusions when replacing (G,u) with any pair producing a factorization of π. Thus, by Tannakian duality, the Hopf image must be given by the fact that the intertwining spaces must be the biggest, subject to the above inclusions.

On the other hand, since u is biunitary, so is U, and it follows that the spaces on the right form a Tannakian category. Thus, we have a quantum group (H,v) given by:

$$Hom(v^{\otimes k},v^{\otimes l})=Hom(U^{\otimes k},U^{\otimes l})$$

By the above discussion, $C(H)$ follows to be the Hopf image of π, as claimed. □

The inner faithful models $\pi:C(G)\to M_K(C(T))$ are a very interesting notion, because they are not subject to the coamenability condition on G, as it was the case with the stationary models, as explained in Theorem 16.3.

In fact, there are no known restrictions on the class of subgroups $G\subset U_N^+$ which can be modelled in an inner faithful way. Thus, our modelling theory applies a priori to any compact quantum group.

Regarding now the study of the inner faithful models, a key problem is that of computing the Haar integration functional. The result here is as follows:

THEOREM 16.8. *Given an inner faithful model $\pi:C(G)\to M_K(C(T))$, we have*

$$\int_G=\lim_{k\to\infty}\frac{1}{k}\sum_{r=1}^k\int_G^r$$

where $\int_G^r=(\varphi\circ\pi)^{*r}$, with $\varphi=tr\otimes\int_T$ being the random matrix trace.

PROOF. As a first observation, there is an obvious similarity here with the Woronowicz construction of the Haar measure, explained before. In fact, the above result holds more generally for any model $\pi:C(G)\to B$, with $\varphi\in B^*$ being a faithful trace. With this picture in hand, the Woronowicz construction simply corresponds to the case $\pi=id$, and the result itself is therefore a generalization of Woronowicz's result.

In order to prove now the result, we can proceed as before. If we denote by $\int_G'$ the limit in the statement, we must prove that this limit converges, and that we have:

$$\int_G'=\int_G$$

It is enough to check this on the coefficients of corepresentations, and if we let $v=u^{\otimes k}$ be one of the Peter-Weyl corepresentations, we must prove that we have:

$$\left(id\otimes\int_G'\right)v=\left(id\otimes\int_G\right)v$$

We know that the matrix on the right is the orthogonal projection onto $Fix(v)$. Regarding now the matrix on the left, this is the orthogonal projection onto the 1-eigenspace of $(id\otimes\varphi\pi)v$. Now observe that, if we set $V_{ij}=\pi(v_{ij})$, we have:

$$(id\otimes\varphi\pi)v=(id\otimes\varphi)V$$

Thus, as before, we conclude that the 1-eigenspace that we are interested in equals $Fix(V)$. But, according to Theorem 16.7, we have:

$$Fix(V)=Fix(v)$$

Thus, we have proved that we have $\int_G'=\int_G$, as desired. □

Summarizing, we have so far a notion of matrix model, and a notion of inner faithfulness which is quite broad, and that we can study via both algebra and analysis.

Before getting into more about inner faithfulness, let us first go back to the stationary models. These models are quite restrictive, because G must be coamenable. However, there are many interesting examples of coamenable compact quantum groups, and in order to better understand these examples, and also in order to construct some new examples, our idea will be that of looking for stationary models for them.

We first have the following useful theoretical result:

Theorem 16.9. *For $\pi:C(G)\to M_K(C(T))$, the following are equivalent:*

1. *$Im(\pi)$ is a Hopf algebra, and $(tr\otimes\int_T)\pi$ is the Haar integration on it.*
2. *$\psi=(tr\otimes\int_X)\pi$ satisfies the idempotent state property $\psi*\psi=\psi$.*
3. *$T_e^2=T_e$, $\forall p\in\mathbb N$, $\forall e\in\{1,*\}^p$, where:*

$$(T_e)_{i_1\ldots i_p,j_1\ldots j_p}=\left(tr\otimes\int_T\right)(U_{i_1j_1}^{e_1}\ldots U_{i_pj_p}^{e_p})$$

If these conditions are satisfied, we say that π is stationary on its image.

Proof. Given a matrix model $\pi:C(G)\to M_K(C(T))$ as in the statement, we can factorize it via its Hopf image, as in Definition 16.6 above:

$$\pi:C(G)\to C(H)\to M_K(C(T))$$

Now observe that the conditions (1,2,3) in the statement depend only on the factorized representation:

$$\nu:C(H)\to M_K(C(T))$$

Thus, we can assume in practice that we have $G=H$, which means that we can assume that π is inner faithful. With this assumption made, the general integration formula from Theorem 16.8 applies to our situation, and the proof of the equivalences goes as follows:

(1) $\implies$ (2) This is clear from definitions, because the Haar integration on any compact quantum group satisfies the idempotent state equation:

$$\psi * \psi = \psi$$

(2) $\implies$ (1) Assuming $\psi * \psi = \psi$, we have, for any $r \in \mathbb{N}$:

$$\psi^{*r} = \psi$$

Thus Theorem 16.8 gives $\int_G = \psi$, and by using Theorem 16.3, we obtain the result.

In order to establish now (2) $\iff$ (3), we use the following elementary formula, which comes from the definition of the convolution operation:

$$\psi^{*r}(u_{i_1j_1}^{e_1} \ldots u_{i_pj_p}^{e_p}) = (T_e^r)_{i_1\ldots i_p,j_1\ldots j_p}$$

(2) $\implies$ (3) Assuming $\psi * \psi = \psi$, by using the above formula at $r = 1, 2$ we obtain that the matrices T_e and T_e^2 have the same coefficients, and so they are equal.

(3) $\implies$ (2) Assuming $T_e^2 = T_e$, by using the above formula at $r = 1, 2$ we obtain that the linear forms ψ and $\psi * \psi$ coincide on any product of coefficients $u_{i_1j_1}^{e_1} \ldots u_{i_pj_p}^{e_p}$. Now since these coefficients span a dense subalgebra of $C(G)$, this gives the result. $\square$

16b. Statistical mechanics

With the above results in hand, we can discuss certain questions related to statistical mechanics, along the lines of the planar algebra approach of Jones.

16c. Quantum mechanics

There are many interesting questions here.

16d. Particle physics

Again, there are many interesting questions here.

16e. Exercises

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