Introduction to quantum groups

Teo Banica

Operator algebras, Quantum groups, Representation theory, Diagrams and easiness, Quantum permutations, Matrix models

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Foreword

This is an introduction to quantum groups, focusing on the most basic examples, namely the closed subgroups $G \subset U_N^+$.

We discuss the foundational aspects, and then a number of more specialized topics, of algebraic and probabilistic nature.

These lecture notes consist of slides written in the Summer 2020. Presentations available at my Youtube channel.

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Operator algebras and noncommutative spaces

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Plan

(1) Hilbert spaces, linear operators

(2) Basic spectral/eigenvalue theory

(3) C^*-algebra theory: Gelfand, GNS, FD

(4) Noncommutative spaces: spheres and tori

$\Longrightarrow$ next lecture: quantum groups

Hilbert spaces

<u>Definition</u>. Complex vector space H with $< x, y >$, satisfying:

(1) $< x, y >$ is linear in x, antilinear in y.
(2) $\overline{< x, y >} =< y, x >$, for any x, y.
(3) $< x, x >> 0$, for any $x \neq 0$.
(4) H is complete with respect to $||x|| = \sqrt{< x, x >}$.

Note that (4) is based on Cauchy-Schwarz. Basic examples:

(1) $H = \mathbb{C}^N$, with $< x, y >= \sum_i x_i \bar{y}_i$.
(2) $H = l^2(\mathbb{N})$, with $< x, y >= \sum_i x_i \bar{y}_i$.
(3) $H = L^2(X)$, with $< f, g >= \int_X f(x)\overline{g(x)}dx$.

<u>Gram-Schmidt</u> $\implies$ $H \simeq l^2(I)$. When I is countable, H is called separable. Example: $H = L^2[0, 1]$, cf. Weierstrass.

Operators

Let H be a Hilbert space, with basis $\{e_i\}_{i \in I}$. We have

$$\mathcal{L}(H) \subset M_I(\mathbb{C})$$

with $T : H \to H$ linear corresponding to the following matrix:

$$M_{ij} = < Te_j, e_i >$$

In particular, when $\dim(H) = N < \infty$, we obtain:

$$\mathcal{L}(H) \simeq M_N(\mathbb{C})$$

Also, in the infinite separable case, we obtain:

$$\mathcal{L}(H) \subset M_\infty(\mathbb{C})$$

$\implies$ However, $H = L^2[0,1]$ suggests not to use all this (..)

Bounded operators 1/2

Theorem. Given a Hilbert space H, the linear operators $T : H \to H$ which are bounded, in the sense that

$$||T|| = \sup_{||x|| \leq 1} ||Tx||$$

is finite, form a complex algebra with unit $B(H)$, which:

(1) is complete with respect to $||.||$ (Banach algebra).

(2) has an involution $T \to T^*$, $< Tx, y >=< x, T^*y >$.

The norm and involution are related by $||TT^*|| = ||T||^2$.

Bounded operators 2/2

Proof. Everything here is quite elementary:

(0) Complex algebra with unit: clear.

(1) Norm closed: set $Tx = \lim_{n\to\infty} T_n x$, for any $x \in H$.

(2) Involution: because $\varphi(x) =< Tx, y >$ is linear.

(3) Formula $||TT^*|| = ||T||^2$: double inequality.

Remark. In the matrix setting, $(M^*)_{ij} = \bar{M}_{ji}$.

C^*-algebras

<u>Definition</u>. A C^*-algebra is a complex algebra with unit A, with:

(1) A norm $a \to ||a||$, making it a Banach algebra.

(2) An involution $a \to a^*$, such that $||aa^*|| = ||a||^2$, $\forall a \in A$.

Basic examples: the closed $*$-subalgebras $A \subset B(H)$.
$\implies$ We'll see that any C^*-algebra is of this form.

Also basic: $C(X)$, with X being a compact space.
$\implies$ We'll see that any commutative C^*-algebra is of this form.

Finite dimensional: sums of matrix algebras, $\oplus_i M_{N_i}(\mathbb{C})$.
$\implies$ We'll see that any FD C^*-algebra is of this form.

Spectral theory

Definition. The spectrum of an element $a \in A$ is the set

$$\sigma(a) = \left\{ \lambda \in \mathbb{C} \middle| a - \lambda \notin A^{-1} \right\}$$

where $A^{-1} \subset A$ is the set of invertible elements.

For the matrices, we obtain the eigenvalue set.
For the continuous functions, we obtain the image.

Theorem. $\sigma(ab) = \sigma(ba)$ outside $\{0\}$.

Proof. Indeed, $c = (1 - ab)^{-1} \implies 1 + cba = (1 - ba)^{-1}$.

Remark: in infinite dimensions, $S^*S = 1, SS^* \neq 1$ (shift).

Rational functions 1/2

Given $a \in A$, and a rational function $f = P/Q$ having poles outside $\sigma(a)$, we can construct $f(a) = P(a)Q(a)^{-1}$. We write:

$$f(a) = \frac{P(a)}{Q(a)}$$

Theorem. We have the "rational functional calculus" formula

$$\sigma(f(a)) = f(\sigma(a))$$

valid for any $f \in \mathbb{C}(X)$ having poles outside $\sigma(a)$.

