Tannakian duality, diagrams and easiness

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"Introduction to quantum groups", 4/6

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Plan

(1) Tensor categories

(2) Tannakian duality

(3) Diagrams, easiness

(4) Free quantum groups

$\Longrightarrow$ next lecture: quantum permutations

Representations

(1) A corepresentation of a Woronowicz algebra A is a biunitary matrix $v \in M_n(\mathcal{A})$ satisfying:

– $\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$
– $\varepsilon(v_{ij}) = \delta_{ij}$
– $S(v_{ij}) = v_{ji}^*$

(2) Basic example: the fundamental corepresentation u. In fact, the axioms state that u must be a faithful corepresentation.

(3) With $A = C(G)$, the corepresentations of A correspond to the FD unitary smooth representations of the quantum group G.

(4) We have a full Peter-Weyl theory for them, the main result stating that $\mathcal{A}$ decomposes as an orthogonal direct sum.

Categories 1/6

Definition. The Tannakian category of a Woronowicz algebra (A, u) is the collection $C = (C(k,l))$ of vector spaces

$$C(k,l) = Hom(u^{\otimes k}, u^{\otimes l})$$

where the corepresentations $u^{\otimes k}$ with $k = \circ\bullet\bullet\circ\ldots$ colored integer are defined by $u^{\otimes\circ} = u$, $u^{\otimes\bullet} = \bar{u}$ and multiplicativity.

Remark 1. We already know that C is a tensor $*$-category, the verification of all conditions being elementary.

Remark 2. In fact, C appears by definition as subcategory of the tensor $*$-category $E(k,l) = \mathcal{L}(H^{\otimes k}, H^{\otimes l})$, where $H = \mathbb{C}^N$.

Categories 2/6

Our purpose will be that of reconstructing (A, u) in terms of $C = (C(k, l))$. Here is a useful preliminary result:

<u>Theorem</u>. Given a morphism $\pi : (A, u) \to (B, v)$ we have

$$Hom(u^{\otimes k}, u^{\otimes l}) \subset Hom(v^{\otimes k}, v^{\otimes l})$$

and if these inclusions are all equalities, π is an isomorphism.

<u>Proof</u>. Follows from Peter-Weyl, by contradiction, because each irreducible corepresentation is contained in some $u^{\otimes k}$.

In order to exploit the fact that u is biunitary, we can use:

Theorem. An matrix $u\in M_N(A)$ is biunitary if and only if

$$R\in Hom(1,u\otimes\bar{u})\quad,\quad R\in Hom(1,\bar{u}\otimes u)$$

$$R^*\in Hom(u\otimes\bar{u},1)\quad,\quad R^*\in Hom(\bar{u}\otimes u,1)$$

where $R:\mathbb{C}\to\mathbb{C}^N\otimes\mathbb{C}^N$ is given by $R(1)=\sum_i e_i\otimes e_i$.

Proof. This follows from some elementary computations.

Categories 4/6

<u>Definition</u>. Let H be a finite dimensional Hilbert space. A tensor category over H is a collection $C = (C(k,l))$ of subspaces

$$C(k,l) \subset \mathcal{L}(H^{\otimes k}, H^{\otimes l})$$

satisfying the following conditions:

(1) $S, T \in C$ implies $S \otimes T \in C$.

(2) If $S, T \in C$ are composable, then $ST \in C$.

(3) $T \in C$ implies $T^* \in C$.

(4) Each $C(k,k)$ contains the identity operator.

(5) $C(\emptyset, \circ\bullet)$ and $C(\emptyset, \bullet\circ)$ contain the map $R : 1 \to \sum_i e_i \otimes e_i$.

Categories 5/6

Theorem. Let (A, u) be a Woronowicz algebra, with fundamental corepresentation $u \in M_N(A)$. The associated Tannakian category

$$C(k, l) = Hom(u^{\otimes k}, u^{\otimes l})$$

is then a tensor category over the Hilbert space $H = \mathbb{C}^N$.

Proof. We already know that axioms (1-4) hold indeed, this being elementary, and (5) is something that we just did, clear too.

Categories 6/6

Theorem. Given a tensor category $C = (C(k,l))$, the following algebra is a Woronowicz algebra:

$$A_C = C(U_N^+) \Big/ \left\langle T \in Hom(u^{\otimes k}, u^{\otimes l}) \Big| \forall k, l, \forall T \in C(k,l) \right\rangle$$

In the case where C comes from a Woronowicz algebra (A, v), we have a quotient map $A_C \to A$.

Proof. We have indeed a Woronowicz algebra, because the relations $T \in Hom(u^{\otimes k}, u^{\otimes l})$ are of "Hopf type", i.e. Δ, ε, S factorize.

The fact that we have a quotient map $A_C \to A$ is clear, because the relations defining A_C are satisfied inside A.

