# Quantum field theory

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ABSTRACT. This is an introduction to quantum field theory and modern particle physics. We start by briefly discussing relativity theory, classical fields, quantum mechanics, and other foundational topics. Then we go on a long discussion on quantum fields and modern particle physics, notably with the aim of understanding the Standard Model for the elementary particles, which was discovered some time already, and to have a look as well into the various proposals for its enhancement, that have emerged since.

# Preface

Welcome to quantum field theory, also known as advanced quantum mechanics, or advanced particle physics. And in the hope, of course, that we will survive. With this being not that an easy task, with the number of students, or even professors and researchers, having decided after a while that all this is too complicated, being hard to count.

Well, so as a first question, why is quantum field theory that much complicated? In answer, I can only recommend doing a bit of scientific research, on whatever mathematics or physics topic that you like. You will find that, as days and weeks start passing by, you will accumulate dozens and dozens of pages of computations, for the most quite meaningless, and with most of them being even wrong. So, this will be your life, daily producing all sorts of wrong or meaningless things, to be added to the pile. And one day, after weeks and months of work, among the few hundred pages that you have accumulated, there will be 3-4 of them making some sense, and somewhat solving something, in relation with your initial question. Which is very good news, "problem solved" in scientific parlance, and all you have to do, at that point, is to quickly expand that 3-4 pages into a 20-30 paper, with preliminaries, details and everything, that you can proudly publish afterwards.

Well, quantum field theory, as we presently know it, is a bit something like this, the result of intense scientific research, actually done by many people, over many years, with the peculiarity, however, that the final 3-4 pages mentioned in the above are in fact missing. So, what we have, in the end, is something of terrible complexity, hundreds and even thousands of pages, without clear conclusion. That you will most likely have to learn, if interested in the subject, in the lack of something better.

But, where does the difficulty come from? This comes from an issue which as old as mankind studying physics, namely understanding the structure of our spacetime, or vacuum, if you prefer. That vacuum is certainly quite smart, allowing the propagation of various forces, and with this being known, and not really understood, by us humans since ages. In addition, in more modern times, people like Einstein and Planck came with the idea that this vacuum, besides being smart, is curved, and quantized too. And on top of this came quantum mechanics, raising the possibility that this vacuum has far more dimensions than those that we are used to, at our usual scales. Which, all in all, makes

## PREFACE

it for a bit too much, go talking about particles living and doing various things that they do, such as colliding, in this vacuum, without even knowing what the vacuum is.

Nevermind. Life goes on, scientific research always advances, slowly but surely, exactly as described in the above, and this on even difficult subjects like quantum field theory. The problem remains the same, learn what has been done, and work some more, matter of adding material to the pile, with the hope that things will evolve soon.

The present book is a brief introduction to this, quantum field theory. We will start by quickly discussing relativity theory, classical fields, quantum mechanics, and other foundational topics. Then we will go on a long discussion on quantum fields and modern particle physics, notably with the aim of understanding the Standard Model for the elementary particles, which was discovered some time already, and to have a look as well into the various proposals for its enhancement, that have emerged since.

Many thanks to my cats, for some help with the organization of the book, and with the selection of the topics to be discussed, which was no easy question to deal with, for a medium sized book as the present one. Some things will be certainly missing, but with the cats saying okay, who knows what is really missing, and what isn't.

Cergy, January 2025 Teo Banica

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Part I

Quantum fields

And still on top of this I'm pretty sure It must have rained The day before you came

## CHAPTER 1

# Space and time

## 1a. Speed addition

Welcome to quantum field theory. We will be talking in this book about modern particle physics, and its mathematical interpretations, and the various speculations that can be made. Before that, however, we need to talk about simpler things, namely space and time. And here, we have the following philosophical question, to start with:

QUESTION 1.1. What is our usual space, the one that we live in?

And here, we have plenty of answers. Normally that is  $\mathbb{R}^3$ , where our three coordinates (x, y, z) live. Or perhaps  $\mathbb{R}^4$ , with one dimension for time t added. But then, for being mathematical and relaxed, why not  $\mathbb{R}^N$ , extra dimensions can only help for lots of tricks. But then, why not saying  $\mathbb{C}^N$ , first because  $\mathbb{R}^4 \simeq \mathbb{C}^2$  is something very useful, and then also because complex numbers in general can only help, again with a lot of tricks.

However, the story is not over here. Algebraic geometry tells us that being projective, and compact, is the way, so the space becomes  $P_{\mathbb{R}}^{N-1}$ , or  $P_{\mathbb{C}}^{N-1}$ . On the other hand, quantum mechanics naturally lives over  $\mathbb{C}^{\infty}$ . But the same quantum mechanics, when examined at very small scales, quarks and below, tends to become "confined", somehow of projective nature too, while remaining complex, and infinite dimensional.

And there is more. Again quantum mechanics, but thermodynamics too, tells us that everything is "quantized", and so discrete, with our usual  $\mathbb{R}^N$  and its various versions above appearing as thermodynamic limits of this, what happens in the discrete setting, at extremely small scales, somewhere between the quarks scale, and the Planck scale.

Not very easy to navigate through all this, hope you agree with me. Getting a bit organized now, let us first try to answer a modest version of Question 1.1, as follows:

QUESTION 1.2. What is the usual space for classical mechanics?

And you would probably say here, of course  $\mathbb{R}^3$ , as learned from Newton, and many others. And so would I, but Einstein discovered a bug with all this.

Indeed, based on experiments by Fizeau, then Michelson-Morley and others, and some physics by Maxwell and Lorentz too, Einstein came upon the following principles:

FACT 1.3 (Einstein principles). The following happen:

- (1) Light travels in vacuum at a finite speed,  $c < \infty$ .
- (2) This speed c is the same for all inertial observers.
- (3) In non-vacuum, the light speed is lower, v < c.
- (4) Nothing can travel faster than light,  $v \geq c$ .

The point now is that, obviously, something is wrong here. Indeed, assuming for instance that we have a train, running in vacuum at speed v > 0, and someone on board lights a flashlight \* towards the locomotive, then an observer  $\circ$  on the ground will see the light travelling at speed c + v > c, which is a contradiction:



Equivalently, with the same train running, in vacuum at speed v > 0, if the observer on the ground lights a flashlight \* towards the back of the train, then viewed from the train, that light will travel at speed c + v > c, which is a contradiction again:



Summarizing, Fact 1.3 implies c + v = c, so contradicts classical mechanics, which therefore needs a fix. By dividing all speeds by c, as to have c = 1, and by restricting the attention to the 1D case, to start with, we are led to the following puzzle:

PUZZLE 1.4. How to define speed addition on the space of 1D speeds, which is

$$I = [-1, 1]$$

with our c = 1 convention, as to have 1 + c = 1, as required by physics?

In view of our geometric knowledge so far, a natural idea here would be that of wrapping [-1, 1] into a circle, and then stereographically projecting on  $\mathbb{R}$ . Indeed, we can then "import" to [-, 1, 1] the usual addition on  $\mathbb{R}$ , via the inverse of this map.

So, let us see where all this leads us. First, the formula of our map is as follows:

PROPOSITION 1.5. The map wrapping [-1,1] into the unit circle, and then stereographically projecting on  $\mathbb{R}$  is given by the formula

$$\varphi(u) = \tan\left(\frac{\pi u}{2}\right)$$

with the convention that our wrapping is the most straightforward one, making correspond  $\pm 1 \rightarrow i$ , with negatives on the left, and positives on the right.

**PROOF.** Regarding the wrapping, as indicated, this is given by:

$$u \to e^{it}$$
 ,  $t = \pi u - \frac{\pi}{2}$ 

Indeed, this correspondence wraps [-1, 1] as above, the basic instances of our correspondence being as follows, and with everything being fine modulo  $2\pi$ :

$$-1 \to \frac{\pi}{2}$$
 ,  $-\frac{1}{2} \to -\pi$  ,  $0 \to -\frac{\pi}{2}$  ,  $\frac{1}{2} \to 0$  ,  $1 \to \frac{\pi}{2}$ 

Regarding now the stereographic projection, the picture here is as follows:



Thus, by Thales, the formula of the stereographic projection is as follows:

$$\frac{\cos t}{x} = \frac{1 - \sin t}{1} \implies x = \frac{\cos t}{1 - \sin t}$$

Now if we compose our wrapping operation above with the stereographic projection, what we get is, via the above Thales formula, and some trigonometry:

$$x = \frac{\cos t}{1 - \sin t}$$

$$= \frac{\cos \left(\pi u - \frac{\pi}{2}\right)}{1 - \sin \left(\pi u - \frac{\pi}{2}\right)}$$

$$= \frac{\cos \left(\frac{\pi}{2} - \pi u\right)}{1 + \sin \left(\frac{\pi}{2} - \pi u\right)}$$

$$= \frac{\sin(\pi u)}{1 + \cos(\pi u)}$$

$$= \frac{2\sin \left(\frac{\pi u}{2}\right) \cos \left(\frac{\pi u}{2}\right)}{2\cos^2 \left(\frac{\pi u}{2}\right)}$$

$$= \tan \left(\frac{\pi u}{2}\right)$$

Thus, we are led to the conclusion in the statement.

The above result is very nice, but when it comes to physics, things do not work, for instance because of the wrong slope of the function  $\varphi(u) = \tan\left(\frac{\pi u}{2}\right)$  at the origin, which makes our summing on [-1, 1] not compatible with the Galileo addition, at low speeds.

So, what to do? Obviously, trash Proposition 1.5, and start all over again. Getting back now to Puzzle 1.4, this has in fact a simpler solution, based this time on algebra, and which in addition is the good, physically correct solution, as follows:

THEOREM 1.6. If we sum the speeds according to the Einstein formula

$$u +_e v = \frac{u + v}{1 + uv}$$

then the Galileo formula still holds, approximately, for low speeds

$$u +_e v \simeq u + v$$

and if we have u = 1 or v = 1, the resulting sum is  $u +_e v = 1$ .

PROOF. All this is self-explanatory, and clear from definitions, and with the Einstein formula of  $u +_e v$  itself being just an obvious solution to Puzzle 1.4, provided that, importantly, we know 0 geometry, and rely on very basic algebra only.

So, very nice, problem solved, at least in 1D. But, shall we give up with geometry, and the stereographic projection? Certainly not, let us try to recycle that material. In

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## 1A. SPEED ADDITION

order to do this, let us recall that the usual trigonometric functions are given by:

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad , \quad \cos x = \frac{e^{ix} + e^{-ix}}{2} \quad , \quad \tan x = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})}$$

The point now is that, and you might know this from calculus, the above functions have some natural "hyperbolic" or "imaginary" analogues, constructed as follows:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$
,  $\cosh x = \frac{e^x + e^{-x}}{2}$ ,  $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ 

But the function on the right, tanh, starts reminding the formula of Einstein addition, from Theorem 1.6. So, we have our idea, and we are led to the following result:

THEOREM 1.7. The Einstein speed summation in 1D is given by

 $\tanh x +_e \tanh y = \tanh(x+y)$ 

with  $tanh: [-\infty, \infty] \rightarrow [-1, 1]$  being the hyperbolic tangent function.

**PROOF.** This follows by putting together our various formulae above, but it is perhaps better, for clarity, to prove this directly. Our claim is that we have:

$$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

But this can be checked via direct computation, from the definitions, as follows:

$$\frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

$$= \left(\frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} + \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}}\right) / \left(1 + \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}} \cdot \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}}\right)$$

$$= \frac{(e^{x} - e^{-x})(e^{y} + e^{-y}) + (e^{x} + e^{-x})(e^{y} - e^{-y})}{(e^{x} + e^{-x})(e^{y} + e^{-y}) + (e^{x} - e^{-x})(e^{y} + e^{-y})}$$

$$= \frac{2(e^{x+y} - e^{-x-y})}{2(e^{x+y} + e^{-x-y})}$$

$$= \tanh(x+y)$$

Thus, we are led to the conclusion in the statement.

Very nice all this, hope you agree. As a conclusion, passing from the Riemann stereographic projection sum to the Einstein summation basically amounts in replacing:

#### $\tan \rightarrow \tanh$

Let us formulate as well this finding more philosophically, as follows:

CONCLUSION 1.8. The Einstein speed summation in 1D is the imaginary analogue of the summation on [-1, 1] obtained via Riemann's stereographic projection.

Very nice all this, and time to formulate a more precise version of Question 1.2:

QUESTION 1.9. What is the Einstein speed summation formula in 3D? And, what does this tell us about our usual spacetime  $\mathbb{R}^4$ , how does this exactly get curved?

And we will stop with the philosophy here, we have a very good and concrete question now, and time to get to work. Let us attempt to construct  $u +_e v$  in arbitrary dimensions, just by using our common sense and intuition. When the vectors  $u, v \in \mathbb{R}^N$  are proportional, we are basically in 1D, and so our addition formula must satisfy:

$$u \sim v \implies u +_e v = \frac{u + v}{1 + \langle u, v \rangle}$$

However, the formula on the right will not work as such in general, for arbitrary speeds  $u, v \in \mathbb{R}^N$ , and this because we have, as main requirement for our operation, in analogy with the 1 + v = 1 formula from 1D, the following condition:

$$||u|| = 1 \implies u +_e v = v$$

Equivalently, in analogy with  $u +_e 1 = 1$  from 1D, we would like to have:

$$||v|| = 1 \implies u +_e v = v$$

Summarizing, our  $u \sim v$  formula above is not bad, as a start, but we must add a correction term to it, for the above requirements to be satisfied, and of course with the correction term vanishing when  $u \sim v$ . So, we are led to a math puzzle:

PUZZLE 1.10. What vanishes when  $u \sim v$ , and then how to correctly define

$$u +_e v = \frac{u + v + \gamma_{uv}}{1 + \langle u, v \rangle}$$

as for the correction term  $\gamma_{uv}$  to vanish when  $u \sim v$ ?

But the solution to the first question is well-known in 3D. Indeed, here we can use the vector product  $u \times v$ , that we met before, which notoriously satisfies:

$$u \sim v \implies u \times v = 0$$

Thus, our correction term  $\gamma_{uv}$  must be something containing  $w = u \times v$ , which vanishes when this vector w vanishes, and in addition arranged such that ||u|| = 1 produces a simplification, with  $u +_e v = u$  as end result, and with ||v|| = 1 producing a simplification too, with  $u +_e v = v$  as end result. Thus, our vector calculus puzzle becomes:

PUZZLE 1.11. How to correctly define the Einstein summation in 3 dimensions,

$$u +_e v = \frac{u + v + \gamma_{uvw}}{1 + \langle u, v \rangle}$$

with  $w = u \times v$ , in such a way as for the correction term  $\gamma_{uvw}$  to satisfy

$$w = 0 \implies \gamma_{uvw} = 0$$

and also such that  $||u|| = 1 \implies u +_e v = u$ , and  $||v|| \implies u +_e v = v$ ?

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In order to solve this latter puzzle, the first observation is that  $\gamma_{uvw} = w$  will not do, and this for several reasons. First, this vector points in the wrong direction, orthogonal to the plane spanned by u, v, and we certainly don't want to leave this plane, with our correction. Also, as a technical remark to be put on top of this, the choice  $\gamma_{uvw} = w$  will not bring any simplifications, as required above, in the cases ||u|| = 1 or ||v|| = 1. Thus, certainly wrong choice, and we must invent something more complicated.

Moving ahead now, as obvious task, we must "transport" the vector w to the plane spanned by u, v. But this is simplest done by taking the vector product with any vector in this plane, and so as a reasonable candidate for our correction term, we have:

$$\gamma_{uvw} = (\alpha u + \beta v) \times w$$

Here  $\alpha, \beta \in \mathbb{R}$  are some scalars to be determined, but let us take a break, and leave the computations for later. We did some good work, time to update our puzzle:

PUZZLE 1.12. How to define the Einstein summation in 3 dimensions,

$$u +_e v = \frac{u + v + \gamma_{uvw}}{1 + \langle u, v \rangle}$$

with the correction term being of the following form, with  $w = u \times v$ , and  $\alpha, \beta \in \mathbb{R}$ ,

 $\gamma_{uvw} = (\alpha u + \beta v) \times w$ 

in such a way as to have  $||u|| = 1 \implies u +_e v = u$ , and  $||v|| \implies u +_e v = v$ ?

In order to investigate what happens when ||u|| = 1 or ||v|| = 1, we must compute the vector products  $u \times w$  and  $v \times w$ . So, pausing now our study for consulting the vector calculus database, and then coming back, here is the formula that we need:

$$u \times (u \times v) = < u, v > u - < u, u > v$$

As for the formula of  $v \times w$ , that I forgot to record, we can recover it from the one above of  $u \times w$ , by using the basic properties of the vector products, as follows:

$$v \times (u \times v) = -v \times (v \times u)$$
$$= -(\langle v, u \rangle v - \langle v, v \rangle u)$$
$$= \langle v, v \rangle u - \langle u, v \rangle v$$

With these formulae in hand, we can now compute the correction term, with the result here, that we will need several times in what comes next, being as follows:

**PROPOSITION 1.13.** The correction term  $\gamma_{uvw} = (\alpha u + \beta v) \times w$  is given by

$$\gamma_{uvw} = (\alpha < u, v > +\beta < v, v >)u - (\alpha < u, u > +\beta < u, v >)v$$

for any values of the scalars  $\alpha, \beta \in \mathbb{R}$ .

