Haar measure and Peter-Weyl theory

Teo Banica

"Introduction to quantum groups", 3/6

06/20

Plan

(1) Representations

(2) The Haar measure

(3) Peter-Weyl theory

(4) Kesten amenability

$\Longrightarrow$ next lecture: Tannakian duality

Quantum groups

Axioms. Let A be a C^*-algebra, with $u \in M_N(A)$ biunitary (u, u^t unitaries), whose entries generate A, such that:

– $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ defines a morphism $\Delta : A \to A \otimes A$
– $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : A \to \mathbb{C}$
– $S(u_{ij}) = u_{ji}^*$ defines a morphism $S : A \to A^{opp}$

We write then $A = C(G) = C^*(\Gamma)$, with G compact quantum group, and Γ discrete quantum group [Woronowicz 87].

Examples. Compact Lie groups, discrete group duals (NC tori), liberations and half-liberations, product operations..

Tools. Comultiplication, counit and antipode Δ, ε, S, in analogy with multiplication, unit and inverse m, u, i.

Representations 1/4

Definition. A corepresentation of a Woronowicz algebra A is a biunitary matrix $v \in M_n(\mathcal{A})$ satisfying

- $\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$
- $\varepsilon(v_{ij}) = \delta_{ij}$
- $S(v_{ij}) = v_{ji}^*$

where $\mathcal{A} \subset A$ is the dense $*$-subalgebra of "smooth elements".

Examples. 1 (trivial), u (fundamental), $\bar{u}$ (conjugate).

Idea. The corepresentations of $A = C(G)$ can be thought of as corresponding to the representations of G.

Representations 2/4

Theorem. Given a closed subgroup $G \subset U_N$, the corepresentations of $C(G)$ are in one-to-one correspondence, given by

$$\pi(g) = \begin{pmatrix} v_{11}(g) & \dots & v_{1n}(g) \\ \vdots & & \vdots \\ v_{n1}(g) & \dots & v_{nn}(g) \end{pmatrix}$$

with the finite dimensional unitary smooth representations of G.

Proof. Same computations as when proving that $A = C(G)$ is a Woronowicz algebra, which was already done.

Representations 3/4

Theorem. The corepresentations of a given Woronowicz algebra A are subject to the following operations:

(1) Making sums, $v + w = diag(v, w)$.

(2) Making tensor products, $(v \otimes w)_{ia,jb} = v_{ij} w_{ab}$.

(3) Taking conjugates, $(\bar{v})_{ij} = v_{ij}^*$.

(4) Spinning, $w = UvU^*$, with $U \in U_n$.

Proof. All this is elementary, coming from definitions.

Representations 4/4

Theorem. Given a discrete group $\Gamma = < g_1, \ldots, g_N >$, the corepresentations of $A = C^*(\Gamma)$ are as follows:

(1) Any group element $h \in \Gamma$ is a 1D corepresentation of A, and the operations are the usual ones on group elements.

(2) Any diagonal matrix of type $v = diag(h_1, \ldots, h_n)$, with $n \in \mathbb{N}$, and with $h_1, \ldots, h_n \in \Gamma$, is a corepresentation of A.

(3) More generally, any matrix $w = Udiag(h_1, \ldots, h_n)U^*$ with $h_1, \ldots, h_n \in \Gamma$ and with $U \in U_n$, is a corepresentation of A.

Proof. Follows from $\Delta(h) = h \otimes h$, $\varepsilon(h) = 1$, $S(h) = h^{-1}$.

Comment. We'll see later that (3) gives all corepresentations.

Theory 1/6

<u>Definition.</u> Given corepresentations $v \in M_n(A), w \in M_m(A)$, we set

$$Hom(v,w) = \left\{ T \in M_{m\times n}(\mathbb{C}) \Big| Tv = wT \right\}$$

and we use the following conventions:

(1) $Fix(v) = Hom(1,v)$ and $End(v) = Hom(v,v)$.

(2) $v \sim w$ when $Hom(v,w)$ contains an invertible element.

(3) v is called irreducible, $v \in Irr(G)$, when $End(v) = \mathbb{C}1$.