Rational functions 2/2

Case $f \in \mathbb{C}[X]$. With $f(X) - \lambda = c(X - r_1)\ldots(X - r_n)$:

$$\begin{aligned}
\lambda \notin \sigma(f(a)) \quad &\iff \quad c(a - r_1)\ldots(a - r_n) \in A^{-1} \\
&\iff \quad a - r_1, \ldots, a - r_n \in A^{-1} \\
&\iff \quad r_1, \ldots, r_n \notin \sigma(a) \\
&\iff \quad \lambda \notin f(\sigma(a))
\end{aligned}$$

Case $f \in \mathbb{C}(X)$. With $f = P/Q$ and $F = P - \lambda Q$:

$$\begin{aligned}
\lambda \in \sigma(f(a)) \quad &\iff \quad 0 \in \sigma(F(a)) \\
&\iff \quad 0 \in F(\sigma(a)) \\
&\iff \quad \exists \mu \in \sigma(a), F(\mu) = 0 \\
&\iff \quad \lambda \in f(\sigma(a))
\end{aligned}$$

Basic spectra 1/2

Given an element $a \in A$, its spectral radius $\rho(a)$ is the radius of the smallest disk centered at 0 containing $\sigma(a)$.

Theorem. Let A be a C^*-algebra.

(1) The spectrum of a norm 1 element is in the unit disk.

(2) The spectrum of a unitary ($a^* = a^{-1}$) is on the unit circle.

(3) The spectrum of a self-adjoint element ($a = a^*$) is real.

(4) ρ of a normal element ($aa^* = a^*a$) equals its norm.

Basic spectra 2/2

(1) Clear from $(1-a)^{-1} = 1 + a + a^2 + \ldots$, for $||a|| < 1$.

(2) Follows by using $f(z) = z^{-1}$. Indeed, we have:

$$\sigma(a)^{-1} = \sigma(a^{-1}) = \sigma(a^*) = \overline{\sigma(a)}$$

(3) Follows from (2), by using $f(z) = (z + it)/(z - it)$.

(4) By (1) we have $\rho(a) \leq ||a||$. Given $\rho > \rho(a)$, we have:

$$\int_{|z|=\rho} \frac{z^n}{z-a}\, dz = \sum_{k=0}^{\infty} \left(\int_{|z|=\rho} z^{n-k-1} dz \right) a^k = a^{n-1}$$

By applying the norm and taking n-th roots we obtain:

$$\rho \geq \lim_{n \to \infty} ||a^n||^{1/n}$$

When $a = a^*$ we're done. In general, use $||aa^*|| = ||a||^2$.

Gelfand

Theorem. Any commutative C^*-algebra is the form $C(X)$, with its "spectrum" $X = Spec(A)$ consisting of the characters $\chi : A \to \mathbb{C}$.

Proof. Set $X = Spec(A)$, with topology making continuous all the evaluation maps $ev_a : \chi \to \chi(a)$. Then X is a compact space, and $a \to ev_a$ is a morphism of algebras $ev : A \to C(X)$.

(1) ev involutive. Using real + imaginary parts, we must prove that $ev_{a^*} = ev_a^*$ when $a = a^*$. But this follows from $\sigma(a) \subset \mathbb{R}$.

(2) ev isometric. Follows from $||ev_a|| = \rho(a) = ||a||$.

(3) ev surjective. Follows from Stone-Weierstrass.

Continuous calculus

Theorem. Assume that $a \in A$ is normal, and let $f \in C(\sigma(a))$.

(1) We can define $f(a) \in A$, with $f \to f(a)$ being a morphism.

(2) We have the formula $\sigma(f(a)) = f(\sigma(a))$.

Proof. Since a is normal, $B =< a >$ is commutative, and the Gelfand theorem gives $B = C(X)$, with $X = Spec(B)$.

The map $X \to \sigma(a)$ given by evaluation at a being bijective, we have $X = \sigma(a)$. Thus $B = C(\sigma(a))$, and we are done.

Positivity

Theorem. For an element $a \in A$, the following are equivalent:

(1) a is positive, in the sense that $\sigma(a) \subset [0, \infty)$.

(2) $a = b^2$, for some $b \in A$ satisfying $b = b^*$.

(3) $a = cc^*$, for some $c \in A$.

(1) $\Longrightarrow$ (2): $\sigma(a) \subset \mathbb{R}$ implies $a = a^*$, so $< a >$ is commutative, and by using the Gelfand theorem, we can set $b = \sqrt{a}$.

(2) $\Longrightarrow$ (3): this is trivial, because we can set $c = b$.

(3) $\Longrightarrow$ (1): by contradition. By multiplying c by a suitable element of $< cc^* >$, we are led to the existence of an element $d \neq 0$ satisfying $-dd^* \geq 0$. With $d = x + iy$ we have:

$$dd^* + d^*d = 2(x^2 + y^2) \geq 0$$

Thus $d^*d \geq 0$, contradicting $\sigma(dd^*) = \sigma(d^*d)$ outside $\{0\}$.