**PROOF.** According to our vector product formulae above, we have:

$$\begin{aligned} \gamma_{uvw} &= (\alpha u + \beta v) \times w \\ &= \alpha(< u, v > u - < u, u > v) + \beta(< v, v > u - < u, v > v) \\ &= (\alpha < u, v > + \beta < v, v >)u - (\alpha < u, u > + \beta < u, v >)v \end{aligned}$$

Thus, we are led to the conclusion in the statement.

Time now to get into the real thing, see what happens when ||u|| = 1 and ||v|| = 1, if we can get indeed  $u +_e v = u$  and  $u +_e v = v$ . It is convenient here to do some reverse engineering. Regarding the first desired formula, namely  $u +_e v = u$ , we have:

$$u +_{e} v = u \iff u + v + \gamma_{uvw} = (1 + \langle u, v \rangle)u$$
$$\iff \gamma_{uvw} = \langle u, v \rangle u - v$$
$$\iff \alpha = 1, \ \beta = 0, \ ||u|| = 1$$

Thus, with the parameter choice  $\alpha = 1, \beta = 0$ , we will have, as desired:

$$||u|| = 1 \implies u +_e v = u$$

In what regards now the second desired formula, namely  $u +_e v = v$ , here the computation is almost identical, save for a sign switch, which after some thinking comes from our choice  $w = u \times v$  instead of  $w = v \times u$ , clearly favoring u, as follows:

$$u +_{e} v = v \iff u + v + \gamma_{uvw} = (1 + \langle u, v \rangle)v$$
$$\iff \gamma_{uvw} = -u + \langle u, v \rangle v$$
$$\iff \alpha = 0, \ \beta = -1, \ ||v|| = 1$$

Thus, with the parameter choice  $\alpha = 0, \beta = -1$ , we will have, as desired:

 $||v|| = 1 \implies u +_e v = v$ 

All this is mixed news, because we managed to solve both our problems, at ||u|| = 1and at ||v|| = 1, but our solutions are different. So, time to breathe, decide that we did enough interesting work for the day, and formulate our conclusion as follows:

**PROPOSITION 1.14.** When defining the Einstein speed summation in 3D as

$$u +_e v = \frac{u + v + u \times (u \times v)}{1 + \langle u, v \rangle}$$

in c = 1 units, the following happen:

- (1) When  $u \sim v$ , we recover the previous 1D formula.
- (2) When ||u|| = 1, speed of light, we have  $u +_e v = u$ .
- (3) However, ||v|| = 1 does not imply  $u +_e v = v$ .
- (4) Also, the formula  $u +_e v = v +_e u$  fails.

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**PROOF.** Here (1) and (2) follow from the above discussion, with the following choice for the correction term, by favoring the ||u|| = 1 problem over the ||v|| = 1 one:

$$\gamma_{uvw} = u \times w$$

In fact, with this choice made, the computation is very simple, as follows:

$$\begin{aligned} ||u|| &= 1 \implies \gamma_{uvw} = \langle u, v \rangle u - v \\ \implies u + v + \gamma_{uvw} = u + \langle u, v \rangle u \\ \implies \frac{u + v + \gamma_{uvw}}{1 + \langle u, v \rangle} = u \end{aligned}$$

As for (3) and (4), these are also clear from the above discussion, coming from the obvious lack of symmetry of our summation formula. 

Looking now at Proposition 1.14 from an abstract, mathematical perspective, there are still many things missing from there, which can be summarized as follows:

QUESTION 1.15. Can we fine-tune the Einstein speed summation in 3D into

$$u +_e v = \frac{u + v + \lambda \cdot u \times (u \times v)}{1 + \langle u, v \rangle}$$

with  $\lambda \in \mathbb{R}$ , chosen such that  $||u|| = 1 \implies \lambda = 1$ , as to have:

- (1)  $||u||, ||v|| < 1 \implies ||u+_e v|| < 1.$ (2)  $||v|| = 1 \implies ||u+_e v|| = 1.$

All this is quite tricky, and deserves some explanations. First, if we add a scalar  $\lambda \in \mathbb{R}$ into our formula, as above, we will still have, exactly as before:

$$u \sim v \implies u +_e v = \frac{1 + uv}{1 + \langle u, v \rangle}$$

On the other hand, we already know from our previous computations, those preceding Proposition 1.14, that if we ask for  $\lambda \in \mathbb{R}$  to be a plain constant, not depending on u, v, then  $\lambda = 1$  is the only good choice, making the following formula happen:

$$||u|| = 1 \implies u +_e v = u$$

But, and here comes our point,  $\lambda = 1$  is not an ideal choice either, because it would be nice to have the properties (1,2) in the statement, and these properties have no reason to be valid for  $\lambda = 1$ , as you can check for instance by yourself by doing some computations. Thus, the solution to our problem most likely involves a scalar  $\lambda \in \mathbb{R}$  depending on u, v, and satisfying the following condition, as to still have  $||u|| = 1 \implies u +_e v = u$ :

$$||u|| = 1 \implies \lambda = 1$$

Obviously, as simplest answer,  $\lambda$  must be some well-chosen function of ||u||, or rather of  $||u||^2$ , because it is always better to use square norms, when possible. But then, with this idea in mind, after a few computations we are led to the following solution:

$$\lambda = \frac{1}{1 + \sqrt{1 - ||u||^2}}$$

Summarizing, final correction done, and with this being the end of mathematics, we did a nice job, and we can now formulate our findings as a theorem, as follows:

THEOREM 1.16. When defining the Einstein speed summation in 3D as

$$u +_{e} v = \frac{1}{1 + \langle u, v \rangle} \left( u + v + \frac{u \times (u \times v)}{1 + \sqrt{1 - ||u||^2}} \right)$$

in c = 1 units, the following happen:

- (1) When  $u \sim v$ , we recover the previous 1D formula.
- (2) We have  $||u||, ||v|| < 1 \implies ||u +_e v|| < 1$ .
- (3) When ||u|| = 1, we have  $u +_e v = u$ .
- (4) When ||v|| = 1, we have  $||u +_e v|| = 1$ .
- (5) However, ||v|| = 1 does not imply  $u +_e v = v$ .
- (6) Also, the formula  $u +_e v = v +_e u$  fails.

**PROOF.** This follows from the above discussion, as follows:

(1) This is something that we know from Proposition 1.14.

(2) In order to simplify notation, let us set  $\delta = \sqrt{1 - ||u||^2}$ , which is the inverse of the quantity  $\gamma = 1/\sqrt{1 - ||u||^2}$ . With this convention, we have:

$$\begin{aligned} u +_e v &= \frac{1}{1 + \langle u, v \rangle} \left( u + v + \frac{\langle u, v \rangle u - ||u||^2 v}{1 + \delta} \right) \\ &= \frac{(1 + \delta + \langle u, v \rangle)u + (1 + \delta - ||u||^2)v}{(1 + \langle u, v \rangle)(1 + \delta)} \end{aligned}$$

Taking now the squared norm and computing gives the following formula:

$$||u +_e v||^2 = \frac{(1+\delta)^2 ||u+v||^2 + (||u||^2 - 2(1+\delta))(||u||^2 ||v||^2 - \langle u, v \rangle^2)}{(1+\langle u, v \rangle)^2 (1+\delta)^2}$$

But this formula can be further processed by using  $\delta = \sqrt{1 - ||u||^2}$ , and by navigating through the various quantities which appear, we obtain, as a final product:

$$||u +_e v||^2 = \frac{||u + v||^2 - ||u||^2 ||v||^2 + \langle u, v \rangle^2}{(1 + \langle u, v \rangle)^2}$$

#### **1B. RELATIVITY THEORY**

But this type of formula is exactly what we need, for what we want to do. Indeed, by assuming ||u||, ||v|| < 1, we have the following estimate:

$$\begin{split} ||u+_e v||^2 < 1 &\iff ||u+v||^2 - ||u||^2 ||v||^2 + \langle u, v \rangle^2 < (1+\langle u, v \rangle)^2 \\ &\iff ||u+v||^2 - ||u||^2 ||v||^2 < 1 + 2 < u, v \rangle \\ &\iff ||u||^2 + ||v||^2 - ||u||^2 ||v||^2 < 1 \\ &\iff (1-||u||^2)(1-||v||^2) > 0 \end{split}$$

Thus, we are led to the conclusion in the statement.

(3) This is something that we know from Proposition 1.14.

(4) This comes from the squared norm formula established in the proof of (2) above, because when assuming ||v|| = 1, we obtain:

$$\begin{aligned} ||u +_e v||^2 &= \frac{||u + v||^2 - ||u||^2 + \langle u, v \rangle^2}{(1 + \langle u, v \rangle)^2} \\ &= \frac{||u||^2 + 1 + 2 \langle u, v \rangle - ||u||^2 + \langle u, v \rangle^2}{(1 + \langle u, v \rangle)^2} \\ &= \frac{1 + 2 \langle u, v \rangle + \langle u, v \rangle^2}{(1 + \langle u, v \rangle)^2} \\ &= 1 \end{aligned}$$

(5) This is clear, from the obvious lack of symmetry of our formula.

(6) This is again clear, from the obvious lack of symmetry of our formula.

That was nice, all this mathematics, and hope you're still with me. And good news, the formula in Theorem 1.16 is the good one, confirmed by experimental physics.

## 1b. Relativity theory

Time now to draw some concrete conclusions, from the above speed computations. Since speed v = d/t is distance over time, we must fine-tune distance d, or time t, or both. Let us first discuss, following as usual Einstein, what happens to time t. Here the result, which might seem quite surprising, at a first glance, is as follows:

THEOREM 1.17. Relativistic time is subject to Lorentz dilation

$$t \to \gamma t$$

where the number  $\gamma \geq 1$ , called Lorentz factor, is given by the formula

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

with v being the moving speed, at which time is measured.

PROOF. Assume indeed that we have a train, moving to the right with speed v, through vacuum. In order to compute the height h of the train, the passenger onboard switches on the ceiling light bulb, measures the time t that the light needs to hit the floor, by travelling at speed c, and concludes that the train height is h = ct:



On the other hand, an observer on the ground will see here something different, namely a right triangle, with on the vertical the height of the train h, on the horizontal the distance vT that the train has travelled, and on the hypotenuse the distance cT that light has travelled, with T being the duration of the event, according to his watch:



Now by Pythagoras applied to this triangle, we have:

$$h^2 + (vT)^2 = (cT)^2$$

Thus, the observer on the ground will reach to the following formula for h:

$$h = \sqrt{c^2 - v^2} \cdot T$$

But h must be the same for both observers, so we have the following formula:

$$\sqrt{c^2 - v^2} \cdot T = ct$$

It follows that the two times t and T are indeed not equal, and are related by:

$$T = \frac{ct}{\sqrt{c^2 - v^2}} = \frac{t}{\sqrt{1 - v^2/c^2}} = \gamma t$$

Thus, we are led to the formula in the statement.

Let us discuss now what happens to length. Intuitively, since speed is distance/time, and since time gets dilated, we can somehow expect distance to get dilated too.

However, and a bit surprisingly, this is wrong, and after due thinking and computations, what we have is in fact the following result:

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## **1B. RELATIVITY THEORY**

THEOREM 1.18. Relativistic length is subject to Lorentz contraction

$$L \to L/\gamma$$

where the number  $\gamma \geq 1$ , called Lorentz factor, is given by the usual formula

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

with v being the moving speed, at which length is measured.

PROOF. As before in the proof of Theorem 1.17, meaning in the same train travelling at speed v, in vacuum, imagine now that the passenger wants to measure the length L of the car. For this purpose he switches on the light bulb, now at the rear of the car, and measures the time t needed for the light to reach the front of the car, and get reflected back by a mirror installed there, according to the following scheme:



He concludes that, as marked above, the length L of the car is given by:

$$L = \frac{ct}{2}$$

Now viewed from the ground, the duration of the event is  $T = T_1 + T_2$ , where  $T_1 > T_2$  are respectively the time needed for the light to travel forward, among others for beating v, and the time for the light to travel back, helped this time by v. More precisely, if l denotes the length of the train car viewed from the ground, the formula of T is:

$$T = T_1 + T_2 = \frac{l}{c - v} + \frac{l}{c + v} = \frac{2lc}{c^2 - v^2}$$

With this data, the formula  $T = \gamma t$  of time dilation established before reads:

$$\frac{2lc}{c^2 - v^2} = \gamma t = \frac{2\gamma L}{c}$$

Thus, the two lengths L and l are indeed not equal, and related by:

$$l = \frac{\gamma L(c^2 - v^2)}{c^2} = \gamma L\left(1 - \frac{v^2}{c^2}\right) = \frac{\gamma L}{\gamma^2} = \frac{L}{\gamma}$$

Thus, we are led to the conclusion in the statement.

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With this discussed, time now to get to the real thing, see what happens to our usual  $\mathbb{R}^4$ . Let us start our discussion with a look at the non-relativistic case. Assuming that the object moves with speed v in the x direction, the frame change is given by:

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

To be more precise, here the first 3 equations come from the law of motion, and t' = t is the old t' = t. In the relativistic setting now, the result is more tricky, as follows:

THEOREM 1.19. In the context of a relativistic object moving with speed v along the x axis, the frame change is given by the Lorentz transformation

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(t - vx/c^{2})$$

with  $\gamma = 1/\sqrt{1 - v^2/c^2}$  being as usual the Lorentz factor.

PROOF. We know that, with respect to the non-relativistic formulae, x is subject to the Lorentz dilation by  $\gamma$ , and we obtain as desired:

$$x' = \gamma(x - vt)$$

Regarding y, z, these are obviously unchanged, so done with these too. Finally, regarding time t, a naive thought would suggest that this is subject to a Lorentz contraction by  $1/\gamma$ , but this is not true, and more thinking leads to the conclusion that we must use the reverse Lorentz transformation, given by the following formulae:

$$x = \gamma(x' + vt')$$
$$y = y'$$
$$z = z'$$

By using the formula of x' we can compute t', and we obtain the following formula:

$$t' = \frac{x - \gamma x'}{\gamma v}$$
$$= \frac{x - \gamma^2 (x - vt)}{\gamma v}$$
$$= \frac{\gamma^2 vt + (1 - \gamma^2) x}{\gamma v}$$

On the other hand, we have the following computation:

$$\gamma^2 = \frac{c^2}{c^2 - v^2} \implies \gamma^2 (c^2 - v^2) = c^2 \implies (\gamma^2 - 1)c^2 = \gamma^2 v^2$$

Thus we can finish the computation of t' as follows:

$$t' = \frac{\gamma^2 v t + (1 - \gamma^2) x}{\gamma v}$$
$$= \frac{\gamma^2 v t - \gamma^2 v^2 x / c^2}{\gamma v}$$
$$= \gamma \left( t - \frac{v x}{c^2} \right)$$

We are therefore led to the conclusion in the statement.

Now since y, z are irrelevant, we can put them at the end, and put the time t first, as to be close to x. By multiplying as well the time equation by c, our system becomes:

$$ct' = \gamma(ct - vx/c)$$
$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$

In linear algebra terms, the result is as follows:

THEOREM 1.20. The Lorentz transformation is given by

$$\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0\\ -\beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct\\ x\\ y\\ z \end{pmatrix} = \begin{pmatrix} ct'\\ x'\\ y'\\ z' \end{pmatrix}$$

where  $\gamma = 1/\sqrt{1-v^2/c^2}$  as usual, and where  $\beta = v/c.$ 

**PROOF.** In terms of  $\beta = v/c$ , replacing v, the system looks as follows:

$$ct' = \gamma(ct - \beta x)$$
$$x' = \gamma(x - \beta ct)$$
$$y' = y$$
$$z' = z$$

But this gives the formula in the statement.

As an illustration, let us verify that the inverse Lorentz transformation is indeed given by reversing the speed,  $v \to -v$ . With notations as in Theorem 1.19, the result is:

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THEOREM 1.21. The inverse of the Lorentz transformation is given by 
$$v \to -v_{z}$$

$$x = \gamma(x' + vt')$$
$$y = y'$$
$$z = z'$$
$$t = \gamma(t' + vx'/c^2)$$

where  $\gamma = 1/\sqrt{1 - v^2/c^2}$  is as usual the Lorentz factor, identical for v and -v.

**PROOF.** In terms of the formalism in Theorem 1.20, reversing the speed  $v \rightarrow -v$  amounts in reversing the  $\beta = v/c$  parameter there:

$$\beta \rightarrow -\beta$$

What we have to prove, in order to establish the result, is that by doing so, we obtain the inverse of the matrix appearing there, namely:

$$L = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0\\ -\beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}$$

That is, we want to prove that the inverse of this matrix is as follows:

$$L^{-1} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0\\ \beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}$$

But here, for the verification of the inversion formula  $LL^{-1} = 1$ , we can restrict the attention to the upper left corner, where the result is clear.

Let us discuss now what happens to momentum, mass and energy. We would like to fix the momentum conservation equations for the plastic collisions, namely:

$$m = m_1 + m_2$$

$$mv = m_1v_1 + m_2v_2$$

However, this cannot really be done with bare hands, and by this meaning with mathematics only. But with some help from experiments, the conclusion is as follows:

FACT 1.22. When defining the relativistic mass of an object of rest mass m > 0, moving at speed v, by the formula

$$M = \gamma m \quad : \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

this relativistic mass M, and the corresponding relativistic momentum P = Mv, are both conserved during collisions.