Theory 2/6

Theorem. We have the following results:

$$T \in Hom(u, v), S \in Hom(v, w) \implies ST \in Hom(u, w)$$

$$S \in Hom(p, q), T \in Hom(v, w) \implies S \otimes T \in Hom(p \otimes v, q \otimes w)$$

$$T \in Hom(v, w) \implies T^* \in Hom(w, v)$$

In other words, the Hom spaces form a tensor $*$-category.

Proof. All this is elementary, coming from definitions.

Comment. We'll be back to this later (Tannakian duality).

Theory 3/6

<u>Theorem.</u> Let $B \subset M_N(\mathbb{C})$ be a C^*-algebra.

(1) We have $1 = p_1 + \ldots + p_k$, with $p_i \in B$ minimal projections.

(2) The spaces $B_i = p_i B p_i$ are non-unital $*$-subalgebras of B.

(3) We have a non-unital $*$-algebra sum $B = B_1 \oplus \ldots \oplus B_k$.

(4) Unital $*$-algebra isomorphisms $B_i \simeq M_{N_i}(\mathbb{C})$, $N_i = rank(p_i)$.

(5) Thus, we can decompose $B \simeq M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$.

(6) This holds in fact for any finite dimensional C^*-algebra.

<u>Proof</u>. This is something that we already know from lecture 1, the idea being (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5) $\implies$ (6).

Theory 4/6

Theorem (PW1). Any corepresentation $v \in M_n(A)$ decomposes as a direct sum of irreducible corepresentations

$$v = v_1 + \ldots + v_k$$

with each v_i being obtained by restricting v to $Im(p_i)$, where $1 = p_1 + \ldots + p_k$ is the partition of unity for $B = End(v)$.

Proof. (1) Let $\Phi : \mathbb{C}^n \to A \otimes \mathbb{C}^n$, $\Phi(e_i) = \sum_j v_{ij} \otimes e_j$. If $V \subset \mathbb{C}^n$ is invariant, $\Phi(V) \subset A \otimes V$, then $\Phi_{|V} : V \to A \otimes V$ is a coaction too, which must come from a subcorepresentation $w \subset v$.

(2) Given $p \in End(v)$, $V = Im(p)$ must be invariant, coming from $w \subset v$, and $p \to w$ maps subprojections to subcorepresentations, and minimal projections to irreducible corepresentations.

(3) With these preliminaries in hand, the result follows.

Theory 5/6

Definition. We denote by $u^{\otimes k}$, with $k = \circ\bullet\bullet\circ\ldots$ being a colored integer, the various tensor products between $u, \bar{u}$, with the rules

$$u^{\otimes\emptyset} = 1 \quad , \quad u^{\otimes\circ} = u \quad , \quad u^{\otimes\bullet} = \bar{u}$$

along with multiplicativity condition

$$u^{\otimes kl} = u^{\otimes k} \otimes u^{\otimes l}$$

and call them Peter-Weyl corepresentations.

Remarks. In the real case, $u = \bar{u}$, we can assume $k \in \mathbb{N}$. In the classical case, we can assume, up to equivalence, $k \in \mathbb{N} \times \mathbb{N}$.

Theory 6/6

Theorem (PW2). Each irreducible corepresentation of A appears inside a Peter-Weyl corepresentation $u^{\otimes k}$.

Proof. Given a corepresentation $v \in M_n(A)$, consider its space of coefficients, $C(v) = span(v_{ij})$. Then $v \to C(v)$ is functorial, mapping subcorepresentations into subspaces. We have:

$$\mathcal{A} = \sum_{k \in \mathbb{N} * \mathbb{N}} C(u^{\otimes k})$$

We have $C(v) \subset \mathcal{A}$, and so, for certain exponents $k_1, \ldots, k_p$:

$$C(v) \subset C(u^{\otimes k_1} \oplus \ldots \oplus u^{\otimes k_p})$$

Thus $v \subset u^{\otimes k_1} \oplus \ldots \oplus u^{\otimes k_p}$, and PW1 gives the result.