NC spaces

Definition. Given an arbitrary C^*-algebra A, we write

$$A = C(X)$$

and call X a "noncommutative compact space".

Equivalently, the category of the noncommutative compact spaces is the category of the C^*-algebras, with the arrows reversed.

The idea is that of studying A, but formulating results in terms of X. For instance whenever we have a morphism $\Phi : A \to B$, we can write $A = C(X), B = C(Y)$, and rather speak of the corresponding morphism $\phi : Y \to X$. And so on, up to technical subtleties.

NC spheres

Definition. We have noncommutative spaces, as follows,

$$C(S^{N-1}_{\mathbb{R},+}) = C^*\left(x_1,\ldots,x_N\Big|x_i = x_i^*, \sum_i x_i^2 = 1\right)$$

$$C(S^{N-1}_{\mathbb{C},+}) = C^*\left(x_1,\ldots,x_N\Big|\sum_i x_i x_i^* = \sum_i x_i^* x_i = 1\right)$$

called free real sphere, and free complex sphere.

Here C^* means "universal C^*-algebra generated by".

These universal algebras are well-defined, because we have

$$\sum_i ||x_i||^2 = \sum_i ||x_i x_i^*|| \leq ||\sum_i x_i x_i^*|| = 1$$

and so the biggest C^*-norms on our algebras exist indeed.

Liberation

Theorem. We have embeddings of NC spaces, as follows,

$$\begin{array}{ccc} S^{N-1}_{\mathbb{C}} & \longrightarrow & S^{N-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{R},+} \end{array}$$

and the free spheres are "liberations" of the classical ones.

Proof. We must establish the following isomorphisms:

$$C(S^{N-1}_{\mathbb{R},+}) = C^*_{comm}\left(x_1,\ldots,x_N\Big|x_i=x_i^*,\sum_i x_i^2=1\right)$$

$$C(S^{N-1}_{\mathbb{C},+}) = C^*_{comm}\left(x_1,\ldots,x_N\Big|\sum_i x_ix_i^*=\sum_i x_i^*x_i=1\right)$$

But these isomorphisms are both clear, by using Gelfand.

Tori

<u>Definition.</u> Given $S \subset S^{N-1}_{\mathbb{C},+}$, the subspace $T \subset S$ given by

$$C(T) = C(S) \Big/ \left\langle x_i x_i^* = x_i^* x_i = \frac{1}{N} \right\rangle$$

is called associated torus. In the real case, we call T cube.

As a basic example, for $S = S^{N-1}_{\mathbb{C}}$ we obtain a <u>torus</u>:

$$S = S^{N-1}_{\mathbb{C}} \implies T = \left\{ x \in \mathbb{C}^N \middle| |x_i| = \frac{1}{\sqrt{N}} \right\}$$

Also, for the real sphere $S = S^{N-1}_{\mathbb{R}}$ we obtain a <u>cube</u>:

$$S = S^{N-1}_{\mathbb{R}} \implies T = \left\{ x \in \mathbb{R}^N \middle| x_i = \pm \frac{1}{\sqrt{N}} \right\}$$

Group algebras

Theorem. Let Γ be a discrete group, and consider the complex group algebra $\mathbb{C}[\Gamma]$, with involution given by:

$$g^* = g^{-1} \quad , \quad \forall g \in \Gamma$$

The maximal C^*-seminorm on $\mathbb{C}[\Gamma]$ is then a C^*-norm, and the corresponding closure of $\mathbb{C}[\Gamma]$ is a C^*-algebra, denoted $C^*(\Gamma)$.

Proof. Let $H = l^2(\Gamma)$, having $\{h\}_{h\in\Gamma}$ as orthonormal basis. Our claim is that we have an embedding, as follows:

$$\pi : \mathbb{C}[\Gamma] \subset B(H) \quad , \quad \pi(g)(h) = gh$$

But this is elementary to check, and gives the result.

Group duals

Theorem. When Γ is abelian, we have an isomorphism

$$C^*(\Gamma) \simeq C(G)$$

where $G = \widehat{\Gamma}$ is its dual, formed by the characters $\chi : \Gamma \to \mathbb{T}$.

Proof. Gelfand gives $A = C(X)$, with $X = Spec(A)$. But the spectrum $X = Spec(A)$, made of characters $\chi : C^*(\Gamma) \to \mathbb{C}$, can be identified with the Pontrjagin dual $G = \widehat{\Gamma}$, as desired.

Definition. Given a discrete group Γ, the space G given by

$$C(G) = C^*(\Gamma)$$

is called abstract dual of Γ, and is denoted $G = \widehat{\Gamma}$.

Back to tori

Theorem. The tori of the basic spheres are all group duals,

$$\begin{array}{ccc}
\mathbb{T}^N & \longrightarrow & \widehat{F_N} \\
\uparrow & & \uparrow \\
\mathbb{Z}_2^N & \longrightarrow & \widehat{\mathbb{Z}_2^{*N}}
\end{array}$$

where F_N is the free group, and $*$ is a free product.