In other words, the situation here is a bit similar to that of the Galileo addition vs Einstein addition for speeds. The collision equations given above are in fact low-speed approximations of the correct, relativistic equations, which are as follows:

$$M = M_1 + M_2$$
$$Mv = M_1v_1 + M_2v_2$$

It remains now to discuss kinetic energy. You have certainly heard of the formula  $E = mc^2$ , which might actually well be on your T-shirt, now as you read this book, and in this case here is the explanation for it, in relation with the above:

THEOREM 1.23. The relativistic energy of an object of rest mass m > 0,

$$\mathcal{E} = Mc^2$$
 :  $M = \gamma m$ 

which is conserved, as being a multiple of M, can be written as  $\mathcal{E} = E + T$ , with

$$E = mc^2$$

being its v = 0 component, called rest energy of m, and with

$$T = (1 - \gamma)mc^2 \simeq \frac{mv^2}{2}$$

being called relativistic kinetic energy of m.

**PROOF.** All this is a bit abstract, coming from Fact 1.22, as follows:

(1) Given an object of rest mass m > 0, consider its relativistic mass  $M = \gamma m$ , as appearing in Fact 1.22, and then consider the following quantity:

$$\mathcal{E} = Mc^2$$

We know from Fact 1.22 that the relativistic mass M is conserved, so  $\mathcal{E} = Mc^2$  is conserved too. In view of this, is makes somehow sense to call  $\mathcal{E}$  energy. There is of course no clear reason for doing that, but let's just do it, and we'll understand later.

(2) Let us compute  $\mathcal{E}$ . This quantity is by definition given by:

$$\mathcal{E} = Mc^2 = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

Since  $1/\sqrt{1-x} \simeq 1 + x/2$  for x small, by calculus, we obtain, for v small:

$$\mathcal{E} \simeq mc^2 \left( 1 + \frac{v^2}{2c^2} \right) = mc^2 + \frac{mv^2}{2}$$

And, good news here, we recognize at right the kinetic energy of m.

(3) But this leads to the conclusions in the statement. Indeed, we are certainly dealing with some sort of energies here, and so calling the above quantity  $\mathcal{E}$  relativistic energy

is legitimate, and calling  $E = mc^2$  rest energy is legitimate too. Finally, the difference between these two energies  $T = \mathcal{E} - E$  follows to be given by:

$$T = (1 - \gamma)mc^2 \simeq \frac{mv^2}{2}$$

Thus, calling T relativistic kinetic energy is legitimate too, and we are done.

## 1c. Curved spacetime

Getting now to spacetime, in non-relativistic physics two events are separated by space  $\Delta x$  and by time  $\Delta t$ , with these two separation variables being independent. In relativistic physics this is no longer true, and the correct analogue of this comes from:

THEOREM 1.24. The following quantity, called relativistic spacetime separation

$$\Delta s^2 = c^2 \Delta t^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2)$$

is invariant under relativistic frame changes.

PROOF. We must prove that the quantity  $K = c^2t^2 - x^2 - y^2 - z^2$  is invariant under Lorentz transformations. For this purpose, observe that we have:

$$K = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \right\rangle$$

Now recall that the Lorentz transformation is given by the following formula, where  $\gamma = 1/\sqrt{1 - v^2/c^2}$  as usual, and where  $\beta = v/c$ :

$$\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0\\ -\beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct\\ x\\ y\\ z \end{pmatrix} = \begin{pmatrix} ct'\\ x'\\ y'\\ z' \end{pmatrix}$$

Thus, if we denote by L the matrix of the Lorentz transformation, and by E the matrix found before, we must prove that for any vector  $\xi$  we have:

$$\langle E\xi, \xi \rangle = \langle EL\xi, L\xi \rangle$$

Since L is symmetric we have  $\langle EL\xi, L\xi \rangle = \langle LEL\xi, \xi \rangle$ , so we must prove:

$$E = LEL$$

But this is the same as proving  $L^{-1}E = EL$ , and by using the fact that  $L \to L^{-1}$  is given by  $\beta \to -\beta$ , what we eventually want to prove is that:

$$L_{-\beta}E = EL_{\beta}$$

So, let us prove this. As usual we can restrict the attention to the upper left corner, call that NW corner, and here we have the following computations:

$$(L_{-\beta}E)_{NW} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ \beta\gamma & -\gamma \end{pmatrix}$$
$$(EL_{\beta})_{NW} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ \beta\gamma & -\gamma \end{pmatrix}$$

The matrices on the right being equal, this gives the result.

## 1d. General relativity

Finally, let us discuss gravity. This can be incorporated too, as follows:

THEOREM 1.25 (Einstein). The theory of gravity can be suitably modified, and merged with relativity, into a theory called general relativity.

**PROOF.** All this is a bit complicated, involving some geometry, as follows:

(1) Before anything, we have seen that in the relativistic context, mass m must be replaced by relativistic mass  $M = \gamma m$ , and momentum p = mv must be replaced by relativistic momentum P = Mv. Thus, as with Galileo and many other things, such as the conservation of mass and of momentum, seen above, there is a bug with the Newton formula  $F = \dot{p}$ , which must be replaced by something of type  $F = \dot{P}$ .

(2) In practice now, as a starting point, let us go back to the formula  $F = -\Delta V$ , that we know well. Geometrically, this suggests looking at the gravitational field of k bodies  $M_1, \ldots, M_k$  as being represented by  $\mathbb{R}^3$  having k holes in it, and with the heavier the  $M_i$ , the bigger the hole, and with poor  $m \simeq 0$  having to roll on all this.

(3) Of course we are here in 4D, for the full picture, that of the potential V, or rather of its graph, and in order to better understand this, it is of help to first consider the question where our bodies  $M_1, \ldots, M_k$  lie in a plane  $\mathbb{R}^2$ .

(4) Still staying inside classical mechanics, it is possible to further build on the above picture in (2), which was something rather intuitive, now with some precise math formulae, relating the geometry of V to the motion of m under its influence.

(5) The point now is that, with (4) done, the passage to relativity can be understood as well, by modifying a bit the geometry there, as to fit with relativistic spacetime, and by having the  $F = \dot{P}$  idea from (1) in mind too. That is the main idea behind general relativity, and in practice, all this needs a bit of technical geometry and formulae.

This was for the basics of Einstein's relativity theory. For more, we refer to his book [29], which is a must-read, for any mathematician, physicist, scientist, or non-scientist.

# 1e. Exercises

Exercises:

EXERCISE 1.26.

EXERCISE 1.27.

EXERCISE 1.28.

EXERCISE 1.29.

EXERCISE 1.30.

EXERCISE 1.31.

Bonus exercise.

## CHAPTER 2

# Quantum physics

## 2a. Atomic theory

Quantum mechanics was born from the study of the hydrogen atom, which consists of a negative charge, the electron, spinning around a positive charge, a proton. Basic electrodynamics, you would say, that I can solve with the Maxwell equations, that I learned the hard way in school. Very good, so let us start indeed with these equations:

**THEOREM** 2.1. Electrodynamics is governed by the formulae

$$\langle \nabla, E \rangle = \frac{\rho}{\varepsilon_0} \quad , \quad \langle \nabla, B \rangle = 0$$
  
 $\nabla \times E = -\dot{B} \quad , \quad \nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \dot{E}$ 

called Maxwell equations.

**PROOF.** This is something fundamental, appearing as a tricky mixture of physics facts and mathematical results, the idea being as follows:

(1) To start with, electrodynamics is the science of moving electrical charges. And this is something quite complicated, because unlike in classical mechanics, where the Newton law is good for both the static and the dynamic setting, the Coulomb law, which is actually very similar to the Newton law, does the job when the charges are static, but no longer describes well the situation when the charges are moving.

(2) The problem comes from the fact that moving charges produce magnetism, and with this being visible when putting together two electric wires, which will attract or repel, depending on orientation. Thus, in contrast with classical mechanics, where static or dynamic problems are described by a unique field, the gravitational one, in electrodynamics we have two fields, namely the electric field E, and the magnetic field B.

(3) Fortunately, there is a full set of equations relating the fields E and B, namely those above. To be more precise, the first formula is the Gauss law,  $\rho$  being the charge, and  $\varepsilon_0$  being a constant, and with this Gauss law more or less replacing the Coulomb law from electrostatics. The second formula is something basic, and anonymous. The third formula is the Faraday law. As for the fourth formula, this is the Ampère law, as modified by Maxwell, with J being the volume current density, and  $\mu_0$  being a constant.

## 2. QUANTUM PHYSICS

In relation now with the hydrogen atom, we have good and bad news. The bad news is that the Maxwell equations, as formulated above, are something rather statistical and macroscopic, and simply do not apply to the hydrogen atom. However, we have good news as well, the point here being that the same Maxwell equations can lead us, in a rather twisted way, via light, spectroscopy, and many more, to the hydrogen atom. So, let us explain this. We first have the following consequence of the Maxwell equations:

THEOREM 2.2. In regions of space where there is no charge or current present the Maxwell equations for electrodynamics read

$$< \nabla, E > = < \nabla, B > = 0$$
  
 $\nabla \times E = -\dot{B}$ ,  $\nabla \times B = \dot{E}/c^2$ 

and both the electric field E and magnetic field B are subject to the wave equation

$$\ddot{\varphi} = c^2 \Delta \varphi$$

where  $\Delta = \sum_i d^2/dx_i^2$  is the Laplace operator, and c is the speed of light.

**PROOF.** Under the circumstances in the statement, namely no charge or current present, the Maxwell equations in Theorem 2.1 simply read:

$$< \nabla, E > = < \nabla, B > = 0$$

$$\nabla \times E = -B$$
 ,  $\nabla \times B = E/c^2$ 

Here we have used a key formula due to Biot-Savart, as follows:

$$\mu_0 \varepsilon_0 = \frac{1}{c^2}$$

Now by applying the curl operator to the last two equations, we obtain:

$$\nabla \times (\nabla \times E) = -\nabla \times \dot{B} = -(\nabla \times B)' = -\ddot{E}/c^2$$
$$\nabla \times (\nabla \times B) = \nabla \times \dot{E}/c^2 = (\nabla \times E)'/c^2 = -\ddot{B}/c^2$$

But the double curl operator is subject to the following formula:

$$\nabla\times(\nabla\times\varphi)=\nabla<\nabla,\varphi>-\Delta\varphi$$

Now by using the first two equations, we are led to the conclusion in the statement.  $\Box$ 

So, what is light? Light is the wave predicted by Theorem 2.2, traveling in vacuum at the maximum possible speed, c, and with an important extra property being that it depends on a real positive parameter, that can be called, upon taste, frequency, wavelength, or color. And in what regards the creation of light, the mechanism here is as follows:

FACT 2.3. An accelerating or decelerating charge produces electromagnetic radiation, called light, whose frequency and wavelength can be explicitly computed.

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This phenomenon can be observed is a variety of situations, such as the usual light bulbs, where electrons get decelerated by the filament, acting as a resistor, or in usual fire, which is a chemical reaction, with the electrons moving around, as they do in any chemical reaction, or in more complicated machinery like nuclear plants, particle accelerators, and so on, leading there to all sorts of eerie glows, of various colors.

In practice now, the classification of light, in a rough form, is as follows:

Frequency	Type	Wavelength
	_	
$10^{18} - 10^{20}$	$\gamma$ rays	$10^{-12} - 10^{-10}$
$10^{16} - 10^{18}$	X - rays	$10^{-10} - 10^{-8}$
$10^{15} - 10^{16}$	UV	$10^{-8} - 10^{-7}$
	_	
$10^{14} - 10^{15}$	blue	$10^{-7} - 10^{-6}$
$10^{14} - 10^{15}$	yellow	$10^{-7} - 10^{-6}$
$10^{14} - 10^{15}$	red	$10^{-7} - 10^{-6}$
	_	
$10^{11} - 10^{14}$	IR	$10^{-6} - 10^{-3}$
$10^9 - 10^{11}$	microwave	$10^{-3} - 10^{-1}$
$1 - 10^9$	radio	$10^{-1} - 10^8$

Observe the tiny space occupied by the visible light, all the colors there, and the many more missing, being squeezed under the  $10^{14} - 10^{15}$  frequency banner. Here is a zoom on that part, with of course the remark that all this, colors, is something subjective:

Frequency $THz = 10^{12} Hz$	Color	Wavelength $nm = 10^{-9} m$
	—	
670 - 790	violet	380 - 450
620 - 670	blue	450 - 485
600 - 620	cyan	485 - 500
530 - 600	green	500 - 565
510 - 530	yellow	565 - 590
480 - 510	orange	590 - 625
400 - 480	red	625 - 750

Many things can be said about light and optics, and going now straight to the point, the idea is that, with this in hand, we can talk about spectroscopy:

FACT 2.4. We can study events via spectroscopy, by capturing the light the event has produced, decomposing it with a prism, carefully recording its "spectral signature", consisting of the wavelenghts present, and their density, and then doing some reverse engineering, consisting in reconstructing the event out of its spectral signature.

## 2. QUANTUM PHYSICS

Going now towards atoms, there is a long story here, involving many discoveries, around 1890-1900, focusing on hydrogen H. We will present here things a bit retrospectively. First on our list is the following discovery, by Lyman in 1906:

FACT 2.5 (Lyman). The hydrogen atom has spectral lines given by the formula

$$\frac{1}{\lambda} = R\left(1 - \frac{1}{n^2}\right)$$

where  $R \simeq 1.097 \times 10^7$  and  $n \ge 2$ , which are as follows,

n	Name	Wavelength	Color
	_	—	
2	$\alpha$	121.567	UV
3	$\beta$	102.572	UV
4	$\gamma$	97.254	UV
÷	:	:	:
$\infty$	limit	91.175	UV

called Lyman series of the hydrogen atom.

Observe that all the Lyman series lies in UV, which is invisible to the naked eye. Due to this fact, this series, while theoretically being the most important, was discovered only second. The first discovery, which was the big one, and the breakthrough, was by Balmer, the founding father of all this, back in 1885, in the visible range, as follows:

FACT 2.6 (Balmer). The hydrogen atom has spectral lines given by the formula

$$\frac{1}{\lambda} = R\left(\frac{1}{4} - \frac{1}{n^2}\right)$$

where  $R \simeq 1.097 \times 10^7$  and  $n \ge 3$ , which are as follows,

n	Name	Wavelength	Color
	—	—	
3	$\alpha$	656.279	red
4	$\beta$	486.135	aqua
5	$\gamma$	434.047	blue
6	$\delta$	410.173	violet
7	ε	397.007	UV
÷	:	:	:
$\infty$	limit	346.600	UV

called Balmer series of the hydrogen atom.

So, this was Balmer's original result, which started everything. As a third main result now, this time in IR, due to Paschen in 1908, we have:

FACT 2.7 (Paschen). The hydrogen atom has spectral lines given by the formula

$$\frac{1}{\lambda} = R\left(\frac{1}{9} - \frac{1}{n^2}\right)$$

where  $R \simeq 1.097 \times 10^7$  and  $n \ge 4$ , which are as follows,

n	Name	Wavelength	Color
	—	—	
4	$\alpha$	1875	$\operatorname{IR}$
5	eta	1282	$\operatorname{IR}$
6	$\gamma$	1094	$\operatorname{IR}$
÷	÷	:	÷
$\infty$	limit	820.4	$\operatorname{IR}$

called Paschen series of the hydrogen atom.

Observe the striking similarity between the above three results. In fact, we have here the following fundamental, grand result, due to Rydberg in 1888, based on the Balmer series, and with later contributions by Ritz in 1908, using the Lyman series as well:

CONCLUSION 2.8 (Rydberg, Ritz). The spectral lines of the hydrogen atom are given by the Rydberg formula, depending on integer parameters  $n_1 < n_2$ ,

$$\frac{1}{\lambda_{n_1 n_2}} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

with R being the Rydberg constant for hydrogen, which is as follows:

$$R \simeq 1.096\ 775\ 83 \times 10^7$$

These spectral lines combine according to the Ritz-Rydberg principle, as follows:

$$\frac{1}{\lambda_{n_1n_2}} + \frac{1}{\lambda_{n_2n_3}} = \frac{1}{\lambda_{n_1n_3}}$$

Similar formulae hold for other atoms, with suitable fine-tunings of R.

Here the first part, the Rydberg formula, generalizes the results of Lyman, Balmer, Paschen, which appear at  $n_1 = 1, 2, 3$ , at least retrospectively. The Rydberg formula predicts further spectral lines, appearing at  $n_1 = 4, 5, 6, \ldots$ , and these were discovered later, by Brackett in 1922, Pfund in 1924, Humphreys in 1953, and others aftwerwards,

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with all these extra lines being in far IR. The simplified complete table is as follows:

$n_1$	$n_2$	Series name	Wavelength $n_2 = \infty$	Color $n_2 = \infty$
		—	—	
1	$2-\infty$	Lyman	91.13  nm	UV
2	$3-\infty$	Balmer	$364.51~\mathrm{nm}$	UV
3	$4-\infty$	Paschen	820.14 nm	IR
		_	_	
4	$5-\infty$	Brackett	$1458.03~\mathrm{nm}$	far IR
5	$6-\infty$	Pfund	$2278.17~\mathrm{nm}$	far IR
6	$7-\infty$	Humphreys	$3280.56~\mathrm{nm}$	far IR
÷		:	÷	:

Regarding the last assertion, concerning other elements, this is something conjectured and partly verified by Ritz, and fully verified and clarified later, via many experiments, the fine-tuning of R being basically  $R \to RZ^2$ , where Z is the atomic number.