Summary

We have so far:

(1) Axioms for A: $N \times N$ Woronowicz algebra

(2) Axioms for C: tensor category over $\mathbb{C}^N$

(3) Correspondence $A \to C$: set $C_A = (Hom(u^{\otimes k}, u^{\otimes l}))_{kl}$

(4) Correspondence $C \to A$: set $A_C = C(U_N^+)/ < C \subset C_A >$

$\implies$ we want to prove that we have a bijection $A \leftrightarrow C$

Step 1

<u>Theorem</u>. Consider the following conditions:

(1) $C = C_{A_C}$, for any Tannakian category C.

(2) $A = A_{C_A}$, for any Woronowicz algebra (A, u).

We have then (1) $\implies$ (2). Also, $C \subset C_{A_C}$ is automatic.

<u>Proof</u>. We know that we have an arrow as follows:

$$A_{C_A} \to A$$

On the other hand, assuming (1), with $C = C_A$ we get:

$$C_A = C_{A_{C_A}}$$

Thus, we can use our quotient map criterion from before, and we get $A_{C_A} = A$, as desired. Finally, the last assertion is clear.

Step 2

Definition. Given a tensor category C over H, we set:

$$E_C = \bigoplus_{k,l} C(k,l) \subset \bigoplus_{k,l} B(H^{\otimes k}, H^{\otimes l}) \subset B\left(\bigoplus_k H^{\otimes k}\right)$$

Also, for any $s \in \mathbb{N}$, we consider the following truncation:

$$E_C^{(s)} = \bigoplus_{|k|,|l| \leq s} C(k,l) \subset \bigoplus_{|k|,|l| \leq s} B(H^{\otimes k}, H^{\otimes l}) = B\left(\bigoplus_{|k| \leq s} H^{\otimes k}\right)$$

Remark. We obtain in this way certain $*$-algebras.

Step 3

Theorem. For any C^*-algebra $B \subset M_n(\mathbb{C})$ we have

$$B = B''$$

where prime denotes the commutant, taken inside $M_n(\mathbb{C})$.

Proof. Let us decompose B as a direct sum of matrix algebras:

$$B = M_{r_1}(\mathbb{C}) \oplus \ldots \oplus M_{r_k}(\mathbb{C})$$

The commutant of this algebra is then as follows:

$$B' = \mathbb{C} \oplus \ldots \oplus \mathbb{C}$$

By taking once again the commutant we obtain B itself.

(This is a particular case of von Neumann's bicommutant theorem)

Step 4

Theorem. Given a category C, the following are equivalent:

(1) $C = C_{A_C}$.

(2) $E_C = E_{C_{A_C}}$.

(3) $E_C^{(s)} = E_{C_{A_C}}^{(s)}$, for any $s \in \mathbb{N}$.

(4) $E_C^{(s)'} = E_{C_{A_C}}^{(s)'}$, for any $s \in \mathbb{N}$.

In addition, $\subset$, $\subset$, $\subset$, $\supset$ respectively are automatically satisfied.

Proof. Here (1) $\iff$ (2) is clear from definitions, (2) $\iff$ (3) is clear from definitions as well, and (3) $\iff$ (4) comes from the bicommutant theorem. As for the last assertion, we have indeed $C \subset C_{A_C}$, and the other inclusions follow from this.

Step 5

Theorem. Given a Woronowicz algebra (A, u), we have

$$E_{C_A}^{(s)} = End\left(\bigoplus_{|k|\leq s} u^{\otimes k}\right)$$

as subalgebras of $B(\oplus_{|k|\leq s} H^{\otimes k})$.

Proof. The algebra $E_{C_A}^{(s)}$ appears by definition as follows:

$$E_{C_A}^{(s)} = \bigoplus_{|k|,|l|\leq s} Hom(u^{\otimes k}, u^{\otimes l}) \subset B\left(\bigoplus_{|k|\leq s} H^{\otimes k}\right)$$

But this is precisely the algebra of intertwiners of $\bigoplus_{|k|\leq s} u^{\otimes k}$.

Step 6

Theorem. For any corepresentation $v \in M_n(A)$, the map

$$\pi_v : A^* \to M_n(\mathbb{C}) \quad , \quad \varphi \to (\varphi(v_{ij}))_{ij}$$

is a representation, having as image $Im(\pi_v) = End(v)'$.

Proof. The first assertion is clear, coming from:

$$\begin{aligned}(\pi_v(\varphi * \psi))_{ij} &= (\varphi \otimes \psi)\Delta(v_{ij}) \\ &= \sum_k \varphi(v_{ik})\psi(v_{kj}) \\ &= \sum_k (\pi_v(\varphi))_{ik}(\pi_v(\psi))_{kj} \\ &= (\pi_v(\varphi)\pi_v(\psi))_{ij}\end{aligned}$$

As for the second assertion, this comes by double inclusion.