Proof. The diagram formed by the algebras $C(T)$ is:

$$\begin{array}{ccc}
C^*(\mathbb{Z}^N) & \longleftarrow & C^*(\mathbb{Z}^{*N}) \\
\downarrow & & \downarrow \\
C^*(\mathbb{Z}_2^N) & \longleftarrow & C^*(\mathbb{Z}_2^{*N})
\end{array}$$

But this gives the result, via some standard identifications.

Summary

(1) C^*-algebras: with norm and involution, $||aa^*|| = ||a||^2$.

(2) Gelfand theorem: commutative case $A = C(X)$.

(3) Noncommutative geometry: write $A = C(X)$ in general.

(4) Examples: NC spheres (real, complex) and tori (group duals).

$\Longrightarrow$ We'll be back to NCG later, doing quantum groups

Embeddings

We want to prove that any C^*-algebra appears as $A \subset B(H)$.

Theorem. Assume that A is commutative, $A = C(X)$, and let μ be a positive measure on X. We have then an embedding

$$A \subset B(H)$$

where $H = L^2(X)$, with $f \in A$ corresponding to $T_f : g \to fg$.

Proof. T_f is well-defined, and bounded as well, because:

$$||fg||_2 = \sqrt{\int_X |f(x)|^2 |g(x)|^2 d\mu(x)} \leq ||f||_\infty ||g||_2$$

We obtain in this way $A \subset B(H)$, as claimed.

Forms

In general, we can replace the positive measures μ with the corresponding integration functionals.

Definition. Consider a linear map $\varphi : A \to \mathbb{C}$.

(1) φ is called positive when $a \geq 0 \implies \varphi(a) \geq 0$.

(2) φ is called faithful and positive if $a \geq 0, a \neq 0 \implies \varphi(a) > 0$.

In the commutative case, $A = C(X)$, we can write:

$$\varphi(f) = \int_X f(x) d\mu(x)$$

In general, the philosophy is similar.

GNS construction

Theorem. Let $\varphi : A \to \mathbb{C}$ be a positive linear form.

(1) $< a, b >= \varphi(ab^*)$ defines a generalized scalar product on A.

(2) By separating and completing we obtain a Hilbert space H.

(3) $\pi(a) : b \to ab$ defines a representation $\pi : A \to B(H)$.

(4) If φ is faithful in the above sense, then π is faithful.

Proof. Almost everything here is straightforward, and the last assertion follows from a positivity trick, namely:

$$a \neq 0 \implies \pi(aa^*) \neq 0 \implies \pi(a) \neq 0$$

Existence

In order to establish the GNS theorem, it remains to prove that any C^*-algebra has a faithful and positive linear form $\varphi : A \to \mathbb{C}$.

Theorem. Let A be a C^*-algebra.

(1) Any positive linear form $\varphi : A \to \mathbb{C}$ is continuous.

(2) φ is positive iff there is a norm one $h \in A_+$, $||\varphi|| = \varphi(h)$.

(3) $\forall a \in A$ there exists φ positive of norm 1, $\varphi(aa^*) = ||a||^2$.

(4) If A is separable there is a faithful positive form $\varphi : A \to \mathbb{C}$.

Proof of (1,2)

(1) This follows from $|\varphi(a)| \leq ||\pi(a)||\varphi(1) \leq ||a||\varphi(1)$.

(2) Let $a \in A_+$, $||a|| \leq 1$. We have then:

$$|\varphi(h) - \varphi(a)| \leq ||\varphi|| \cdot ||h - a|| \leq \varphi(h)1 = \varphi(h)$$

Thus $Re(\varphi(a)) \geq 0$. We must prove $a = a^* \implies \varphi(a) \in \mathbb{R}$.
We can assume $h = 1$. With $a = a^*$, for $t \in \mathbb{R}$ we have:

$$|\varphi(1 + ita)|^2 \leq \varphi(1)^2(1 + t^2||a||^2)$$

On the other hand with $\varphi(a) = x + iy$ we have:

$$|\varphi(1 + ita)| \geq (\varphi(1) - ty)^2$$

We therefore obtain that for any $t \in \mathbb{R}$ we have:

$$\varphi(1)^2(1 + t^2||a||^2) \geq (\varphi(1) - ty)^2$$

Thus we have $y = 0$, and this finishes the proof.

Proof of (3,4)

(3) This follows from (2), and from Hahn-Banach.

(4) Let (a_n) be a dense sequence inside A. For any n we construct a positive form satisfying $\varphi_n(a_n a_n^*) = ||a_n||^2$, and then we set:

$$\varphi = \sum_{n=1}^{\infty} \frac{\varphi_n}{2^n}$$

Let $a \in A$ be a nonzero element. Pick a_n close to a and consider the GNS pair (H, π) associated to (A, φ_n). We have:

$$\begin{aligned}
\varphi_n(aa^*) &= ||\pi(a)1|| \\
&\geq ||\pi(a_n)1|| - ||a - a_n|| \\
&= ||a_n|| - ||a - a_n|| \\
&> 0
\end{aligned}$$

Thus $\varphi_n(aa^*) > 0$, and so $\varphi(aa^*) > 0$, and we are done.