From a theoretical physics viewpoint, the main result remains the middle assertion, called Ritz-Rydberg combination principle, which is something quite puzzling. But this combination principle reminds the formula  $e_{n_1n_2}e_{n_2n_3} = e_{n_1n_3}$  for the usual matrix units  $e_{ij}: e_j \to e_i$ . Thus, we are in familiar territory here, and we can start dreaming of:

THOUGHT 2.9. Observables in quantum mechanics should be some sort of infinite matrices, generalizing the Lyman, Balmer, Paschen lines of the hydrogen atom, and multiplying between them as the matrices do, as to produce further observables.

Now back to more concrete things, as a main problem that we would like to solve, we have the understanding the intimate structure of matter, at the atomic level. There is of course a long story here, regarding the intimate structure of matter, going back centuries and even millennia ago, and our presentation here will be quite simplified. As a starting point, since we need a starting point for all this, let us agree on:

CLAIM 2.10. Ordinary matter is made of small particles called atoms, with each atom appearing as a mix of even smaller particles, namely protons +, neutrons 0 and electrons -, with the same number of protons + and electrons -.

As a first observation, this is something which does not look obvious at all, with probably lots of work, by many people, being involved, as to lead to this claim. And so it is. The story goes back to the discovery of charges and electricity, which were attributed to a small particle, the electron -. Now since matter is by default neutral, this naturally leads to the consideration to the proton +, having the same charge as the electron.

But, as a natural question, why should be these electrons - and protons + that small? And also, what about the neutron 0? These are not easy questions, and the fact that it
### **2B. QUANTUM MECHANICS**

is so came from several clever experiments. Let us first recall that careful experiments with tiny particles are practically impossible. However, all sorts of brutal experiments, such as bombarding matter with other pieces of matter, accelerated to the extremes, or submitting it to huge electric and magnetic fields, do work. And it is such kind of experiments, due to Thomson, Rutherford and others, "peeling off" protons +, neutrons 0 and electrons – from matter, and observing them, that led to the conclusion that these small beasts +, 0, - exist indeed, in agreement with Claim 2.10.

So, taking now Claim 2.10 for granted, how are then the atoms organized, as mixtures of protons +, neutrons 0 and electrons -? The answer here lies again in the abovementioned "brutal" experiments of Thomson, Rutherford and others, which not only proved Claim 2.10, but led to an improved version of it, as follows:

CLAIM 2.11. The atoms are formed by a core of protons + and neutrons 0, surrounded by a cloud of electrons -, gravitating around the core.

This is a considerable advance, because we are now into familiar territory, namely some kind of mechanics. And with this in mind, all the pieces of our puzzle start fitting together, and we are led to the following grand conclusion:

CLAIM 2.12 (Bohr and others). The atoms are formed by a core of protons and neutrons, surrounded by a cloud of electrons, basically obeying to a modified version of electromagnetism. And with a fine mechanism involved, as follows:

- (1) The electrons are free to move only on certain specified elliptic orbits, labelled  $1, 2, 3, \ldots$ , situated at certain specific heights.
- (2) The electrons can jump or fall between orbits  $n_1 < n_2$ , absorbing or emitting light and heat, that is, electromagnetic waves, as accelerating charges.
- (3) The energy of such a wave, coming from  $n_1 \rightarrow n_2$  or  $n_2 \rightarrow n_1$ , is given, via the Planck viewpoint, by the Rydberg formula, applied with  $n_1 < n_2$ .
- (4) The simplest such jumps are those observed by Lyman, Balmer, Paschen. And multiple jumps explain the Ritz-Rydberg formula.

And isn't this beautiful. Moreover, following now Heisenberg, the next claim is that the underlying mathematics in all the above can lead to a beautiful axiomatization of quantum mechanics, as a "matrix mechanics", along the lines of Thought 2.9.

## 2b. Quantum mechanics

Before explaining what Heisenberg was saying, based on Lyman, Balmer, Paschen, namely developing some sort of "matrix mechanics", let us hear as well the point of view of Schrödinger, which came a few years later. His idea was to forget about exact things, and try to investigate the hydrogen atom statistically. Let us start with:

QUESTION 2.13. In the context of the hydrogen atom, assuming that the proton is fixed, what is the probability density  $\varphi_t(x)$  of the position of the electron e, at time t,

$$P_t(e \in V) = \int_V \varphi_t(x) dx$$

as function of an initial probability density  $\varphi_0(x)$ ? Moreover, can the corresponding equation be solved, and will this prove the Bohr claims for hydrogen, statistically?

In order to get familiar with this question, let us first look at examples coming from classical mechanics. In the context of a particle whose position at time t is given by  $x_0 + \gamma(t)$ , the evolution of the probability density will be given by:

$$\varphi_t(x) = \varphi_0(x) + \gamma(t)$$

However, such examples are somewhat trivial, of course not in relation with the computation of  $\gamma$ , usually a difficult question, but in relation with our questions, and do not apply to the electron. The point indeed is that, in what regards the electron, we have:

FACT 2.14. In respect with various simple interference experiments:

- (1) The electron is definitely not a particle in the usual sense.
- (2) But in most situations it behaves exactly like a wave.
- (3) But in other situations it behaves like a particle.

Getting back now to the Schrödinger question, all this suggests to use, as for the waves, an amplitude function  $\psi_t(x) \in \mathbb{C}$ , related to the density  $\varphi_t(x) > 0$  by the formula  $\varphi_t(x) = |\psi_t(x)|^2$ . Not that a big deal, you would say, because the two are related by simple formulae as follows, with  $\theta_t(x)$  being an arbitrary phase function:

$$\varphi_t(x) = |\psi_t(x)|^2$$
,  $\psi_t(x) = e^{i\theta_t(x)}\sqrt{\varphi_t(x)}$ 

However, such manipulations can be crucial, raising for instance the possibility that the amplitude function satisfies some simple equation, while the density itself, maybe not. And this is what happens indeed. Schrödinger was led in this way to:

CLAIM 2.15 (Schrödinger). In the context of the hydrogen atom, the amplitude function of the electron  $\psi = \psi_t(x)$  is subject to the Schrödinger equation

$$ih\dot{\psi} = -\frac{h^2}{2m}\Delta\psi + V\psi$$

m being the mass,  $h = h_0/2\pi$  the reduced Planck constant, and V the Coulomb potential of the proton. The same holds for movements of the electron under any potential V.

Observe the similarity with the wave equation  $\ddot{\varphi} = v^2 \Delta \varphi$ , and with the heat equation  $\dot{\varphi} = \alpha \Delta \varphi$  too. Many things can be said here. Following now Heisenberg and Schrödinger, and then especially Dirac, who did the axiomatization work, we have:

DEFINITION 2.16. In quantum mechanics the states of the system are vectors of a Hilbert space H, and the observables of the system are linear operators

$$T: H \to H$$

which can be densely defined, and are taken self-adjoint,  $T = T^*$ . The average value of such an observable T, evaluated on a state  $\xi \in H$ , is given by:

$$< T > = < T\xi, \xi >$$

In the context of the Schrödinger mechanics of the hydrogen atom, the Hilbert space is the space  $H = L^2(\mathbb{R}^3)$  where the wave function  $\psi$  lives, and we have

$$< T > = \int_{\mathbb{R}^3} T(\psi) \cdot \bar{\psi} \, dx$$

which is called "sandwiching" formula, with the operators

$$x$$
 ,  $-\frac{i\hbar}{m}\nabla$  ,  $-i\hbar\nabla$  ,  $-\frac{\hbar^2\Delta}{2m}$  ,  $-\frac{\hbar^2\Delta}{2m}+V$ 

representing the position, speed, momentum, kinetic energy, and total energy.

In other words, we are doing here two things. First, we are declaring by axiom that various "sandwiching" formulae found before by Heisenberg, involving the operators at the end, that we will not get into in detail here, hold true. And second, we are raising the possibility for other quantum mechanical systems, more complicated, to be described as well by the mathematics of the operators on a certain Hilbert space H, as above.

Now, let us go back to the Schrödinger equation from Claim 2.15. We have:

**PROPOSITION 2.17.** We have the following formula,

$$\dot{\varphi} = \frac{i\hbar}{2m} \left( \Delta \psi \cdot \bar{\psi} - \Delta \bar{\psi} \cdot \psi \right)$$

for the time derivative of the probability density function  $\varphi = |\psi|^2$ .

**PROOF.** According to the Leibnitz product rule, we have the following formula:

$$\dot{\varphi} = \frac{d}{dt}|\psi|^2 = \frac{d}{dt}(\psi\bar{\psi}) = \dot{\psi}\bar{\psi} + \psi\dot{\bar{\psi}}$$

On the other hand, the Schrödinger equation and its conjugate read:

$$\dot{\psi} = \frac{ih}{2m} \left( \Delta \psi - \frac{2m}{h^2} V \psi \right) \quad , \quad \dot{\bar{\psi}} = -\frac{ih}{2m} \left( \Delta \bar{\psi} - \frac{2m}{h^2} V \bar{\psi} \right)$$

By plugging this data, we obtain the following formula:

$$\dot{\varphi} = \frac{i\hbar}{2m} \left[ \left( \Delta \psi - \frac{2m}{\hbar^2} V \psi \right) \bar{\psi} - \left( \Delta \bar{\psi} - \frac{2m}{\hbar^2} V \bar{\psi} \right) \psi \right]$$

But this gives, after simplifying, the formula in the statement.

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As an important application of Proposition 2.17, we have:

THEOREM 2.18. The Schrödinger equation conserves probability amplitudes,

$$\int_{\mathbb{R}^3} |\psi_0|^2 = 1 \implies \int_{\mathbb{R}^3} |\psi_t|^2 = 1$$

in agreement with the basic probabilistic requirement, P = 1 overall.

**PROOF.** According to the formula in Proposition 2.17, we have:

$$\frac{d}{dt} \int_{\mathbb{R}^3} |\psi|^2 dx = \int_{\mathbb{R}^3} \frac{d}{dt} |\psi|^2 dx$$
$$= \int_{\mathbb{R}^3} \dot{\varphi} dx$$
$$= \frac{ih}{2m} \int_{\mathbb{R}^3} \left( \Delta \psi \cdot \bar{\psi} - \Delta \bar{\psi} \cdot \psi \right) dx$$

Now by remembering the definition of the Laplace operator, we have:

$$\frac{d}{dt} \int_{\mathbb{R}^3} |\psi|^2 dx = \frac{i\hbar}{2m} \int_{\mathbb{R}^3} \sum_i \left( \frac{d^2\psi}{dx_i^2} \cdot \bar{\psi} - \frac{d^2\bar{\psi}}{dx_i^2} \cdot \psi \right) dx$$
$$= \frac{i\hbar}{2m} \sum_i \int_{\mathbb{R}^3} \frac{d}{dx_i} \left( \frac{d\psi}{dx_i} \cdot \bar{\psi} - \frac{d\bar{\psi}}{dx_i} \cdot \psi \right) dx$$
$$= \frac{i\hbar}{2m} \sum_i \int_{\mathbb{R}^2} \left[ \frac{d\psi}{dx} \cdot \bar{\psi} - \frac{d\bar{\psi}}{dx} \cdot \psi \right]_{-\infty}^{\infty} \frac{dx}{dx_i}$$
$$= 0$$

Thus, we are led to the conclusion in the statement.

Moving now towards hydrogen, we have here the following result:

THEOREM 2.19. In the case of time-independent potentials V, including the Coulomb potential of the proton, the solutions of the Schrödinger equation

$$ih\dot{\psi} = -\frac{h^2}{2m}\Delta\psi + V\psi$$

which are of the following special form, with the time and space variables separated,

$$\psi_t(x) = w_t \phi(x)$$

are given by the following formulae, with E being a certain constant,

$$w = e^{-iEt/h}w_0$$
 ,  $E\phi = -\frac{h^2}{2m}\Delta\phi + V\phi$ 

with the equation for  $\phi$  being called time-independent Schrödinger equation.

**PROOF.** By dividing by  $\psi$ , the Schrödinger equation becomes:

$$ih \cdot \frac{\dot{w}}{w} = -\frac{h^2}{2m} \cdot \frac{\Delta\phi}{\phi} + V$$

Now since the left-hand side depends only on time, and the right-hand side depends only on space, both quantities must equal a constant E, and this gives the result.  $\Box$ 

Moving ahead with theory, we can further build on Theorem 2.19, with a number of key observations on the time-independent Schrödinger equation, as follows:

THEOREM 2.20. In the case of time-independent potentials V, the Schrödinger equation and its time-independent version have the following properties:

- (1) For solutions of type  $\psi = w_t \phi(x)$ , the density  $\varphi = |\psi|$  is time-independent, and more generally, all quantities of type  $\langle T \rangle$  are time-independent.
- (2) The time-independent Schrödinger equation can be written as  $H\phi = E\phi$ , with H = T + V being the total energy, of Hamiltonian.
- (3) For solutions of type  $\psi = w_t \phi(x)$  we have  $\langle H^k \rangle = E^k$  for any k. In particular we have  $\langle H \rangle = E$ , and the variance is  $\langle H^2 \rangle \langle H \rangle^2 = 0$ .

**PROOF.** All the formulae are clear indeed from the fact that, when using the sandwiching formula for computing averages, the phases will cancel:

$$< T > = \int_{\mathbb{R}^3} \bar{\psi} \cdot T \cdot \psi \, dx = \int_{\mathbb{R}^3} \bar{\phi} \cdot T \cdot \phi \, dx$$

Thus, we are led to the various conclusions in the statement.

We have as well the following key result, mathematical this time:

THEOREM 2.21. The solutions of the Schrödinger equation with time-independent potential V appear as linear combinations of separated solutions

$$\psi = \sum_{n} c_n e^{-iE_n t/h} \phi_h$$

with the absolute values of the coefficients being given by

$$\langle H \rangle = \sum_{n} |c_n|^2 E_n$$

 $|c_n|$  being the probability for a measurement to return the energy value  $E_n$ .

PROOF. This is something standard, which follows from Fourier analysis, which allows the decomposition of  $\psi$  as in the statement, and that we will not really need, in what follows next. As before, for a physical discussion here, we refer to Griffiths [43].

## 2c. Hydrogen atom

In order to solve now the hydrogen atom, by using the Schrödinger equation, the idea will be that of reformulating this equation in spherical coordinates. We have:

THEOREM 2.22. The time-independent Schrödinger equation in spherical coordinates separates, for solutions of type  $\phi = \rho(r)\alpha(s,t)$ , into two equations, as follows,

$$\frac{d}{dr}\left(r^2 \cdot \frac{d\rho}{dr}\right) - \frac{2mr^2}{h^2}(V - E)\rho = K\rho$$
$$\sin s \cdot \frac{d}{ds}\left(\sin s \cdot \frac{d\alpha}{ds}\right) + \frac{d^2\alpha}{dt^2} = -K\sin^2 s \cdot \alpha$$

with K being a constant, called radial equation, and angular equation.

**PROOF.** We use the following well-known formula for the Laplace operator in spherical coordinates, whose proof can be found in any advanced calculus book:

$$\Delta = \frac{1}{r^2} \cdot \frac{d}{dr} \left( r^2 \cdot \frac{d}{dr} \right) + \frac{1}{r^2 \sin s} \cdot \frac{d}{ds} \left( \sin s \cdot \frac{d}{ds} \right) + \frac{1}{r^2 \sin^2 s} \cdot \frac{d^2}{dt^2}$$

By using this formula, the time-independent Schrödinger equation reformulates as:

$$(V-E)\phi = \frac{h^2}{2m} \left[ \frac{1}{r^2} \cdot \frac{d}{dr} \left( r^2 \cdot \frac{d\phi}{dr} \right) + \frac{1}{r^2 \sin s} \cdot \frac{d}{ds} \left( \sin s \cdot \frac{d\phi}{ds} \right) + \frac{1}{r^2 \sin^2 s} \cdot \frac{d^2\phi}{dt^2} \right]$$

Let us look now for separable solutions for this latter equation, consisting of a radial part and an angular part, as in the statement, namely:

$$\phi(r, s, t) = \rho(r)\alpha(s, t)$$

By plugging this function into our equation, we obtain:

$$(V-E)\rho\alpha = \frac{h^2}{2m} \left[ \frac{\alpha}{r^2} \cdot \frac{d}{dr} \left( r^2 \cdot \frac{d\rho}{dr} \right) + \frac{\rho}{r^2 \sin s} \cdot \frac{d}{ds} \left( \sin s \cdot \frac{d\alpha}{ds} \right) + \frac{\rho}{r^2 \sin^2 s} \cdot \frac{d^2\alpha}{dt^2} \right]$$

By multiplying everything by  $2mr^2/(h^2\rho\alpha)$ , and then moving the radial terms to the left, and the angular terms to the right, this latter equation can be written as follows:

$$\frac{2mr^2}{h^2}(V-E) - \frac{1}{\rho} \cdot \frac{d}{dr} \left( r^2 \cdot \frac{d\rho}{dr} \right) = \frac{1}{\alpha \sin^2 s} \left[ \sin s \cdot \frac{d}{ds} \left( \sin s \cdot \frac{d\alpha}{ds} \right) + \frac{d^2\alpha}{dt^2} \right]$$

Since this latter equation is now separated between radial and angular variables, both sides must be equal to a certain constant -K, and this gives the result.