Summary

We are interested in the FD unitary smooth representations of G. These come from the biunitary matrices $v \in M_n(\mathcal{A})$ satisfying:

– $\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$
– $\varepsilon(v_{ij}) = \delta_{ij}$
– $S(v_{ij}) = v_{ji}^*$

As basic examples we have $1, u, \bar{u}$, and more generally the PW corepresentations $u^{\otimes k}$, with k colored integer.

The corepresentations decompose into irreducibles (PW1) and the irreducibles can be obtained by splitting the $u^{\otimes k}$ (PW2).

Haar measure 1/8

Theorem. The algebra $A = C(G)$ with $G \subset U_N$, has a unique faithful positive unital linear form $\int_G : A \to \mathbb{C}$ satisfying:

$$\int_G f(xy)dx = \int_G f(yx)dx = \int_G f(x)dx$$

This can be constructed by starting with any faithful positive unital form $\varphi \in A^*$, and taking the Cesàro limit

$$\int_G = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \varphi^{*k}$$

where the convolution operation is $\phi * \psi = (\phi \otimes \psi)\Delta$.

Proof. Well-known, and we'll reprove it anyway.

Haar measure 2/8

Definition. Given a Woronowicz algebra $A = C(G)$, a positive unital tracial state $\int_G : A \to \mathbb{C}$ subject to the conditions

$$\left(\int_G \otimes id\right)\Delta = \left(id \otimes \int_G\right)\Delta = \int_G (.)1$$

called left/right invariance, is called Haar integration over G.

Remark. In the classical case, $G \subset U_N$, we know that $\int_G$ exists, is unique, and can be constructed via a Cesàro limit.

Haar measure 3/8

<u>Theorem</u>. Given a discrete group $\Gamma =< g_1, \ldots, g_N >$, the algebra $A = C^*(\Gamma)$ has a Haar functional, given on group elements by:

$$\int_{\widehat{\Gamma}} g = \delta_{g,1}$$

This functional is faithful on the image on $C^*(\Gamma)$ in the regular representation. In the abelian case, this is the counit of $C(\widehat{\Gamma})$.

<u>Proof</u>. Consider indeed the left regular representation:

$$\pi : C^*(\Gamma) \to B(l^2(\Gamma)) \quad , \quad \pi(g)(h) = gh$$

The composition $\int_{\widehat{\Gamma}}$ of π with $T \to < T1, 1 >$ is given by:

$$\int_{\widehat{\Gamma}} g =< g1, 1 >= \delta_{g,1}$$

But this gives all the assertions, the last one being clear too.

Haar measure 4/8

Theorem. Given an arbitrary unital linear form $\varphi \in A^*$, the limit

$$\int_\varphi a = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \varphi^{*k}(a)$$

exists, and for a corepresentation coefficient $a = (\tau \otimes id)v$, we have

$$\int_\varphi a = \tau(P)$$

where P is the projection onto the 1-eigenspace of $(id \otimes \varphi)v$.

Proof. This is linear algebra, on the space of coefficients of v.

Haar measure 5/8

Theorem. Given a faithful unital linear form $\varphi \in A^*$, the limit

$$\int_\varphi a = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi^{*k}(a)$$

exists, and is independent of φ, given on coefficients by

$$\left(id \otimes \int_\varphi \right) v = P$$

where P is the projection onto $Fix(v) = \{\xi \in \mathbb{C}^n | v\xi = \xi\}$.

Proof. With $M = (id \otimes \varphi)v$ we must prove that $M\xi = \xi$ implies $v\xi = \xi$. But this follows via a standard positivity trick.

Haar measure 6/8

Assume indeed $M\xi = \xi$, and consider the following element:

$$a = \sum_i \left(\sum_j v_{ij}\xi_j - \xi_i \right) \left(\sum_k v_{ik}\xi_k - \xi_i \right)^*$$

We must prove that $a = 0$. Since v is biunitary, we have:

$$\begin{aligned} a &= \sum_i \left(\sum_j \left(v_{ij}\xi_j - \frac{1}{N}\xi_i \right) \right) \left(\sum_k \left(v_{ik}^*\bar{\xi}_k - \frac{1}{N}\bar{\xi}_i \right) \right) \\ &= 2(||\xi||^2 - Re(< v\xi, \xi >)) \end{aligned}$$

By using now $M\xi = \xi$, we obtain from this $\varphi(a) = 0$. Now since φ is faithful, this gives $a = 0$, and so $v\xi = \xi$, as desired.