GNS theorem

Theorem. Let A be a C^*-algebra.

(1) A appears as $A \subset B(H)$, for some Hilbert space H.

(2) When A is separable, H can be chosen to be separable.

(3) When A is FD, the space H can be chosen to be FD.

Proof. Follows indeed by performing the GNS construction. $\square$

Finite dimensions

Theorem. Let $A \subset M_N(\mathbb{C})$ be a C^*-algebra.

(1) We have $1 = p_1 + \ldots + p_k$, with $p_i \in A$ minimal projections.

(2) The spaces $A_i = p_i A p_i$ are non-unital $*$-subalgebras of A.

(3) We have a non-unital $*$-algebra sum $A = A_1 \oplus \ldots \oplus A_k$.

(4) Unital $*$-algebra isomorphisms $A_i \simeq M_{N_i}(\mathbb{C})$, $N_i = rank(p_i)$.

(5) Thus, we can decompose $A \simeq M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$.

(6) This holds in fact for any finite dimensional C^*-algebra.

Proof. (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5) $\implies$ (6).

Conclusions

C^*-algebras: algebras with norm and involution, $||aa^*|| = ||a||^2$.

(1) Gelfand theorem: commutative case $A = C(X)$.

(2) Gelfand-Naimark-Segal theorem: $A \subset B(H)$.

(3) Finite dimensions: $A = M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$.

$\Longrightarrow$ More basic theory: von Neumann algebras.

$\Longrightarrow$ $A = C(X)$. Spheres and tori. What about groups?

Compact and discrete quantum groups

Teo Banica

"Introduction to quantum groups", 2/6

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Plan

(1) Compact quantum groups

(2) Discrete quantum groups

(3) Basic examples, operations

(4) Quantum isometry groups

$\Longrightarrow$ next lecture: representations

Operator algebras

C^*-algebras: with norm and involution, $||aa^*|| = ||a||^2$.

(1) Gelfand theorem: commutative case $A = C(X)$.

(2) Gelfand-Naimark-Segal theorem: $A \subset B(H)$.

(3) Finite dimensions: $A = M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$.

$\implies$ $A = C(X)$, with X "noncommutative compact space"

$\implies$ NC spheres, NC tori. What about quantum groups?

Classical groups

Let G be a compact Lie group. Then $G \subset U_N$. Multiplication:

$$(UV)_{ij} = \sum_k U_{ik} V_{kj}$$

By Stone-Weierstrass we have $C(G) = < u_{ij} >$, where:

$$u_{ij}(U) = U_{ij}$$

The multiplication $G \times G \to G$ transposes as:

$$u_{ij} \to \sum_k u_{ik} \otimes u_{kj}$$

Thus G is well described by $C(G)$, together with $u = (u_{ij})$.

Axioms

Let A be a C^*-algebra, with $u \in M_N(A)$ biunitary (u, u^t unitaries), whose entries generate A, such that:

– $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ defines a morphism $\Delta : A \to A \otimes A$
– $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : A \to \mathbb{C}$
– $S(u_{ij}) = u_{ji}^*$ defines a morphism $S : A \to A^{opp}$

We write then $A = C(G) = C^*(\Gamma)$, and call:

– G a compact quantum group
– Γ a discrete quantum group

[axioms due to Woronowicz, 1987, slightly modified here]

Compact groups 1/2

Theorem. For a closed subgroup $G \subset U_N$, the algebra $A = C(G)$, with the matrix formed by the standard coordinates

$$u_{ij}(g) = g_{ij}$$

is a Woronowicz algebra, with structural maps given by

$$\Delta = m^T \quad , \quad \varepsilon = u^T \quad , \quad S = i^T$$

where m, u, i are the multiplication, unit and inverse of G.

Any commutative Woronowicz algebra appears in this way.

Compact groups 2/2

Proof. We compute m^T, u^T, i^T. We have:

$$m^T(u_{ij})(U\otimes V)=(UV)_{ij}=\sum_k U_{ik}V_{kj}=\sum_k(u_{ik}\otimes u_{kj})(U\otimes V)$$

Regarding now u^T, here we have:

$$u^T(u_{ij})=1_{ij}=\delta_{ij}$$

As for the map i^T, this is given by:

$$i^T(u_{ij})(U)=(U^{-1})_{ij}=\bar{U}_{ji}=u_{ji}^*(U)$$

Thus the axioms are satisfied, with $\Delta=m^T,\varepsilon=u^T,S=i^T$.

Finally, the last assertion follows by applying Gelfand.

Group duals 1/2

Theorem. For a discrete group $\Gamma = < g_1, \ldots, g_N >$, the algebra $A = C^*(\Gamma)$, with the diagonal matrix formed by the generators

$$u = diag(g_1, \ldots, g_N)$$

is a Woronowicz algebra, with structural maps given by

$$\Delta(g) = g \otimes g \quad , \quad \varepsilon(g) = 1 \quad , \quad S(g) = g^{-1}$$

for any group element $g \in \Gamma$. This algebra is cocommutative, in the sense that $\Sigma\Delta = \Delta$, where $\Sigma(a \otimes b) = b \otimes a$ is the flip.