Let us first study the angular equation. The result here is as follows:

THEOREM 2.23. The separated solutions  $\alpha = \sigma(s)\theta(t)$  of the angular equation,

$$\sin s \cdot \frac{d}{ds} \left( \sin s \cdot \frac{d\alpha}{ds} \right) + \frac{d^2 \alpha}{dt^2} = -K \sin^2 s \cdot \alpha$$

are given by the following formulae, where  $l \in \mathbb{N}$  is such that K = l(l+1),

$$\sigma(s) = P_l^m(\cos s) \quad , \quad \theta(t) = e^{imt}$$

and where  $m \in \mathbb{Z}$  is a constant, and with  $P_l^m$  being the Legendre function,

$$P_l^m(x) = (-1)^m (1 - x^2)^{m/2} \left(\frac{d}{dx}\right)^m P_l(x)$$

where  $P_l$  are the Legendre polynomials, given by the following formula:

$$P_l(x) = \frac{1}{2^l l!} \left(\frac{d}{dx}\right)^l (x^2 - 1)^l$$

These solutions  $\alpha = \sigma(s)\theta(t)$  are called spherical harmonics.

**PROOF.** This follows indeed from all the above, and with the comment that everything is taken up to linear combinations. We will normalize the wave function later.  $\Box$ 

In order to finish our study, it remains to solve the radial equation, for the Coulomb potential V of the proton. As a first manipulation on the radial equation, we have:

**PROPOSITION 2.24.** The radial equation, written with K = l(l+1),

$$(r^{2}\rho')' - \frac{2mr^{2}}{h^{2}}(V-E)\rho = l(l+1)\rho$$

takes with  $\rho = u/r$  the following form, called modified radial equation,

$$Eu = -\frac{h^2}{2m} \cdot u'' + \left(V + \frac{h^2 l(l+1)}{2mr^2}\right)u$$

which is a time-independent 1D Schrödinger equation.

**PROOF.** With  $\rho = u/r$  as in the statement, we have:

$$\rho = \frac{u}{r} \quad , \quad \rho' = \frac{u'r - u}{r^2} \quad , \quad (r^2 \rho')' = u''r$$

By plugging this data into the radial equation, this becomes:

$$u''r - \frac{2mr}{h^2}(V - E)u = \frac{l(l+1)}{r} \cdot u$$

By multiplying everything by  $h^2/(2mr)$ , this latter equation becomes:

$$\frac{h^2}{2m} \cdot u'' - (V - E)u = \frac{h^2 l(l+1)}{2mr^2} \cdot u$$

But this gives the formula in the statement. As for the interpretation, as time-independent 1D Schrödinger equation, this is clear as well, and with the comment here that the term added to the potential V is some sort of centrifugal term.

It remains to solve the above equation, for the Coulomb potential of the proton. And we have here the following result, which proves the original claims by Bohr:

THEOREM 2.25 (Schrödinger). In the case of the hydrogen atom, where V is the Coulomb potential of the proton, the modified radial equation, which reads

$$Eu = -\frac{h^2}{2m} \cdot u'' + \left(-\frac{Ke^2}{r} + \frac{h^2l(l+1)}{2mr^2}\right)u$$

leads to the Bohr formula for allowed energies,

$$E_n = -\frac{m}{2} \left(\frac{Ke^2}{h}\right)^2 \cdot \frac{1}{n^2}$$

with  $n \in \mathbb{N}$ , the binding energy being

$$E_1 \simeq -2.177 \times 10^{-18}$$

with means  $E_1 \simeq -13.591$  eV.

**PROOF.** This is again something non-trivial, the idea being as follows:

(1) By dividing our modified radial equation by E, this becomes:

$$-\frac{h^2}{2mE} \cdot u'' = \left(1 + \frac{Ke^2}{Er} - \frac{h^2l(l+1)}{2mEr^2}\right)u$$

In terms of  $\alpha = \sqrt{-2mE}/h$ , this equation takes the following form:

$$\frac{u''}{\alpha^2} = \left(1 + \frac{Ke^2}{Er} + \frac{l(l+1)}{(\alpha r)^2}\right)u$$

In terms of the new variable  $p = \alpha r$ , this latter equation reads:

$$u'' = \left(1 + \frac{\alpha K e^2}{Ep} + \frac{l(l+1)}{p^2}\right)u$$

Now let us introduce a new constant S for our problem, as follows:

$$S = -\frac{\alpha K e^2}{E}$$

In terms of this new constant, our equation reads:

$$u'' = \left(1 - \frac{S}{p} + \frac{l(l+1)}{p^2}\right)u$$

(2) The idea will be that of looking for a solution written as a power series, but before that, we must "peel off" the asymptotic behavior. Which is something that can be done,

of course, heuristically. With  $p \to \infty$  we are led to u'' = u, and ignoring the solution  $u = e^p$  which blows up, our approximate asymptotic solution is:

$$u \sim e^{-p}$$

Similarly, with  $p \to 0$  we are led to  $u'' = l(l+1)u/p^2$ , and ignoring the solution  $u = p^{-l}$  which blows up, our approximate asymptotic solution is:

$$u \sim p^{l+1}$$

(3) The above heuristic considerations suggest writing our function u as follows:

$$u = p^{l+1}e^{-p}v$$

So, let us do this. In terms of v, we have the following formula:

$$u' = p^{l} e^{-p} \left[ (l+1-p)v + pv' \right]$$

Differentiating a second time gives the following formula:

$$u'' = p^{l}e^{-p}\left[\left(\frac{l(l+1)}{p} - 2l - 2 + p\right)v + 2(l+1-p)v' + pv''\right]$$

Thus the radial equation, as modified in (1) above, reads:

$$pv'' + 2(l+1-p)v' + (S-2(l+1))v = 0$$

(4) We will be looking for a solution v appearing as a power series:

$$v = \sum_{j=0}^{\infty} c_j p^j$$

But our equation leads to the following recurrence formula for the coefficients:

$$c_{j+1} = \frac{2(j+l+1) - S}{(j+1)(j+2l+2)} \cdot c_j$$

(5) We are in principle done, but we still must check that, with this choice for the coefficients  $c_j$ , our solution v, or rather our solution u, does not blow up. And the whole point is here. Indeed, at j >> 0 our recurrence formula reads, approximately:

$$c_{j+1} \simeq \frac{2c_j}{j}$$

But, surprisingly, this leads to  $v \simeq c_0 e^{2p}$ , and so to  $u \simeq c_0 p^{l+1} e^p$ , which blows up.

(6) As a conclusion, the only possibility for u not to blow up is that where the series defining v terminates at some point. Thus, we must have for a certain index j:

$$2(j+l+1) = S$$

In other words, we must have, for a certain integer n > l:

$$S = 2n$$

(7) We are almost there. Recall from (1) above that S was defined as follows:

$$S = -\frac{\alpha K e^2}{E} \quad : \quad \alpha = \frac{\sqrt{-2mE}}{h}$$

Thus, we have the following formula for the square of S:

$$S^{2} = \frac{\alpha^{2}K^{2}e^{4}}{E^{2}} = -\frac{2mE}{h^{2}} \cdot \frac{K^{2}e^{4}}{E^{2}} = -\frac{2mK^{2}e^{4}}{h^{2}E}$$

Now by using the formula S = 2n from (6), the energy E must be of the form:

$$E = -\frac{2mK^2e^4}{h^2S^2} = -\frac{mK^2e^4}{2h^2n^2}$$

Calling this energy  $E_n$ , depending on  $n \in \mathbb{N}$ , we have, as claimed:

$$E_n = -\frac{m}{2} \left(\frac{Ke^2}{h}\right)^2 \cdot \frac{1}{n^2}$$

(8) Thus, we proved the Bohr formula. Regarding numerics, the data is as follows:

$$K = 8.988 \times 10^9$$
 ,  $e = 1.602 \times 10^{-19}$   
 $h = 1.055 \times 10^{-34}$  ,  $m = 9.109 \times 10^{-31}$ 

But this gives the formula of  $E_1$  in the statement.

As a first remark, all this agrees with the Rydberg formula, due to:

THEOREM 2.26. The Rydberg constant for hydrogen is given by

$$R = -\frac{E_1}{h_0 c}$$

where  $E_1$  is the Bohr binding energy, and the Rydberg formula itself, namely

$$\frac{1}{\lambda_{n_1 n_2}} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

simply reads, via the energy formula in Theorem 2.25,

$$\frac{1}{\lambda_{n_1 n_2}} = \frac{E_{n_2} - E_{n_1}}{h_0 c}$$

which is in agreement with the Planck formula  $E = h_0 c / \lambda$ .

**PROOF.** Here the first assertion is something numeric, coming from the fact that the formula in the statement gives, when evaluated, the Rydberg constant:

$$R = \frac{-E_1}{h_0 c} = \frac{2.177 \times 10^{-18}}{6.626 \times 10^{-34} \times 2.998 \times 10^8} = 1.096 \times 10^7$$

#### 2C. HYDROGEN ATOM

Regarding now the second assertion, by dividing  $R = -E_1/(h_0c)$  by any number of type  $n^2$  we obtain, according to the energy convention in Theorem 2.25:

$$\frac{R}{n^2} = -\frac{E_n}{h_0 c}$$

But these are exactly the numbers which are subject to substraction in the Rydberg formula, and so we are led to the conclusion in the statement.  $\hfill \Box$ 

In order to investigate heavier atoms, we need to know more about hydrogen. So, let us go back to our study of the Schrödinger equation for it. Our conclusions so far are:

THEOREM 2.27. The wave functions of the hydrogen atom are the following functions, labelled by three quantum numbers, n, l, m,

$$\phi_{nlm}(r,s,t) = \rho_{nl}(r)\alpha_l^m(s,t)$$

where  $\rho_{nl}(r) = p^{l+1}e^{-p}v(p)/r$  with  $p = \alpha r$  as before, with the coefficients of v subject to

$$c_{j+1} = \frac{2(j+l+1-n)}{(j+1)(j+2l+2)} \cdot c_j$$

and  $\alpha_l^m(s,t)$  being the spherical harmonics found before.

**PROOF.** This follows indeed by putting together all the results obtained so far, and with the remark that everything is up to the normalization of the wave function.  $\Box$ 

In what regards the main wave function, that of the ground state, we have:

THEOREM 2.28. With the hydrogen atom in its ground state, the wave function is

$$\phi_{100}(r,s,t) = \frac{1}{\sqrt{\pi a^3}} e^{-r/a}$$

where  $a = 1/\alpha$  is the inverse of the parameter appearing in our computations above,

$$\alpha = \frac{\sqrt{-2mE}}{h}$$

called Bohr radius of the hydrogen atom. This Bohr radius is given by

$$a = \frac{h^2}{mKe^2}$$

which numerically means  $a \simeq 5.291 \times 10^{-11}$ .

**PROOF.** According to our formulae above, the parameter  $\alpha$  there is given by:

$$\alpha = \frac{\sqrt{-2mE}}{h} = \frac{1}{h} \cdot m \cdot \frac{Ke^2}{h} = \frac{mKe^2}{h^2}$$

Thus, the inverse  $\alpha = 1/a$  is indeed given by the formula in the statement. Regarding now the wave function, we know from the above that this consists of:

$$\rho_{10}(r) = \frac{2e^{-r/a}}{\sqrt{a^3}} \quad , \quad \alpha_0^0(s,t) = \frac{1}{2\sqrt{\pi}}$$

By making the product, we obtain the formula of  $\phi_{100}$  in the statement. Finally, in what regards the numerics, these are as follows:

$$a = \frac{1.055^2 \times 10^{-68}}{9.109 \times 10^{-31} \times 8.988 \times 10^9 \times 1.602^2 \times 10^{-38}} = 5.297 \times 10^{-11}$$

Thus, we are led to the conclusions in the statement.

In order to improve our results, we will need the following standard fact:

**PROPOSITION 2.29.** The polynomials v(p) are given by the formula

$$v(p) = L_{n-l-1}^{2l+1}(p)$$

where the polynomials on the right, called associated Laguerre polynomials, are given by

$$L_q^p(x) = (-1)^p \left(\frac{d}{dx}\right)^p L_{p+q}(x)$$

with  $L_{p+q}$  being the Laguerre polynomials, given by the following formula,

$$L_q(x) = \frac{e^x}{q!} \left(\frac{d}{dx}\right)^q \left(e^{-x}x^q\right)$$

called Rodrigues formula for the Laguerre polynomials.

**PROOF.** The story here is very similar to that of the Legendre polynomials. Consider the Hilbert space  $H = L^2[0, \infty)$ , with the following scalar product on it:

$$\langle f,g \rangle = \int_0^\infty f(x)g(x)e^{-x}\,dx$$

The orthogonal basis obtained by applying Gram-Schmidt to the Weierstrass basis  $\{x^q\}$  is then formed by the Laguerre polynomials  $\{L_q\}$ , and this gives the results.  $\Box$ 

With the above result in hand, we can now improve our main results, as follows:

THEOREM 2.30. The wave functions of the hydrogen atom are given by

$$\phi_{nlm}(r,s,t) = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n(n+l)!}} e^{-r/na} \left(\frac{2r}{na}\right)^l L_{n-l-1}^{2l+1} \left(\frac{2r}{na}\right) \alpha_l^m(s,t)$$

with  $\alpha_l^m(s,t)$  being the spherical harmonics found before.

PROOF. This follows indeed by putting together what we have, and then doing some remaining work, concerning the normalization of the wave function.  $\Box$ 

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# 2d. Fine structure

What is next? All sorts of corrections to the solution that we found, due to various phenomena that we neglected in our computations, or rather in our modeling of the problem, which can be both of electric and relativistic nature.

So, let us explain now the standard corrections to the Schrödinger solution to the hydrogen atom. We will focus on energy only, so let us start by recalling:

THEOREM 2.31 (Schrödinger). The energy of the  $\phi_{nlm}$  state of the hydrogen atom is independent on the quantum numbers l, m, given by the Bohr formula

$$E_n = -\frac{\alpha^2}{n^2} \cdot \frac{mc^2}{2}$$

where  $\alpha$  is a dimensionless constant, called fine structure constant, given by

$$\alpha = \frac{Ke^2}{hc}$$

which in practice means  $\alpha \simeq 1/137$ .

**PROOF.** This is the Bohr energy formula that we know, proved by Schrödinger, and reformulated in terms of Sommerfeld's fine structure constant:

(1) We know from Theorem 2.25 that we have the following formula, which can be written as in the statement, by using the fine structure constant  $\alpha$ :

$$E_n = -\frac{m}{2} \left(\frac{Ke^2}{h}\right)^2 \cdot \frac{1}{n^2}$$

(2) Observe now that our modified Bohr formula can be further reformulated as follows, with  $T_c$  being the kinetic energy of the electron traveling at speed c:

$$E_n = -\frac{\alpha^2}{n^2} \cdot T_c$$

Thus  $\alpha^2$ , and so  $\alpha$  too, is dimensionless, as being a quotient of energies.

(3) Let us doublecheck however this latter fact, the check being instructive. With respect to the SI system that we use, the units for K, e, h, c are:

$$U_K = \frac{m^3 \cdot kg}{s^2 \cdot C^2} \quad , \quad U_e = C \quad , \quad U_h = \frac{m^2 \cdot kg}{s} \quad , \quad U_c = \frac{m}{s}$$

Thus the units for the fine structure constant  $\alpha$  are, as claimed:

$$U_{\alpha} = U_C \cdot U_e^2 \cdot U_h^{-1} \cdot U_c^{-1} = \frac{m^3 \cdot kg}{s^2 \cdot C^2} \cdot C^2 \cdot \frac{s}{m^2 \cdot kg} \cdot \frac{s}{m} = 1$$

(4) In what regards now the numerics, these are as follows:

$$\alpha = \frac{Ke^2/h}{c} \simeq \frac{2.186 \times 10^6}{2.998 \times 10^8} = 7.291 \times 10^{-3} \simeq \frac{1}{137}$$

Here we used a standard estimate for  $Ke^2/h$ , from the proof of Theorem 2.25.

The fine structure constant  $\alpha$  is a remarkable quantity, as obvious from the above, and more on it in a moment. Among its other magic features, it manages well  $2\pi$  factors. Indeed, by using  $K = 1/(4\pi\varepsilon_0)$  and  $h = h_0/2\pi$ , we can write this constant as:

$$\alpha = \frac{e^2}{2\varepsilon_0 h_0 c}$$

Finally, let us record the complete official data for  $\alpha$  and its inverse  $\alpha^{-1}$ :

 $\alpha = 0.007 \ 297 \ 352 \ 5693(11)$ 

 $\alpha^{-1} = 137.035\ 999\ 084(21)$ 

As a final comment here, all this lengthy discussion about  $\alpha$  might sound a bit like mania, or mysticism. But wait for it. Sometimes soon  $\alpha$  will be part of your life.

Moving ahead now with corrections to Theorem 2.31, we will be quite brief, and for further details, we refer as usual to our favorite books, Feynman [35], Griffiths [43] and Weinberg [93]. We first have the following result, which is something non-trivial:

THEOREM 2.32. There is a relativistic correction to be made to the Bohr energy  $E_n$  of the state  $\phi_{nlm}$ , depending on the quantum number l, given by

$$\mathcal{E}_{nl} = \frac{\alpha^2 E_n}{n^2} \left( \frac{n}{l+1/2} - \frac{3}{4} \right)$$

coming by replacing the kinetic energy by the relativistic kinetic energy.