Haar measure 7/8

Theorem. Any Woronowicz algebra has a unique Haar integration functional, which can be constructed by starting with any faithful positive unital state $\varphi \in A^*$, and setting

$$\int_G = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \varphi^{*k}$$

where $\phi * \psi = (\phi \otimes \psi)\Delta$. Moreover, for any corepresentation v,

$$\left(id \otimes \int_G\right) v = P$$

where P is the projection onto $Fix(v) = \{\xi \in \mathbb{C}^n | v\xi = \xi\}$.

Proof. We already know all this, modulo a few extra minor things.

Haar measure 8/8

Theorem. We have the following results:

(1) For a product $G\times H$, we have $\int_{G\times H}=\int_G\otimes\int_H$.

(2) For a dual free product $G\,\hat{*}\,H$, we have $\int_{G\,\hat{*}\,H}=\int_G*\int_H$.

(3) For a quotient $G\to H$, we have $\int_H=\left(\int_G\right)_{|C(H)}$.

(4) For a projective version $G\to PG$, we have $\int_{PG}=\left(\int_G\right)_{|C(PG)}$.

Proof. All these results follow from uniqueness.

Theory 1/4

Theorem. We have a Frobenius type isomorphism

$$Hom(v,w)\simeq Fix(v\otimes\bar{w})$$

valid for any two corepresentations v,w.

Proof. We have the following equivalences:

$$T\in Hom(v,w)\iff Tv=wT\iff\sum_j T_{aj}v_{ji}=\sum_b w_{ab}T_{bi}$$

$$T\in Fix(v\otimes\bar{w})\iff(v\otimes\bar{w})T=\xi\iff\sum_{jb}v_{ij}w_{ab}^*T_{bj}=T_{ai}$$

With this, both inclusions follow from the biunitarity of v,w.

Theory 2/4

Theorem (PW3). The dense subalgebra $\mathcal{A} \subset A$ decomposes as

$$\mathcal{A} = \bigoplus_{v \in Irr(A)} M_{\dim(v)}(\mathbb{C})$$

isomorphism of $*$-coalgebras, with the summands being pairwise orthogonal with respect to $< a, b >= \int_G ab^*$.

Proof. We must prove that for $v, w \in Irr(A)$ we have:

$$v \not\sim w \implies C(v) \perp C(w)$$

The matrix P given by $P_{ia,jb} = \int_G v_{ij} w^*_{ab}$ is the projection onto:

$$Fix(v \otimes \bar{w}) \simeq Hom(v, w) = \{0\}$$

Thus we have $P = 0$, and this gives the result.

Theory 3/4

<u>Theorem</u>. The characters of the corepresentations, given by

$$\chi_v = \sum_i v_{ii}$$

behave as follows, in respect to the various operations:

$$\chi_{v+w} = \chi_v + \chi_w \quad , \quad \chi_{v\otimes w} = \chi_v \chi_w \quad , \quad \chi_{\bar{v}} = \chi_v^*$$

In addition, assuming $v \sim w$, we have $\chi_v = \chi_w$.

<u>Proof</u>. All this is clear, coming from definitions.

Theory 4/4

Theorem (PW4). The characters of irreducible corepresentations belong to the algebra of "smooth central functions"

$$\mathcal{A}_{central} = \left\{ a \in \mathcal{A} \middle| \Sigma\Delta(a) = \Delta(a) \right\}$$

and form an orthonormal basis of it.

Proof. The only tricky assertion is the norm 1 one. But:

$$\int_G \chi_v \chi_v^* = \sum_{ij} \int_G v_{ii} v_{jj}^* = \sum_i \frac{1}{N} = 1$$

Here we have used the fact that the integrals $\int_G v_{ij} v_{kl}^*$ form the orthogonal projection onto $Fix(v \otimes \bar{v}) \simeq End(v) = \mathbb{C}1$.