Remark. We'll see later that any cocommutative Woronowicz algebra appears in this way (needs representation theory).

Group duals 2/2

Proof. Consider the following unitary representation:

$$\Gamma \to C^*(\Gamma) \otimes C^*(\Gamma) \quad , \quad g \to g \otimes g$$

This produces a map $\Delta : C^*(\Gamma) \to C^*(\Gamma) \otimes C^*(\Gamma)$, given by:

$$\Delta(g) = g \otimes g$$

Similarly, ε comes from the trivial representation:

$$\Gamma \to \{1\} \quad , \quad g \to 1$$

As for S, this comes from the following representation:

$$\Gamma \to C^*(\Gamma)^{opp} \quad , \quad g \to g^{-1}$$

Remark. Note that the use of the opposite algebra is needed.

Comments 1/4

Assume that Γ is abelian, and let $G = \widehat{\Gamma}$ be its Pontrjagin dual, formed by the characters $\chi : \Gamma \to \mathbb{T}$. The isomorphism

$$C^*(\Gamma) \simeq C(G)$$

transforms the structural maps of $C^*(\Gamma)$, given by

$$\Delta(g) = g \otimes g \quad , \quad \varepsilon(g) = 1 \quad , \quad S(g) = g^{-1}$$

into the structural maps of $C(G)$, given by:

$$\Delta\varphi(g, h) = \varphi(gh) \quad , \quad \varepsilon(\varphi) = \varphi(1) \quad , \quad S\varphi(g) = \varphi(g^{-1})$$

Thus, $G = \widehat{\Gamma}$ is a compact quantum group isomorphism.

Comments 2/4

Motivated by this, given a Woronowicz algebra

$$A = C(G) = C^*(\Gamma)$$

we say that G, Γ are dual to each other, and write:

$$G = \widehat{\Gamma} \quad , \quad \Gamma = \widehat{G}$$

This duality extends the usual Pontrjagin duality.

Comments 3/4

Motivated by the compact Lie group case, we have:

Definition. Given $A = C(G)$, we denote by $\mathcal{A} \subset A$ the dense $*$-algebra generated by the coordinates u_{ij}, and we write

$$\mathcal{A} = C^\infty(G)$$

and call it "algebra of smooth functions" on G.

Example. For $A = C^*(\Gamma)$ we have $\mathcal{A} = \mathbb{C}[\Gamma]$.

Comments 4/4

Motivated by the group dual case, we have:

Definition. We agree to identify (A, u) and (B, v) when we have a $*$-algebra isomorphism

$$\mathcal{A} \simeq \mathcal{B}$$

mapping standard coordinates to standard coordinates, $u_{ij} \to v_{ij}$.

Example. This identifies for instance $C^*(\Gamma)$ with $C^*_{red}(\Gamma)$.

Summary

(1) We are looking at pairs (A,u), with $u\in M_N(A)$ biunitary, with:

$$\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}\quad,\quad\varepsilon(u_{ij})=\delta_{ij}\quad,\quad S(u_{ij})=u_{ji}^*$$

(2) We have compact and discrete quantum groups, given by:

$$A=C(G)=C^*(\Gamma)$$

(3) These quantum groups are dual to each other, and we write:

$$G=\widehat{\Gamma}\quad,\quad\Gamma=\widehat{G}$$

(4) We set $C^\infty(G)=<u_{ij}>$, and we use the identifications:

$$C^\infty(G)\simeq C^\infty(H)\quad,\quad u_{ij}\to v_{ij}$$

(5) All this is supported by C^*-algebras, and the above results.

Tech 1/2

Theorem. The comultiplication Δ, counit ε and antipode S satisfy the following conditions,

(1) Coassociativity: $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$.

(2) Counitality: $(id \otimes \varepsilon)\Delta = (\varepsilon \otimes id)\Delta = id$.

(3) Coinversality: $m(id \otimes S)\Delta = m(S \otimes id)\Delta = \varepsilon(.)1$.

on the dense $*$-subalgebra $\mathcal{A} \subset A$ generated by the variables u_{ij}.

Proof. Clear on coordinates, and so on the $*$-algebra $\mathcal{A}$.

Tech 2/2

Remark. In the commutative case, $G \subset U_N$, we have

$$\Delta = m^T \quad , \quad \varepsilon = u^T \quad , \quad S = i^T$$

and the 3 conditions satisfied by Δ, ε, S come by transposition from the basic 3 conditions satisfied by m, u, i, namely

$$m(m \times id) = m(id \times m)$$

$$m(id \times u) = m(u \times id) = id$$

$$m(id \otimes i)\delta = m(i \otimes id)\delta = 1$$

whre $\delta(g) = (g, g)$. In general, the philosophy is the same.