**PROOF.** According to Einstein, the relativistic kinetic energy is given by:

$$T = \frac{p^2}{2m} - \frac{p^4}{8m^3c^2} + \dots$$

The Schrödinger equation, based on  $T = p^2/2m$ , must be therefore corrected with a  $\mathcal{T} = -p^4/(8m^3c^2)$  term, and this leads to the above correction term  $\mathcal{E}_{nl}$ .

Equally non-trivial is the following correction, independent from the above one:

THEOREM 2.33. There is a spin-related correction to be made to the Bohr energy  $E_n$  of the state  $\phi_{nlm}$ , depending on the number  $j = l \pm 1/2$ , given by

$$\mathcal{E}_{nj} = -\frac{\alpha^2 E_n}{n^2} \cdot \frac{n(j-l)}{(l+1/2)(j+1/2)}$$

coming from the torque of the proton on the magnetic moment of the electron.

PROOF. As we will explain later, the electron has a spin  $\pm 1/2$ , which is naturally associated to the quantum number l, leading to the parameter  $j = l \pm 1/2$ . But, knowing now that the electron has a spin, the proton which moves around it certainly acts on its magnetic moment, and this leads to the above correction term  $\mathcal{E}_{nj}$ .

So, these are the first two corrections to be made, and again, we refer to Feynman [35], Griffiths [43], Weinberg [93] for details. Obviously we don't quite know what we're doing here, but let us add now the above corrections to  $E_n$ , and see what we get. We obtain in this way one of the most famous formulae in quantum mechanics, namely:

THEOREM 2.34. The energy levels of the hydrogen atom, taking into account the fine structure coming from the relativistic and spin-related correction, are given by

$$E_{nj} = E_n \left[ 1 + \frac{\alpha^2}{n^2} \left( \frac{n}{j+1/2} - \frac{3}{4} \right) \right]$$

with  $j = l \pm 1/2$  being as above, and with  $\alpha$  being the fine structure constant.

**PROOF.** We have the following computation, based on the above formulae:

$$\mathcal{E}_{nl} + \mathcal{E}_{nj} = \frac{\alpha^2 E_n}{n^2} \left( \frac{n}{l+1/2} - \frac{3}{4} - \frac{n(j-l)}{(l+1/2)(j+1/2)} \right)$$
$$= \frac{\alpha^2 E_n}{n^2} \left( \frac{n}{l+1/2} \left( 1 - \frac{j-l}{j+1/2} \right) - \frac{3}{4} \right)$$
$$= \frac{\alpha^2 E_n}{n^2} \left( \frac{n}{j+1/2} - \frac{3}{4} \right)$$

Thus the corrected formula of the energy is as follows:

$$E_{nj} = E_n + \mathcal{E}_{nl} + \mathcal{E}_{nj}$$
$$= E_n + \frac{\alpha^2 E_n}{n^2} \left( \frac{n}{j+1/2} - \frac{3}{4} \right)$$

We are therefore led to the conclusion in the statement.

Summarizing, quantum mechanics is more complicated than what originally appears from Schrödinger's solution of the hydrogen atom. Which was something quite complicated too, we must admit that. And the story is not over here, because on top of the above fine structure correction, which is of order  $\alpha^2$ , we have afterwards the Lamb shift, which is an order  $\alpha^3$  correction, then the hyperfine splitting, and more.

Quite remarkably, Feynman and others managed to find a global way of viewing all the phenomena that can appear, corresponding to an infinite series in  $\alpha$ . To be more precise, their theory, called quantum electrodynamics (QED), is an advanced version of quantum mechanics, still used nowadays for any delicate computation.

# 2e. Exercises

Exercises:

EXERCISE 2.35.

EXERCISE 2.36.

EXERCISE 2.37.

EXERCISE 2.38.

EXERCISE 2.39.

EXERCISE 2.40.

Bonus exercise.

# CHAPTER 3

# **Dirac** equation

# 3a. Dirac equation

We have seen in previous chapter that quantum mechanics provides explanations and equations for all the basic phenomena appearing at the atomic level. Among others, we have reached to a quite decent level of understanding of the hydrogen atom.

In the remainder of this Part I of the present book we will be interested in rather abstract aspects, and more specifically in "fixing quantum mechanics". And by this we mean not that quantum mechanics is wrong, but that there are certainly a few things that we came upon, which are not very clear, and need to be fixed, as follows:

(1) We would like our theory to be relativistic. Among others, for getting rid of the "relativistic correction" to the hydrogen atom, a correction never being a good thing.

(2) In fact, we would like to have a conceptual understanding of the spin correction too, as to get rid of the whole "fine structure correction" to the hydrogen atom.

(3) We would like our electrons to be joined by more particles, with the minimum here including the protons, the neutrons, and also the photons, representing light.

(4) And then, why not looking too into phenomena that we have not investigated yet, such as radioactivity. Or splitting protons and neutrons into smaller particles.

We will discuss here all these questions. Quite remarkably, there is a common mathematical framework for investigating all these questions, called quantum field theory (QFT). So, we will develop QFT, and then we will turn to questions (1,2) above, and present an amazing answer to them, involving a QFT called quantum electrodynamics (QED). Then we will turn also to questions (3,4), and discuss a bit the status here, notably with a few words on quantum chromodynamics (QCD), which is the quantum field theory obtained by splitting protons and neutrons into smaller particles.

Before starting, let us mention that things won't be easy. Our present level in quantum mechanics, now at this page 50 of the present book, corresponds more or less to things

known since the 1920s. We will of course make big efforts for understanding what happened in the 1930s, then 1940s, then 1950s and so on, but so many things that happened, and the remainder to this Part I will be just a modest introduction to all this.

Getting started, let us formulate a clear objective:

OBJECTIVE 3.1. We would like to have a relativistic version of quantum mechanics, and with the electron being joined by the photon, representing light. If possible, we would like our theory to cover as well the proton, and the neutron.

Here the relativistic requirement is very natural in regards with all that has been said above, this being certainly the gate towards a better quantum mechanics.

Regarding the other particles, intuition and common sense would dictate to go first towards the proton and neutron, because aren't these, along with the electron, the constituents of normal matter, that we are normally interested in. However, and here comes our point, mathematically speaking, the electron can certainly live without protons and neutrons, because in order to move, it just needs a positive charge attracting it, and this positive charge can be well something abstract, as per general field theory philosophy.

In contrast, however, the electron cannot live without the photon. The point is that in the context of the basic physics of atoms, electrons can jump between energy levels, emitting or absorbing photons, and with this being known to happen even in the absence of external stimuli. Thus, and for concluding, the true "brother" of the electron is not the proton or the neutron, but rather the photon. And so, the minimal extension of quantum mechanics that we are trying to build should deal with electrons and photons.

Let us first look into the photon, try to understand how to make it fit into our theory, and leave the electron for later. As a starting point, we have:

FACT 3.2. The master equation for free electromagnetic radiation, that is, for free photons, is the wave equation at speed v = c, namely:

$$\ddot{\varphi} = c^2 \Delta \varphi$$

This equation can be reformulated in the more symmetric form

$$\left(\frac{1}{c^2} \cdot \frac{d^2}{dt^2} - \Delta\right)\varphi = 0$$

with the operator on the left being called the d'Alembertian.

To be more precise here, these are things that we know well, from chapter 1, and from chapter 2, when first talking about the wave equation, and radiation. In addition, and importantly, we also know from there that the wave equation, at any speed v, is

relativistic, in the sense that it is invariant under Lorentz transformations, which are as follows, with  $\gamma = 1/\sqrt{1 - v^2/c^2}$  being as usual the Lorentz factor:

$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma(t - vx/c^{2})$$

So far, so good. In relation now with the electron, there is an obvious similarity here with the free Schrödinger equation, without potential V, which reads:

$$\left(i\frac{d}{dt} + \frac{h}{2m}\,\Delta\right)\psi = 0$$

This similarity suggests looking for a relativistic version of the Schrödinger equation, which is compatible with the wave equation at v = c. And coming up with such an equation is not very complicated, the straightforward answer being as follows:

**DEFINITION 3.3.** The following abstract mathematical equation,

$$\left(-\frac{1}{c^2} \cdot \frac{d^2}{dt^2} + \Delta\right)\psi = \frac{m^2c^2}{h^2}\psi$$

on a function  $\psi = \psi_t(x)$ , is called the Klein-Gordon equation.

To be more precise, what we have here is some sort of a speculatory equation, formally obtained from the Schrödinger equation, via a few simple manipulations, as to make it relativistic. And with the relation with photons being something very simple, the thing being that at zero mass, m = 0, we obtain precisely the wave equation at v = c.

All this is very nice, looks like we have a beginning of theory here, both making the electrons relativistic, and unifying them with photons. And isn't this too beautiful to be true. Going ahead now with physics, the following question appears:

# QUESTION 3.4. What does the Klein-Gordon equation really describe?

And here, unfortunately, bad news all the way. A closer look at the Klein-Gordon equation reveals all sorts of bugs, making it unusable for anything reasonable. And with the main bug, which is enough for disqualifying it, being that, unlike the Schrödinger equation which preserves probability amplitudes  $|\psi|^2$ , the Klein-Gordon equation does not have this property. Thus, even before trying to understand what the Klein-Gordon equation really describes, we are left with the conclusion that this equation cannot really describe anything reasonable, due to the formal nature of the function  $\psi$  involved.

So, this was for the story of the Klein-Gordon equation. Actually this equation was first discovered by Schrödinger himself, in the context of his original work on the Schrödinger equation. But noticing the above bugs with it, Schrödinger dismissed it right

way, and then downgraded his objectives, looking for something non-relativistic instead, and then found the Schrödinger equation, leading to the story that we know.

This being said, the Klein-Gordon equation found later a number of interesting applications, the continuation of the story being as follows:

(1) Dirac found a clever way of extracting the "square root" of the Klein-Gordon equation. And this square root equation, called Dirac equation, turned out to be the correct one, making exactly what the Klein-Gordon equation was supposed to do.

(2) Technically speaking, the Klein-Gordon equation is very useful for investigating the Dirac equation, because the components of the solutions of the Dirac equation satisfy the Klein-Gordon equation. More on this later, when discussing the Dirac equation.

(3) Finally, the Klein-Gordon equation was later recognized to describe well the spin 0 particles. But with these particles being something specialized, including the unstable and sowewhat fringe "pions", and the Higgs boson, which is something complicated.

We will discuss all this, in what follows. In any case, we have here a beginning of good discussion, with our cocktail of thoughts and ideas including electrons, photons, relativity and spin, which are exactly the things that we wanted to include in our discussion. So, all that is left is to clarify all this, and we will do so, following Dirac. Dirac came upon the idea of extracting the square root of the Klein-Gordon operator, as follows:

**PROPOSITION 3.5.** We can extract the square root of the Klein-Gordon operator, via a formula as follows,

$$-\frac{1}{c^2} \cdot \frac{d^2}{dt^2} + \Delta = \left(\frac{i}{c} \cdot \frac{Pd}{dt} + \frac{Qd}{dx} + \frac{Rd}{dy} + \frac{Sd}{dz}\right)^2$$

by using matrices P, Q, R, S which anticommute, AB = -BA, and whose squares equal one,  $A^2 = 1$ .

**PROOF.** We have the following computation, valid for any matrices P, Q, R, S, with the notation  $\{A, B\} = AB + BA$ :

$$\begin{pmatrix} \frac{i}{c} \cdot \frac{Pd}{dt} + \frac{Qd}{dx} + \frac{Rd}{dy} + \frac{Sd}{dz} \end{pmatrix}^2 = -\frac{1}{c^2} \cdot \frac{P^2 d^2}{dt^2} + \frac{Q^2 d^2}{dx^2} + \frac{R^2 d^2}{dy^2} + \frac{S^2 d^2}{dz^2} \\ + \frac{i}{c} \left( \frac{\{P, Q\}d^2}{dtdx} + \frac{\{P, R\}d^2}{dtdy} + \frac{\{P, S\}d^2}{dtdz} \right) \\ + \frac{\{Q, R\}d^2}{dxdy} + \frac{\{Q, S\}d^2}{dxdz} + \frac{\{R, S\}d^2}{dydz}$$

Thus, in order to obtain in this way the Klein-Gordon operator, the conditions in the statement must be satisfied.  $\hfill \Box$ 

As a technical comment here, normally when extracting a square root, we should look for a hermitian operator. In view of this, observe that we have:

$$\left(\frac{i}{c} \cdot \frac{Pd}{dt} + \frac{Qd}{dx} + \frac{Rd}{dy} + \frac{Sd}{dz}\right)^* = -\frac{i}{c} \cdot \frac{P^*d}{dt} + \frac{Q^*d}{dx} + \frac{R^*d}{dy} + \frac{S^*d}{dz}$$

Thus, we should normally add the conditions  $P^* = -P$  and  $Q^* = Q$ ,  $R^* = R$ ,  $S^* = S$  to those above. But, the thing is that due to some subtle reasons, the natural square root of the Klein-Gordon operator is not hermitian. More on this later.

Looking for matrices P, Q, R, S as above is not exactly trivial, and the simplest solutions appear in  $M_4(\mathbb{C})$ , in connection with the Pauli matrices, as follows:

**PROPOSITION 3.6.** The simplest matrices P, Q, R, S as above appear as

$$P = \gamma_0$$
 ,  $Q = i\gamma_1$  ,  $R = i\gamma_2$  ,  $S = i\gamma_3$ 

with  $\gamma_0, \gamma_1, \gamma_2, \gamma_3$  being the Dirac matrices, given by

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

where  $\sigma_1, \sigma_2, \sigma_3$  are the Pauli spin matrices, given by:

$$\sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
,  $\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $\sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ 

**PROOF.** We have  $\gamma_0^2 = 1$ , and by using  $\sigma_i^2 = 1$  for any i = 1, 2, 3, we have as well the following formula, which shows that we have  $(i\gamma_i)^2 = 1$ , as needed:

$$\gamma_i^2 = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

As in what regards the commutators, we first have, for any i = 1, 2, 3, the following equalities, which show that  $\gamma_0$  anticommutes indeed with  $\gamma_i$ :

$$\gamma_0 \gamma_i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$
$$\gamma_i \gamma_0 = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

Regarding now the remaining commutators, observe here that we have:

$$\gamma_i \gamma_j = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_i \sigma_j & 0 \\ 0 & -\sigma_i \sigma_j \end{pmatrix}$$

Now since the Pauli matrices anticommute, we obtain  $\gamma_i \gamma_j = -\gamma_j \gamma_i$ , as desired.  $\Box$ We can now put everything together, and we obtain:

THEOREM 3.7. The following operator, called Dirac operator,

$$D = i\left(\frac{\gamma_0 d}{cdt} + \frac{\gamma_1 d}{dx} + \frac{\gamma_2 d}{dy} + \frac{\gamma_3 d}{dz}\right)$$

has the property that its square is the Klein-Gordon operator.

**PROOF.** With notations from Proposition 3.5 and Proposition 3.6, and by making the choices in Proposition 3.6, we have:

$$\frac{i}{c} \cdot \frac{Pd}{dt} + \frac{Qd}{dx} + \frac{Rd}{dy} + \frac{Sd}{dz} = \frac{i}{c} \cdot \frac{\gamma_0 d}{dt} + \frac{i\gamma_1 d}{dx} + \frac{i\gamma_2 d}{dy} + \frac{i\gamma_3 d}{dz}$$
$$= i\left(\frac{\gamma_0 d}{cdt} + \frac{\gamma_1 d}{dx} + \frac{\gamma_2 d}{dy} + \frac{\gamma_3 d}{dz}\right)$$

Thus, we have here a square root of the Klein-Gordon operator, as desired.

We can now extract the square root of the Klein-Gordon equation, as follows:

DEFINITION 3.8. We have the following equation, called Dirac equation,

$$ih\left(\frac{\gamma_0 d}{cdt} + \frac{\gamma_1 d}{dx} + \frac{\gamma_2 d}{dy} + \frac{\gamma_3 d}{dz}\right)\psi = mc\psi$$

obtained by extracting the square root of the Klein-Gordon equation.

As usual with such theoretical physics equations, extreme caution is recommended, at least to start with. We will slowly examine this equation, in what follows, and the good news will be that, passed a few difficulties, this will turn to be a true, magic equation.

As a first observation, all this is very related to spin. In fact, as we will see later, the Dirac equation is the correct relativistic equation describing the spin 1/2 particles.

The Dirac equation comes with a price to pay, which is that of opening Pandora's box of particles. To be more precise, once we adopt this equation, we must surely adopt all its free solutions. And bad news here, the solution which is complementary to the electron is not the proton, but rather a weird new particle, called the positron. In order to explain all this, which is something quite tricky, let us start with the following observation:

**PROPOSITION 3.9.** For a particle at rest, meaning under the assumption

$$\frac{d\psi}{dx} = \frac{d\psi}{dy} = \frac{d\psi}{dz} = 0$$

the Dirac equation takes the form

$$\frac{ih}{c} \cdot \gamma_0 \cdot \frac{d\psi}{dt} = mc\psi$$

with  $\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  being as usual the first Dirac matrix.