Conclusion

$\Longrightarrow$ We want to prove Tannakian duality, $A \leftrightarrow C$. Passed a few trivialities, this amounts in proving that:

$$C_{A_C} \subset C$$

$\Longrightarrow$ By using the $C \to E_C$ construction, truncated at $s \in \mathbb{N}$, and then a bicommutant trick, this is the same as proving that:

$$E_C^{(s)'} \subset E_{C_{A_C}}^{(s)'}$$

$\Longrightarrow$ We know that for any A we have $E_{C_A}^{(s)'} = Im(\pi_v)$, where

$$v = \bigoplus_{|k| \leq s} u^{\otimes k}$$

and where $\pi_v : A^* \to M_n(\mathbb{C})$ is given by $\varphi \to (\varphi(v_{ij}))_{ij}$.

Modelling 1/4

In order to model A_C, and to fine-tune the results that we have, consider the following pair of dual vector spaces:

$$F = \bigoplus_k B\left(H^{\otimes k}\right) \quad , \quad F^* = \bigoplus_k B\left(H^{\otimes k}\right)^*$$

Let $f_{ij}, f_{ij}^* \in F^*$ be the standard generators of $B(H)^*, B(\bar{H})^*$.

(1) F^* is a $*$-algebra, with multiplication $\otimes$ and involution $f_{ij} \leftrightarrow f_{ij}^*$.

(2) F^* is a $*$-bialgebra, with $\Delta(f_{ij}) = \sum_k f_{ik} \otimes f_{kj}$ and $\varepsilon(f_{ij}) = \delta_{ij}$.

(3) We have a $*$-bialgebra isomorphism $< u_{ij} > \simeq F^*$, $u_{ij} \to f_{ij}$.

Modelling 2/4

Theorem. The smooth part of the algebra A_C is given by

$$\mathcal{A}_C \simeq F^*/J$$

where $J \subset F^*$ is the ideal coming from the following relations,

$$\sum_{p_1,\ldots,p_k} T_{i_1\ldots i_l,p_1\ldots p_k} f_{p_1 j_1} \otimes \ldots \otimes f_{p_k j_k}$$

$$= \sum_{q_1,\ldots,q_l} T_{q_1\ldots q_l,j_1\ldots j_k} f_{i_1 q_1} \otimes \ldots \otimes f_{i_l q_l} \quad , \quad \forall i, j$$

one for each pair of colored integers k, l, and each $T \in C(k, l)$.

Proof. This is indeed clear from definitions.

Modelling 3/4

Theorem. The linear space $\mathcal{A}_C^*$ is given by the formula

$$\mathcal{A}_C^* = \left\{ a \in F \middle| T a_k = a_l T, \forall T \in C(k,l) \right\}$$

and its representation constructed before, namely

$$\pi_v : \mathcal{A}_C^* \to B(\oplus_{|k| \leq s} H^{\otimes k})$$

appears diagonally, by truncating, $\pi_v : a \to (a_k)_{kk}$.

Proof. Once again, this an elementary computation.

Modelling 4/4

In order to conclude, consider the following spaces:

$$F_s = \bigoplus_{|k| \leq s} B\left(H^{\otimes k}\right) \quad , \quad F_s^* = \bigoplus_{|k| \leq s} B\left(H^{\otimes k}\right)^*$$

We denote by $a \to a_s$ the truncation $F \to F_s$. We have:

(1) $E_C^{(s)'} \subset F_s$.

(2) $E_C' \subset F$.

(3) $\mathcal{A}_C^* = E_C'$.

(4) $Im(\pi_v) = (E_C')_s$.

Indeed, all this follows from the above interpretation of $\mathcal{A}_C^*$.

Duality

Theorem. We have a Tannakian duality correspondence

$$A \leftrightarrow C$$

between Woronowicz algebras and tensor categories, given by

$$C_A = (Hom(u^{\otimes k}, u^{\otimes l}))_{kl}$$

in one sense, from algebras to categories, and by

$$A_C = C(U_N^+)/ < C \subset C_A >$$

in the other sense, from categories to algebras.