1. Products

Given two compact quantum groups G, H, so is their product $G \times H$, constructed as follows:

$$C(G \times H) = C(G) \otimes C(H)$$

Equivalently, at the level of the associated discrete quantum groups Γ, Λ, which are dual to G, H, we have:

$$C^*(\Gamma \times \Lambda) = C^*(\Gamma) \otimes C^*(\Lambda)$$

As an illustration, we have things of type $G \times \widehat{\Lambda}$, with G, Λ both classical, which are not classical, nor group duals.

2. Dual free products

Given two compact quantum groups G, H, so is their dual free product $G \,\hat{*}\, H$, constructed as follows:

$$C(G \,\hat{*}\, H) = C(G) * C(H)$$

Equivalently, at the level of the associated discrete quantum groups Γ, Λ, which are dual to G, H, we have a usual free product:

$$C^*(\Gamma * \Lambda) = C^*(\Gamma) * C^*(\Lambda)$$

This construction always produces non-classical quantum groups, unless of course $G = \{1\}$ or $H = \{1\}$.

3. Free complexification

Given a compact quantum group G, we can construct its free complexification $\widetilde{G}$ as follows, where $z = id \in C(\mathbb{T})$:

$$C(\widetilde{G}) \subset C(\mathbb{T}) * C(G) \quad , \quad \tilde{u} = zu$$

Equivalently, at the level of the associated discrete duals $\Gamma, \widetilde{\Gamma}$, we have the following formula, where $z = 1 \in \mathbb{Z}$:

$$C^*(\widetilde{\Gamma}) \subset C^*(\mathbb{Z}) * C^*(\Gamma) \quad , \quad \tilde{u} = zu$$

We'll see later that the "free analogues" of O_N, U_N are related by free complexification. Simpler than for O_N, U_N themselves (!)

4. Subgroups

Let G be compact quantum group, and let $I \subset C(G)$ be a closed $*$-ideal satisfying the following "Hopf ideal" condition:

$$\Delta(I) \subset C(G) \otimes I + I \otimes C(G)$$

We have then a closed subgroup $H \subset G$, as follows:

$$C(H) = C(G)/I$$

Dually, we obtain a quotient of discrete quantum groups:

$$\widehat{\Gamma} \to \widehat{\Lambda}$$

In all this the Hopf ideal condition is needed for Δ to factorize.

5. Quotients

Let us call "corepresentation" of a Woronowicz algebra $A = C(G)$ any unitary matrix $w \in M_n(\mathcal{A})$ satisfying:

$$\Delta(w_{ij}) = \sum_k w_{ik} \otimes w_{kj} \quad , \quad \varepsilon(w_{ij}) = \delta_{ij} \quad , \quad S(w_{ij}) = w_{ji}^*$$

In this situation, we have a quotient group $G \to H$, given by:

$$C(H) = < w_{ij} >$$

At the dual level we obtain a discrete quantum subgroup:

$$\widehat{\Lambda} \subset \widehat{\Gamma}$$

We will be back later to corepresentations, with a full theory.

6. Projective version

Given a quantum group G, with fundamental corepresentation $u = (u_{ij})$, the $N^2 \times N^2$ matrix given in double indices by

$$w_{ia,jb} = u_{ij} u_{ab}^*$$

is a corepresentation, and the following happen:

(1) The corresponding quotient $G \to PG$ is a quantum group.

(2) In the classical case, $G \subset U_N$, we have $PG = G/(G \cap \mathbb{T}^N)$.

(3) For the group duals, $\Gamma =< g_i >$, we have $\widehat{P\Gamma} =< g_i g_j^{-1} >$.

Summary

The compact quantum groups are subject to making:

1. Products $G \times H$
2. Dual free products $G \,\hat{*}\, H$
3. Free complexification $G \rightsquigarrow \widetilde{G}$
4. Subgroups $H \subset G$
5. Quotients $G \to H$
6. Projective versions $G \to PG$

However, as "basic input" we only have groups, and group duals.

Liberations 1/4

Theorem. We have quantum groups defined via

$$C(O_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big| u=\bar{u}, u^t=u^{-1}\right)$$

$$C(U_N^+) = C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big| u^*=u^{-1}, u^t=\bar{u}^{-1}\right)$$

called free orthogonal, and free unitary quantum groups.

Proof. If u is biunitary/orthogonal, so are the matrices

$$(u^\Delta)_{ij}=\sum_k u_{ik}\otimes u_{kj} \quad , \quad (u^\varepsilon)_{ij}=\delta_{ij} \quad , \quad (u^S)_{ij}=u_{ji}^*$$

and so we can construct Δ, ε, S, by universality.

Liberations 2/4

The quantum groups O_N^+, U_N^+ have the following properties:

(1) The closed subgroups $G \subset U_N^+$ are exactly the $N \times N$ compact quantum groups.

(2) As for the closed subgroups $G \subset O_N^+$, these are exactly those satisfying $u = \bar{u}$.

(3) We have embeddings $O_N \subset O_N^+$ and $U_N \subset U_N^+$, obtained by dividing $C(O_N^+), C(U_N^+)$ by their commutator ideals.