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**PROOF.** Consider indeed the Dirac equation, as formulated in Definition 3.8:

$$ih\left(\frac{\gamma_0 d}{cdt} + \frac{\gamma_1 d}{dx} + \frac{\gamma_2 d}{dy} + \frac{\gamma_3 d}{dz}\right)\psi = mc\psi$$

With the above rest assumption, we are led to the equation in the statement. The above equation at rest is very easy to solve, the result being as follows:

THEOREM 3.10. The solutions of the Dirac equation for particles at rest are

$$\psi = \begin{pmatrix} e^{-imc^2t/h}\xi\\ e^{imc^2t/h}\eta \end{pmatrix}$$

with  $\xi, \eta \in \mathbb{R}^2$  being arbitrary vectors.

**PROOF.** In order to solve the Dirac equation in Proposition 3.9, let us write:

$$\psi = \begin{pmatrix} \varphi \\ \phi \end{pmatrix}$$

With this notation, the Dirac equation at rest takes the following form:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} d\varphi/dt \\ d\phi/dt \end{pmatrix} = -\frac{imc^2}{h} \begin{pmatrix} \varphi \\ \phi \end{pmatrix}$$

Now by looking at the components, the equations are as follows:

$$\frac{d\varphi}{dt} = -\frac{imc^2}{h}\varphi \quad , \quad \frac{d\phi}{dt} = \frac{imc^2}{h}\phi$$

But the solutions of these latter equations are as follows, with  $\xi, \eta \in \mathbb{R}^2$ :

$$\varphi = e^{-imc^2t/h}\xi \quad , \quad \phi = e^{imc^2t/h}\eta$$

Thus, we are led to the conclusion in the statement.

The question is now, is the above result good news or not? Not really, because in view of what we know from quantum mechanics, an  $e^{-iEt/h}$  factor should correspond to the time dependence of a quantum state with energy E, which at rest is  $E = mc^2$ . And from this perspective, while the above  $\varphi$  functions look very good, the other components, the  $\phi$  functions, look bad, seemingly coming from particles having "negative energy".

So, what to do? In order to avoid particles with negative energy, which is something that definitely looks very bad, the solution is that of talking about antiparticles with positive energy, and to formulate, as a continuation of Theorem 3.10:

THEOREM 3.11. The basic solutions of the Dirac equation for particles at rest are

$$\psi^{1} = e^{-imc^{2}t/h} \begin{pmatrix} 1\\0\\0\\0 \end{pmatrix} , \qquad \psi^{2} = e^{-imc^{2}t/h} \begin{pmatrix} 0\\1\\0\\0 \end{pmatrix}$$

corresponding to the electron with spin up, and spin down, plus

$$\psi^{3} = e^{imc^{2}t/h} \begin{pmatrix} 0\\0\\1\\0 \end{pmatrix} , \qquad \psi^{4} = e^{imc^{2}t/h} \begin{pmatrix} 0\\0\\0\\1 \end{pmatrix}$$

corresponding to a new particle, the positron, with spin up, and spin down.

PROOF. Here the mathematics comes from what we found in Theorem 3.10, and the terminology and philosophy comes from the above discussion. With the remark that the newly introduced positron is rather an antiparticle, but more on this later.  $\Box$ 

Not very good, all this. Dirac himself could not believe it, and it took some joint effort of Weyl, Pauli, Oppenheimer and others to convince him that yes, unfortunately the positrons predicted by his equation are not the usual protons. And so that goodbye reasonable physics, goodbye common sense, and welcome positrons.

In what concerns us, we have been extremely reluctant, throughout this book, to talk about new particles, but no choice now, we will have to back up, and adopt the positrons. But, passed this, we will slam down the cover of Pandora's box, right away. We definitely don't want all sorts of fringe, short-lived particles to invade our theory, and multiply like mushrooms, and transform our carefully built theory into something apocalyptic.

Be said in passing, after some thinking, positrons are not that bad, as particles. If there's one sort of bad particles in this life, these are the short-lived ones, which appear as some sort of "mathematical complications", which do not really exist in the real, statistical life, which takes place over substantial time t > 0. And positrons are not like this, they are nice and stable, exactly as the electrons. Their only fault is that of not being very frequent, a positron's fate being that of being quickly eaten by an electron passing by. But to be blamed for this lack of symmetry is not quantum mechanics, but rather the mechanism of the Big Bang, and once we're fine with this, we're fine with positrons.

More about positrons later, when talking about Feynman diagrams and QED. We will see at that time that positrons are in fact something very natural, and we will get to know and love them on the same level as the usual electrons.

Moving now forward, let us attempt to solve the following question:

QUESTION 3.12. What are the plane wave solutions

$$\psi(s) = ae^{-i < k, s > u}$$

of the Dirac equation?

To be more precise, we are using here, as argument of the function  $\psi$ , the standard relativistic space-time position  $s \in \mathbb{R}^4$  of our particle, namely:

$$s = \begin{pmatrix} ct \\ r \end{pmatrix}$$
,  $r = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ 

Next, we have in the above a constant  $a \in \mathbb{R}$ , which will be quite irrelevant to our computations, the Dirac equation being linear. Regarding now k, it is convenient to write this vector split over components, as we did in the above with s, as follows:

$$k = \begin{pmatrix} f \\ g \end{pmatrix} \qquad , \qquad g = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$$

With these conventions, along with the standard relativistic convention that the space coordinates contribute with - signs, the scalar product in Question 3.12 is given by:

$$< k, s >= cft - < g, r >$$

Now observe that the real part of the exponential in Question 3.12 is given by:

$$Re(e^{-i < k, s >}) = \cos\left(cft - < g, r >\right)$$

Thus, what we have here, justifying the terminology, is a sinusoidal wave propagating in the direction g, with angular frequency and wavelenght as follows:

$$\omega = cf \quad , \quad \lambda = 2\pi/||g||$$

In order to answer Question 3.12, we must first plug into the Dirac equation our special function  $\psi$ . We are led in this way to a quite simple equation, as follows:

**PROPOSITION 3.13.** The Dirac equation for plane wave functions

$$\psi(s) = ae^{-i \langle k, s \rangle} u$$

takes the following special form, no longer involving derivatives,

$$h(\gamma_0 f - \gamma_1 g_1 - \gamma_2 g_2 - \gamma_3 g_3)u = mcu$$

with the above conventions for indices and vectors.

**PROOF.** Consider indeed the Dirac equation, as formulated in Definition 3.8:

$$ih\left(\frac{\gamma_0 d}{c dt} + \frac{\gamma_1 d}{dx} + \frac{\gamma_2 d}{dy} + \frac{\gamma_3 d}{dz}\right)\psi = mc\psi$$

For the function  $\psi$  in the statement, the derivatives are given by:

$$\frac{d\psi}{ds_i} = -ik_i\psi$$

Thus, with our above conventions for indices and vectors, we have:

$$rac{d\psi}{cdt}=-if\psi$$
 ,  $rac{d\psi}{dr_i}=ig_i\psi$ 

By plugging these quantities in the Dirac equation, this equation becomes:

$$h(\gamma_0 f - \gamma_1 g_1 - \gamma_2 g_2 - \gamma_3 g_3)\psi = mc\psi$$

Now by using again  $\psi = ae^{-i \langle k, s \rangle}u$ , this equation takes the following form:

$$h(\gamma_0 f - \gamma_1 g_1 - \gamma_2 g_2 - \gamma_3 g_3)ae^{-i < k, s > u} = mcae^{-i < k, s > u}$$

Thus, by simplifying, we are led to the equation in the statement.

Let us study now the equation that we found. As a first observation, we can further fine-tune the equation in Proposition 3.13, via some simple manipulations, as follows:

**PROPOSITION 3.14.** In the context of Proposition 3.13, with the notation

$$u = \begin{pmatrix} v \\ w \end{pmatrix}$$

the Dirac equation takes the following form, in terms of the components v, w,

$$v = \frac{\langle g, \sigma \rangle}{f - mc/h} w$$
 ,  $w = \frac{\langle g, \sigma \rangle}{f + mc/h} v$ 

where  $\sigma_1, \sigma_2, \sigma_3$  stand as usual for the Pauli spin matrices.

**PROOF.** According to the definition of the Dirac matrices, in terms of the Pauli ones, we have the following computation, for the operator appearing in Proposition 3.13:

$$\begin{aligned} \gamma_0 f - \gamma_1 g_1 - \gamma_2 g_2 - \gamma_3 g_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} f - \sum_{i=1}^3 \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} g_i \\ &= \begin{pmatrix} f & 0 \\ 0 & -f \end{pmatrix} - \begin{pmatrix} 0 & \langle g, \sigma \rangle \\ -\langle g, \sigma \rangle & 0 \end{pmatrix} \\ &= \begin{pmatrix} f & -\langle g, \sigma \rangle \\ \langle g, \sigma \rangle & -f \end{pmatrix} \end{aligned}$$

Thus, the quantity which must vanish in Proposition 3.13 is given by:

$$\begin{pmatrix} h(\gamma_0 f - \gamma_1 g_1 - \gamma_2 g_2 - \gamma_3 g_3) - mc \end{pmatrix} u$$

$$= \begin{pmatrix} hf - mc & -h < g, \sigma > \\ h < g, \sigma > & -hf - mc \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}$$

$$= \begin{pmatrix} (hf - mc)v - h < g, \sigma > w \\ h < g, \sigma > v - (hf + mc)w \end{pmatrix}$$

We therefore conclude that, in our case, the Dirac equation reads:

$$(hf - mc)v = h < g, \sigma > w$$
$$h < g, \sigma > v = (hf + mc)w$$

Thus, we are led to the conclusion in the statement.

In order to solve now our equation, let us make the following observation:

**PROPOSITION 3.15.** In the context of Proposition 3.14 we must have

$$||g||^2 = f^2 - \left(\frac{mc}{h}\right)^2$$

under the assumption that the solution is nonzero,  $u \neq 0$ .

**PROOF.** Consider the equations found in Proposition 3.14, namely:

$$v = \frac{\langle g, \sigma \rangle}{f - mc/h} w$$
 ,  $w = \frac{\langle g, \sigma \rangle}{f + mc/h} v$ 

By substituting, we are led to the following formulae:

$$v = \frac{\langle g, \sigma \rangle^2}{f^2 - (mc/h)^2} v$$
 ,  $w = \frac{\langle g, \sigma \rangle^2}{f^2 - (mc/h)^2} w$ 

Thus, assuming that the solution is nonzero,  $u \neq 0$ , we must have:

$$\frac{\langle g, \sigma \rangle^2}{f^2 - (mc/h)^2} = \begin{pmatrix} 1 & 0\\ 0 & 1 \end{pmatrix}$$

Now, let us compute the left term. According to our various conventions above, and to the formulae for the Pauli matrices, we have the following formula:

$$\langle g, \sigma \rangle = g_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + g_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + g_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} g_3 & g_1 - ig_2 \\ g_1 + ig_2 & -g_3 \end{pmatrix}$$

By raising this quantity to the square, we obtain:

$$\langle g, \sigma \rangle^{2} = \begin{pmatrix} g_{3} & g_{1} - ig_{2} \\ g_{1} + ig_{2} & -g_{3} \end{pmatrix} \begin{pmatrix} g_{3} & g_{1} - ig_{2} \\ g_{1} + ig_{2} & -g_{3} \end{pmatrix}$$

$$= \begin{pmatrix} g_{3}^{2} + (g_{1} - ig_{2})(g_{1} + ig_{2}) & g_{3}(g_{1} - ig_{2}) - (g_{1} - ig_{2})g_{3} \\ (g_{1} + ig_{2})g_{3} - g_{3}(g_{1} + ig_{2}) & (g_{1} + ig_{2})(g_{1} - ig_{2}) + g_{3}^{2} \end{pmatrix}$$

$$= \begin{pmatrix} g_{1}^{2} + g_{2}^{2} + g_{3}^{2} & 0 \\ 0 & g_{1}^{2} + g_{2}^{2} + g_{3}^{2} \end{pmatrix}$$

$$= ||g||^{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, the condition that we found above, coming from  $u \neq 0$ , reads:

$$\frac{||g||^2}{f^2 - (mc/h)^2} \begin{pmatrix} 1 & 0\\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0\\ 0 & 1 \end{pmatrix}$$

We conclude that we must have the following equality:

$$||g||^2 = f^2 - \left(\frac{mc}{h}\right)^2$$

Thus, we are led to the conclusion in the statement.

The point now is that the above result invites us to use the rescaled energy-momentum four-vector as variable,  $k = \pm p/h$ , and we are led in this way to the following result:

THEOREM 3.16. The basic plane wave solutions, of type

$$\psi(s) = ae^{-i \langle k, s \rangle} u$$

of the Dirac equation, come from the functions

$$u^{1} = \frac{1}{E + mc^{2}} \begin{pmatrix} E + mc^{2} \\ 0 \\ cp_{z} \\ cp_{x} + icp_{y} \end{pmatrix} , \qquad u^{2} = \frac{1}{E + mc^{2}} \begin{pmatrix} 0 \\ E + mc^{2} \\ cp_{x} - icp_{y} \\ -cp_{z} \end{pmatrix}$$

corresponding to particle solutions, plus from the functions

$$u^{3} = \frac{1}{E + mc^{2}} \begin{pmatrix} cp_{z} \\ cp_{x} + icp_{y} \\ E + mc^{2} \\ 0 \end{pmatrix} , \qquad u^{4} = \frac{1}{E + mc^{2}} \begin{pmatrix} cp_{x} - icp_{y} \\ -cp_{z} \\ 0 \\ E + mc^{2} \end{pmatrix}$$

corresponding to antiparticle solutions.

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**PROOF.** This comes by putting together all the above. Indeed, with  $k = \pm p/h$ , as suggested above, we have four choices, which are as follows:

$$v = \begin{pmatrix} 1\\0 \end{pmatrix} , \quad w = \frac{c}{E + mc^2} \begin{pmatrix} p_z\\p_x + ip_y \end{pmatrix}$$
$$v = \begin{pmatrix} 0\\1 \end{pmatrix} , \quad w = \frac{c}{E + mc^2} \begin{pmatrix} p_x - ip_y\\-p_z \end{pmatrix}$$
$$w = \begin{pmatrix} 1\\0 \end{pmatrix} , \quad v = \frac{c}{E + mc^2} \begin{pmatrix} p_z\\p_x + ip_y \end{pmatrix}$$
$$w = \begin{pmatrix} 0\\1 \end{pmatrix} , \quad v = \frac{c}{E + mc^2} \begin{pmatrix} p_z\\p_x - ip_y\\-p_z \end{pmatrix}$$

Thus, we are led to the solutions in the statement.

Regarding the exact physical interpretation of the above plane wave solutions that we found, this is something quite tricky, and we will discuss this later.

In any case, we have now in our theory the electron accompanied by the positron and the photon. There are in fact many other particles which satisfy the Dirac equation, with this equation being in fact the one which describes the spin 1/2 particles. More on this later, when we will know more about the various particles that can appear.

As a last topic, from this preliminary discussion on the Dirac equation, let us discuss now the normalization of the solutions that we found above. We will need:

**PROPOSITION 3.17.** For the basic plane wave solutions found above, we have

$$||u||^2 = \frac{2E}{E + mc^2}$$

with the norm being computed with respect to the usual complex scalar product.

**PROOF.** According to our formulae above, for  $u = u^1, u^2, u^3, u^4$  we have:

$$||u||^{2} = \frac{1}{(E+mc^{2})^{2}} \left( (E+mc^{2})^{2} + c^{2}(p_{x}^{2} + p_{y}^{2} + p_{z}^{2}) \right)$$
$$= \frac{1}{(E+mc^{2})^{2}} \left( (E+mc^{2})^{2} + c^{2}||p||^{2} \right)$$

Now recall that for the energy-momentum vector  $\tilde{p} = (E/c, p)$  we have  $||\tilde{p}|| = mc$ . Thus, the norm of the momentum vector component is given by:

$$||p||^{2} = \left(\frac{E}{c}\right)^{2} - ||\tilde{p}||^{2} = \frac{E^{2}}{c^{2}} - m^{2}c^{2}$$

With this formula in hand, we can finish our computation, as follows:

$$||u||^{2} = \frac{1}{(E+mc^{2})^{2}} \left( (E+mc^{2})^{2} + c^{2} \left( \frac{E^{2}}{c^{2}} - m^{2}c^{2} \right) \right)$$
  
$$= \frac{1}{(E+mc^{2})^{2}} \left( E^{2} + m^{2}c^{4} + 2Emc^{2} + E^{2} - m^{2}c^{4} \right)$$
  
$$= \frac{1}{(E+mc^{2})^{2}} \left( 2E^{2} + 2Emc^{2} \right)$$
  
$$= \frac{2E}{E+mc^{2}}$$

Thus, we are led to the conclusion in the statement.

In what regards now the normalization of the solutions u found in Theorem 3.16, there are several possible useful conventions here, as follows:

$$||Nu||^2 = \frac{2E}{c}$$
 ,  $||Nu||^2 = \frac{E}{mc^2}$  ,  $||Nu||^2 = 1$ 

The corresponding normalizations constants N can be computed by using Proposition 3.17, and are respectively given by the following formulae:

$$N = \sqrt{\frac{E + mc^2}{c}} \quad , \quad N = \sqrt{\frac{E + mc^2}{2mc^2}} \quad , \quad N = \sqrt{\frac{E + mc^2}{2E}}$$

As before with the exact physical interpretation of the plane wave solutions that we found, their normalization is also something quite tricky, and we will discuss this later.