Proof 1/2

We have to prove that, for any category C, and any $s \in \mathbb{N}$:

$$E_C^{(s)'} = (E'_C)_s$$

By taking duals, this is the same as proving that:

$$\left\{ f \in F_s^* \middle| f_{|(E'_C)_s} = 0 \right\} = \left\{ f \in F_s^* \middle| f_{|E_C^{(s)'}} = 0 \right\}$$

We use $\mathcal{A}_C^* = E'_C$. Since we have $\mathcal{A}_C = F^*/J$, we conclude that the ideal $J \subset F^*$ previously constructed is given by:

$$J = \left\{ f \in F^* \middle| f_{|E'_C} = 0 \right\}$$

Proof 2/2

The point now is that we have, for any $s \in \mathbb{N}$:

$$J \cap F_s^* = \left\{ f \in F_s^* \middle| f_{|E_C^{(s)\prime}} = 0 \right\}$$

On the other hand, we have as well:

$$\begin{aligned} J \cap F_s^* &= \left\{ f \in F^* \middle| f_{|E_C'} = 0 \right\} \cap F_s^* \\ &= \left\{ f \in F_s^* \middle| f_{|E_C'} = 0 \right\} \\ &= \left\{ f \in F_s^* \middle| f_{|(E_C')_s} = 0 \right\} \end{aligned}$$

Thus, we are led to the equality that we wanted to prove.

Applications

Many applications, and to begin with, we have as plan:

(1) The biggest quantum group, namely U_N^+, must correspond to the smallest tensor category, namely $< R >$.

(2) It is well-known that $R : 1 \to \sum_i e_i \otimes e_i$ can be pictured as a semicircle $\cap$, so we have to get into diagrams.

(3) We will reach in this way to a notion of "easy quantum group", covering O_N, O_N^+, U_N, U_N^+, and many other examples.

(4) As a main application, we will solve the problem of computing the law of the main character for O_N, O_N^+, U_N, U_N^+.

Easiness 1/3

Let $P(k,l)$ be the set of partitions between an upper colored integer k, and a lower colored integer l.

Definition. A collection of subsets $D(k,l) \subset P(k,l)$ is called a category of partitions when it satisfies:

(1) Stability under the horizontal concatenation, $(\pi,\sigma) \to [\pi\sigma]$.

(2) Stability under vertical concatenation $(\pi,\sigma) \to [^{\sigma}_{\pi}]$ (matching).

(3) Stability under the upside-down turning $*$, with $\circ \leftrightarrow \bullet$.

(4) Each $P(k,k)$ contains the identity partition $||\ldots||$.

(5) Both $P(\emptyset,\circ\bullet)$ and $P(\emptyset,\bullet\circ)$ contain the semicircle $\cap$.

Easiness 2/3

<u>Definition.</u> A closed subgroup $G \subset U_N^+$ is called easy when

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \middle| \pi \in D(k,l)\right)$$

for a certain category of partitions $D \subset P$, where

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with $\delta_\pi \in \{0, 1\}$ depending on whether the indices fit or not.

Easiness 3/3

Theorem. The basic unitary quantum groups, namely

$$\begin{array}{ccc} U_N & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & O_N^+ \end{array}$$

are all easy, coming from the following categories of pairings:

$$\begin{array}{ccc} \mathcal{P}_2 & \longleftarrow & \mathcal{NC}_2 \\ \downarrow & & \downarrow \\ P_2 & \longleftarrow & NC_2 \end{array}$$

Proof. This comes from Tannaka (classical case: Brauer).

Applications 1/3

Theorem. We have the following free complexification formula,

$$\widetilde{O}_N^+ = U_N^+$$

and for projective versions we have the following isomorphism,

$$PO_N^+ = PU_N^+$$

by identifying as usual the full and reduced versions.

Proof. We know that we have $\widetilde{O}_N^+ \subset U_N^+$, and since the Tannakian categories coincide, this is an isomorphism.

Applications 2/3

<u>Theorem.</u> The moments of the main characters for

$$\begin{array}{ccc} U_N & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & O_N^+ \end{array}$$

are, in the $N \to \infty$ limit, as follows:

(1) On the bottom, with $k = 2l$, we have $(2l)!!$ and $\frac{1}{l+1}\binom{2l}{l}$.

(2) On top we have similar numbers, with k being now colored.

<u>Proof</u>. This follows by counting the pairings, with $N \to \infty$ being needed as for $\{T_\pi\}$ to be linearly independent.

Applications 3/3

Theorem. The asymptotic laws of the main characters for

$$\begin{array}{ccc} U_N & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & O_N^+ \end{array}$$

are the basic measures in probability and free probability:

$$\begin{array}{ccc} \textit{Complex Gaussian} & \longrightarrow & \textit{Voiculescu circular} \\ \uparrow & & \uparrow \\ \textit{Real Gaussian} & \longrightarrow & \textit{Wigner semicircular} \end{array}$$

Proof. Calculus if we guess the answer, Stieltjes inversion otherwise.

Conclusion

We have a theory of easy quantum groups, featuring:

(1) Simple axioms: "C must come from partitions".

(2) The quantum groups O_N, O_N^+, U_N, U_N^+ as main examples.

(3) Many other potential examples, e.g. coming from P, NC.

(4) Interesting connections with probability/free probability.

$\Longrightarrow$ next lecture: quantum permutations