Examples 1/2

<u>Theorem</u>. Let $\Gamma =< g_1, \ldots, g_N >$ be a discrete group.

(1) The 1D corepresentations of $C^*(\Gamma)$ are the elements $g \in \Gamma$.

(2) The corepresentations of $C^*(\Gamma)$ are sums of group elements.

<u>Theorem</u>. The cocommutative Woronowicz algebras appear as

$$C^*(\Gamma) \to A \to C^*_{red}(\Gamma)$$

with Γ being a discrete group, $A = C^*_\pi(\Gamma)$ with $\pi \otimes \pi \subset \pi$.

<u>Proofs</u>. All this is clear from the Peter-Weyl theory.

Examples 2/2

Theorem. We have the following results:

(1) The irreps of a product $G \times H$ are the tensor products of the form $\pi \otimes \nu$, with π, ν being irreps of G, H.

(2) The irreps of a dual free product $G \,\hat{*}\, H$ appear as alternating tensor products of irreps of G, H.

(3) The irreps of a quotient $G \to H$ are the irreps of G whose coefficients belong to $C(H)$.

(4) The irreps of $G \to PG$ are the irreps of G which appear by decomposing the tensor powers of $ad(\pi) = \pi \otimes \bar{\pi}$.

Proofs. Once again, all this is clear from the Peter-Weyl theory.

Amenability 1/3

Theorem. Let A_{full} be the enveloping C^*-algebra of $\mathcal{A}$, and let A_{red} be the quotient of A by the null ideal of the Haar integration. The following are then equivalent:

(1) The Haar functional of A_{full} is faithful.

(2) The projection map $A_{full} \to A_{red}$ is an isomorphism.

(3) The counit map $\varepsilon : A_{full} \to \mathbb{C}$ factorizes through A_{red}.

(4) We have $N \in \sigma(Re(\chi_u))$, the spectrum being taken inside A_{red}.

If this is the case, we say that G is coamenable, and Γ is amenable.

Amenability 2/3

(1) $\iff$ (2) This follows from the fact that the GNS construction for the algebra A_{full} produces the algebra A_{red}.

(2) $\iff$ (3) Here $\implies$ is trivial. Conversely, the comultiplication of $\mathcal{A}$ can be extended into a map $\Phi : A_{red} \to A_{red} \otimes A_{full}$, and the composition $(\varepsilon \otimes id)\Phi$ is then our desired isomorphism.

(3) $\iff$ (4) The implication $\implies$ is clear, because from $\varepsilon(u_{ii}) = 1$ for any i, we obtain the following formula:

$$\varepsilon(N - Re(\chi(u))) = 0$$

Thus $N - Re(\chi(u))$ is not invertible in A_{red}, as claimed.

Amenability 3/3

(4) $\implies$ (3) With $v = u \oplus \bar{u}$, our assumption reads:

$$\dim v \in \sigma(\chi_v)$$

By functional calculus the same holds for $w = v + 1$, and in fact for any tensor power $w_k = w^{\otimes k}$. Now choose for each $k \in \mathbb{N}$ a state $\varepsilon_k \in A_{red}^*$ having the following property:

$$\varepsilon_k(w_k) = \dim w_k$$

By Peter-Weyl we must have $\varepsilon_k(v) = \dim v$, for any $v \leq w_k$, and since each irreducible corepresentation of A appears in this way, the sequence ε_k converges to a counit map $\varepsilon : A_{red} \to \mathbb{C}$, as desired.

Conclusion

We have a fully working Haar integration theory and Peter-Weyl theory, the applications of all this being, so far:

(1) Representations of group duals, and of various products.

(2) A fully satisfactory notion of amenability/coamenability.

(3) In particular, a Kesten amenability criterion, $N \in \sigma(Re(\chi_u))$.

(4) Suggesting that computing $law(\chi_u)$ is the "main problem".

$\Longrightarrow$ next lecture: Tannakian duality, easiness