Let us discuss now invariance questions for the solutions of the Dirac equation. As already mentioned in the above, this equation was meant to be a relativistic version of the Schrödinger equation, but the fact that this equation is indeed relativistic, from the point of view of the invariance of solutions, is still something that we must establish.

We recall that the relativistic frame change, with respect to moving with speed v along Ox, is given by the following formulae, where  $\beta = v/c$  and  $\gamma = 1/\sqrt{1-\beta^2}$ :

$$ct' = \gamma(ct - \beta x)$$
$$x' = \gamma(x - \beta ct)$$
$$y' = y$$
$$z' = z$$

Equivalently, in matrix form, we have the following formula:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

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Regarding the reverse frame change, this is obtained via  $v \to -v$ , which gives the following formulae, with  $\beta = v/c$  and  $\gamma = 1/\sqrt{1-\beta^2}$  as before:

$$ct = \gamma(ct' + \beta x')$$
$$x = \gamma(x' + \beta ct')$$
$$y = y'$$
$$z = z'$$

Equivalently, in matrix form, we have the following formula:

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

We refer to the above for more on these formulae, and also for a proof of the fact that the Maxwell equations are indeed invariant under these transformations.

In what regards now the Dirac equation, we have the following result:

THEOREM 3.18. A solution  $\psi$  of the Dirac equation leads, infinitesimally, to the following solution of the same equation, with respect to a frame change as above,

 $\psi' = A\psi$ 

with the matrix A being given by the following formula,

$$A = \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ b & 0 & 0 & a \end{pmatrix}$$

where the parameters are given by the following formulae,

$$a = \sqrt{\frac{\gamma+1}{2}}$$
,  $b = -\sqrt{\frac{\gamma-1}{2}}$ 

with  $\gamma = 1/\sqrt{1 - v^2/c^2}$  being the Lorentz factor.

**PROOF.** This is something quite tricky, the idea being as follows:

(1) Consider indeed the Dirac equation, as formulated in Definition 3.8:

$$ih\left(\frac{\gamma_0 d}{c dt} + \frac{\gamma_1 d}{dx} + \frac{\gamma_2 d}{dy} + \frac{\gamma_3 d}{dz}\right)\psi = mc\psi$$

It is convenient to use the relativistic space-time position vector, given by:

$$s = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

With this convention, the Dirac equation, as formulated above, becomes:

$$ih\sum_{i=0}^{3}\gamma_{i}\frac{d\psi}{ds_{i}}=mc\psi$$

(2) Now let us write as well this equation in the new frame, as follows:

$$ih\sum_{i=0}^{3}\gamma_{i}\frac{d\psi'}{ds'_{i}} = mc\psi'$$

We can compute the derivation operators  $d/ds'_i$  in terms of the original derivation operators  $d/ds_i$  by using the chain rule, starting from:

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

Indeed, if we denote by  $L^{-1}$  the 4 × 4 matrix appearing above, that of the reverse frame change, then the above formula reads, in terms of space-time position vectors:

$$s = L^{-1}s'$$

Now by using the chain rule, we obtain from this the following formula:

$$\frac{d}{ds'_i} = \sum_j \frac{ds_j}{ds'_i} \cdot \frac{d}{ds_j}$$

$$= \sum_j \frac{d(L^{-1}s')_j}{ds'_i} \cdot \frac{d}{ds_j}$$

$$= \sum_{jk} \frac{d((L^{-1})_{jk}s'_k)}{ds'_i} \cdot \frac{d}{ds_j}$$

$$= \sum_{jk} (L^{-1})_{jk} \frac{ds'_k}{ds'_i} \cdot \frac{d}{ds_j}$$

$$= \sum_j (L^{-1})_{ji} \frac{d}{ds_j}$$

$$= \sum_j (L^{-1})_{ij} \frac{d}{ds_j}$$

Here we have used at the end the fact that  $L^{-1}$  is symmetric. In vector notation now, the conclusion is that we have the following formula:

$$\frac{d}{ds'} = L^{-1} \frac{d}{ds}$$

(3) With this formula in hand, let us go back to the Dirac equation in the new frame, and try to find a solution of type  $\psi' = A\psi$  for it. Our equation reads:

$$ih\sum_{i=0}^{3}\gamma_{i}\frac{dA\psi}{ds_{i}'}=mcA\psi$$

By using the linearity of the derivatives, and then the formula found in (2), the left term of this new Dirac equation is given by the following formula:

$$ih\sum_{i=0}^{3}\gamma_{i}\frac{dA\psi}{ds'_{i}} = ih\sum_{i=0}^{3}\gamma_{i}A\frac{d\psi}{ds'_{i}}$$
$$= ih\sum_{i=0}^{3}\gamma_{i}AL^{-1}\frac{d\psi}{ds_{i}}$$

Summarizing, with  $\psi' = A\psi$ , our equation takes the following form:

$$ih\sum_{i=0}^{3}\gamma_iAL^{-1}\frac{d\psi}{ds_i} = mcA\psi$$

Equivalently, by multiplying everything by  $A^{-1}$ , our equation becomes:

$$ih\sum_{i=0}^{3}A^{-1}\gamma_{i}AL^{-1}\frac{d\psi}{ds_{i}}=mc\psi$$

(4) Now let us compare this new equation that we found with the original Dirac equation, from (1), which was as follows:

$$ih\sum_{i=0}^{3}\gamma_{i}\frac{d\psi}{ds_{i}}=mc\psi$$

In order to have solutions  $\psi' = A\psi$  as above, in a plain, non-infinitesimal sense, the obvious possibility is that when we have the following formulae, for any *i*:

$$A^{-1}\gamma_i A L^{-1} = \gamma_i$$

Thus, as a conclusion to this discussion, in order to prove our theorem, in a plain formulation, it would be enough to establish the following formulae, for any i:

$$A^{-1}\gamma_i A = \gamma_i L$$

(5) With this done, let us have a look at the matrix A in the statement. That matrix is constructed by using two numbers a, b, which are given by:

$$a = \sqrt{rac{\gamma+1}{2}}$$
 ,  $b = -\sqrt{rac{\gamma-1}{2}}$ 

Our first claim is that we have the following useful formulae, relating a, b:

$$a^{2} - b^{2} = 1$$
$$a^{2} + b^{2} = \gamma$$
$$2ab = -\gamma\beta$$

Indeed, the first two formulae are clear, and the third formula comes from:

$$2ab = -\sqrt{\gamma^2 - 1}$$
$$= -\sqrt{\frac{1}{1 - \beta^2} - 1}$$
$$= -\sqrt{\frac{\beta^2}{1 - \beta^2}}$$
$$= -\frac{\beta}{\sqrt{1 - \beta^2}}$$
$$= -\gamma\beta$$

Observe also that the above formula  $a^2 - b^2 = 1$  suggests using a notation of type  $a = \cosh p, b = \sinh p$ , but we will not need this here.

(6) Before getting to the matrix A in the statement, let us further study the above numbers a, b. With the help of the formulae connecting them, from (5), we obtain:

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & 2ab \\ 2ab & a^2 + b^2 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}$$

We recognize here the upper left block of L, and so we have:

$$L = \begin{pmatrix} a & b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^2$$

A similar discussion goes for the inverse Lorentz matrix. Indeed, we have:

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} a & -b \\ -b & a \end{pmatrix} = \begin{pmatrix} a^2 - b^2 & 0 \\ 0 & a^2 - b^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, we have the following matrix inversion formula:

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}^{-1} = \begin{pmatrix} a & -b \\ -b & a \end{pmatrix}$$

We conclude that the inverse of the Lorentz matrix is given by:

$$L^{-1} = \begin{pmatrix} a & -b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^2$$

(7) Now let us look at the matrix in the statement, namely:

$$A = \begin{pmatrix} a & 0 & 0 & b \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ b & 0 & 0 & a \end{pmatrix}$$

This matrix, and its inverse, are then given by the following formulae:

$$A = a + b\gamma_0\gamma_2$$

$$A^{-1} = a - b\gamma_0\gamma_2$$

Indeed, in what regards the formula of A, this comes from:

$$\begin{aligned} \gamma_0 \gamma_2 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

As for the formula of  $A^{-1}$ , this comes from the following computation, with  $J = \gamma_0 \gamma_2$ , which satisfies  $J^2 = 1$ , by using the formula  $a^2 - b^2 = 1$  from (5):

$$(a+bJ)(a-bJ) = a^2 + abJ - abJ - b^2J^2$$
$$= a^2 - b^2$$
$$= 1$$

(8) In relation now with the formulae needed in (4), our first claim is that:

$$A\gamma_0 A = \gamma_0$$
$$A^{-1}\gamma_1 A = \gamma_1$$
$$A\gamma_2 A = \gamma_2$$
$$A^{-1}\gamma_3 A = \gamma_3$$

(9) Indeed, the first formula comes from the following computation:

$$A\gamma_0 A = (a + b\gamma_0\gamma_2)\gamma_0(a + b\gamma_0\gamma_2)$$
  
=  $a^2\gamma_0 + ab\gamma_0\gamma_0\gamma_2 + ab\gamma_0\gamma_2\gamma_0 + b^2\gamma_0\gamma_2\gamma_0\gamma_0\gamma_2$   
=  $a^2\gamma_0 - b^2\gamma_0$   
=  $\gamma_0$ 

The second formula comes from a similar computation, as follows:

$$A^{-1}\gamma_1 A = (a + b\gamma_0\gamma_2)\gamma_1(a - b\gamma_0\gamma_2)$$
  
=  $a^2\gamma_1 - ab\gamma_1\gamma_0\gamma_2 + ab\gamma_0\gamma_2\gamma_1 - b^2\gamma_0\gamma_2\gamma_1\gamma_0\gamma_2$   
=  $a^2\gamma_1 - b^2\gamma_1$   
=  $\gamma_1$
The third formula again comes from a similar computation, as follows:

$$A\gamma_2 A = (a + b\gamma_0\gamma_2)\gamma_2(a + b\gamma_0\gamma_2)$$
  
=  $a^2\gamma_2 + ab\gamma_2\gamma_0\gamma_2 + ab\gamma_0\gamma_2\gamma_2 + b^2\gamma_0\gamma_2\gamma_2\gamma_0\gamma_2$   
=  $a^2\gamma_2 - b^2\gamma_2$   
=  $\gamma_2$ 

As for the fourth formula, this comes again from a similar computation, namely:

$$A^{-1}\gamma_{3}A = (a + b\gamma_{0}\gamma_{2})\gamma_{3}(a - b\gamma_{0}\gamma_{2})$$
  
$$= a^{2}\gamma_{3} - ab\gamma_{3}\gamma_{0}\gamma_{2} + ab\gamma_{0}\gamma_{2}\gamma_{3} - b^{2}\gamma_{0}\gamma_{2}\gamma_{3}\gamma_{0}\gamma_{2}$$
  
$$= a^{2}\gamma_{3} - b^{2}\gamma_{3}$$
  
$$= \gamma_{3}$$

(10) Now observe that, with respect to the formulae needed in (4), the second and the fourth formulae found in (8) are what we need. As for the first and third formulae, these are not exactly what we need, and we must fine-tune them. We first have:

$$A^{-1}\gamma_0 A = (a - b\gamma_0\gamma_2)\gamma_0(a + b\gamma_0\gamma_2)$$
  
=  $a^2\gamma_0 + ab\gamma_0\gamma_0\gamma_2 - ab\gamma_0\gamma_2\gamma_0 - b^2\gamma_0\gamma_2\gamma_0\gamma_0\gamma_2$   
=  $(a^2 + b^2)\gamma_0 + 2ab\gamma_2$   
=  $\gamma \cdot \gamma_0 - \gamma\beta \cdot \gamma_2$ 

Similarly, we have the following computation:

$$A^{-1}\gamma_2 A = (a - b\gamma_0\gamma_2)\gamma_2(a + b\gamma_0\gamma_2)$$
  
=  $a^2\gamma_2 + ab\gamma_2\gamma_0\gamma_2 - ab\gamma_0\gamma_2\gamma_2 - b^2\gamma_0\gamma_2\gamma_2\gamma_0\gamma_2$   
=  $(a^2 + b^2)\gamma_2 - 2ab\gamma_0$   
=  $\gamma \cdot \gamma_2 + \gamma\beta \cdot \gamma_0$ 

(11) Time now to review the conditions found in (4). These conditions, corresponding to the plain Lorentz invariance of the solutions of the Dirac equation, were  $A^{-1}\gamma_i A = \gamma_i L$ . But because of  $\gamma_i^2 = 1$ , we can reformulate them in the following way:

$$L = \gamma_i A^{-1} \gamma_i A$$

Now in view of the above, it makes sense to introduce the following matrices:

$$L_i = \gamma_i A^{-1} \gamma_i A$$

According to the computations in (9), we have the following formulae:

$$L_1 = L_3 = 1$$

On the other hand, according to the computations in (10), we have as well:

$$L_0 = \gamma_0(\gamma \cdot \gamma_0 - \gamma \beta \cdot \gamma_2) = \gamma - \gamma \beta \cdot \gamma_0 \gamma_2$$

#### 3. DIRAC EQUATION

$$L_2 = \gamma_2(\gamma \cdot \gamma_2 + \gamma \beta \cdot \gamma_0) = \gamma - \gamma \beta \cdot \gamma_0 \gamma_2$$
  
in usual matrix form, we have the following formulae:

Thus, in usual matrix form, we have the following formulae:

$$L_{0} = L_{2} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & \gamma & -\gamma\beta & 0 \\ 0 & -\gamma\beta & \gamma & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix}$$

(12) The point now is that, based on what we found above, we can say that  $\psi' = A\psi$  satisfies the Dirac equation in the new frame, in an infinitesimal sense, as claimed.  $\Box$ 

#### **3**b.

3c.

3d.

#### 3e. Exercises

Exercises:

EXERCISE 3.19.

Exercise 3.20.

EXERCISE 3.21.

EXERCISE 3.22.

EXERCISE 3.23.

EXERCISE 3.24.

Bonus exercise.

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# Quantum fields

#### 4a. Quantum fields

4b.

4c.

4d.

#### 4e. Exercises

Exercises:

EXERCISE 4.1.

EXERCISE 4.2.

EXERCISE 4.3.

EXERCISE 4.4.

EXERCISE 4.5.

EXERCISE 4.6.

Part II

Standard Model

There's a place out on the edge of town sir Rising above the factories and the fields And ever since I was a child, I can remember That mansion on the hill

5a.

**5**b.

5c.

5d.

5e. Exercises

Exercises:

Exercise 5.1.

Exercise 5.2.

Exercise 5.3.

EXERCISE 5.4.

EXERCISE 5.5.

Exercise 5.6.

6a.

6b.

6c.

6d.

### 6e. Exercises

Exercises:

EXERCISE 6.1.

Exercise 6.2.

Exercise 6.3.

EXERCISE 6.4.

EXERCISE 6.5.

Exercise 6.6.

7a.

7b.

7c.

7d.

7e. Exercises

Exercises:

Exercise 7.1.

Exercise 7.2.

Exercise 7.3.

EXERCISE 7.4.

Exercise 7.5.

Exercise 7.6.

8a.

**8**b.

8c.

8d.

8e. Exercises

Exercises:

EXERCISE 8.1.

Exercise 8.2.

Exercise 8.3.

EXERCISE 8.4.

Exercise 8.5.

EXERCISE 8.6.

# Part III

# Symmetry groups

And the sky was made of amethyst And all the stars were just like little fish You should learn when to go You should learn how to say no

9a.

9b.

9c.

9d.

9e. Exercises

Exercises:

Exercise 9.1.

Exercise 9.2.

Exercise 9.3.

Exercise 9.4.

Exercise 9.5.

Exercise 9.6.

10a.

10b.

10c.

10d.

#### 10e. Exercises

Exercises:

Exercise 10.1.

EXERCISE 10.2.

Exercise 10.3.

EXERCISE 10.4.

Exercise 10.5.

EXERCISE 10.6.

11a.

11b.

11c.

11d.

#### 11e. Exercises

Exercises:

EXERCISE 11.1.

Exercise 11.2.

Exercise 11.3.

EXERCISE 11.4.

EXERCISE 11.5.

EXERCISE 11.6.

12a.

12b.

12c.

12d.

12e. Exercises

Exercises:

Exercise 12.1.

EXERCISE 12.2.

Exercise 12.3.

EXERCISE 12.4.

EXERCISE 12.5.

EXERCISE 12.6.

Part IV

Lattice theory

In the bleak midwinter Frosty wind made moan Earth stood hard as iron Water like a stone

13a.

13b.

13c.

13d.

#### 13e. Exercises

Exercises:

Exercise 13.1.

EXERCISE 13.2.

EXERCISE 13.3.

EXERCISE 13.4.

EXERCISE 13.5.

EXERCISE 13.6.

14a.

14b.

14c.

14d.

14e. Exercises

Exercises:

EXERCISE 14.1.

EXERCISE 14.2.

Exercise 14.3.

EXERCISE 14.4.

EXERCISE 14.5.

EXERCISE 14.6.

15a.

15b.

15c.

15d.

#### 15e. Exercises

Exercises:

Exercise 15.1.

EXERCISE 15.2.

EXERCISE 15.3.

Exercise 15.4.

EXERCISE 15.5.

EXERCISE 15.6.

16a. 16b. 16c. 16d.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter.

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