Liberations 3/4

Theorem. The following inclusions are proper, at any $N \geq 2$:

$$\begin{array}{ccc} U_N & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & O_N^+ \end{array}$$

Proof. Follows by looking at group dual subgroups. Indeed, we have

$$\widehat{L}_N \subset O_N^+ \quad , \quad \widehat{F}_N \subset U_N^+$$

where $L_N = \mathbb{Z}_2^{*N}$, and where $F_N = \mathbb{Z}^{*N}$ is the free group.

Remark. We have a connection here with the "free tori".

Liberations 4/4

Theorem. We have intermediate liberations as follows,

$$\begin{array}{ccccc}
U_N & \longrightarrow & U_N^* & \longrightarrow & U_N^+\\
\uparrow & & \uparrow & & \uparrow\\
O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+
\end{array}$$

with $*$ meaning that u_{ij}, u_{ij}^* must satisfy the relations $abc = cba$.

Proof. If the entries of u "half-commute", so do the entries of

$$(u^\Delta)_{ij} = \sum_k u_{ik} \otimes u_{kj} \quad , \quad (u^\varepsilon)_{ij} = \delta_{ij} \quad , \quad (u^S)_{ij} = u_{ji}^*$$

so we can construct indeed Δ, ε, S. More can be said here (..)

Affine isometries

Question. Are our quantum groups compatible with the spheres?

Definition. Given an algebraic manifold $X \subset S^{N-1}_{\mathbb{C}}$, the formula

$$G(X) = \left\{ U \in U_N \Big| U(X) = X \right\}$$

defines a compact group of unitary matrices (a.k.a. isometries), called affine isometry group of X.

$\implies$ For the classical spheres $S^{N-1}_{\mathbb{R}}, S^{N-1}_{\mathbb{C}}$ we obtain in this way the classical groups O_N, U_N.

Quantum isometries

Given an algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$, the category of the closed subgroups $G \subset U_N^+$ acting affinely on X, in the sense that

$$\Phi(x_i) = \sum_a u_{ia} \otimes x_a$$

defines a morphism of C^*-algebras, as follows,

$$\Phi : C(X) \to C(G) \otimes C(X)$$

has a universal object, denoted $G^+(X)$, and called "affine quantum isometry group" of X. This is indeed routine algebra.

Rotations and spheres 1/2

Theorem. The quantum isometry groups of the basic spheres,

$$\begin{array}{ccccc}
S^{N-1}_{\mathbb{C}} & \to & S^{N-1}_{\mathbb{C},*} & \to & S^{N-1}_{\mathbb{C},+}\\
\uparrow & & \uparrow & & \uparrow\\
S^{N-1}_{\mathbb{R}} & \to & S^{N-1}_{\mathbb{R},*} & \to & S^{N-1}_{\mathbb{R},+}
\end{array}$$

are the basic orthogonal and unitary quantum groups, namely:

$$\begin{array}{ccccc}
U_N & \to & U_N^* & \to & U_N^+\\
\uparrow & & \uparrow & & \uparrow\\
O_N & \to & O_N^* & \to & O_N^+
\end{array}$$

Rotations and spheres, 2/2

<u>Proof</u>. The variables $X_i = \sum_a u_{ia} \otimes x_a$ satisfy

$$\sum_i X_i X_i^* = \sum_{iab} u_{ia} u_{ib}^* \otimes x_a x_b^* = \sum_a 1 \otimes x_a x_a^* = 1 \otimes 1$$

$$\sum_i X_i^* X_i = \sum_{iab} u_{ia}^* u_{ib} \otimes x_a^* x_b = \sum_a 1 \otimes x_a^* x_a = 1 \otimes 1$$

so we have an action $U_N^+ \curvearrowright S^{N-1}_{\mathbb{C},+}$.

If the variables are u_{ij} are real, or half-commute, or commute, so do the variables X_i. Thus, we have actions everywhere.

Some routine work shows that all these actions are universal.

Conclusion

We have a theory of compact/discrete quantum groups, featuring:

(1) Simple axioms for the algebras $A = C(G) = C^*(\Gamma)$.

(2) The duality formulae $G = \widehat{\Gamma}$ and $\Gamma = \widehat{G}$ well understood.

(3) Manipulations with Δ, ε, S as our main tool, at least so far.

(4) Many examples (various liberations, standard operations).

(5) Compatibility of all this with the noncommutative tori/spheres.

$\Longrightarrow$ next lecture: representation theory

Haar measure and Peter-Weyl theory

Teo Banica

"Introduction to quantum groups", 3/6

06/20

Plan

(1) Representations

(2) The Haar measure

(3) Peter-Weyl theory

(4) Kesten amenability

$\Longrightarrow$ next lecture: Tannakian duality

Quantum groups

Axioms. Let A be a C^*-algebra, with $u \in M_N(A)$ biunitary (u, u^t unitaries), whose entries generate A, such that:

– $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ defines a morphism $\Delta : A \to A \otimes A$
– $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : A \to \mathbb{C}$
– $S(u_{ij}) = u_{ji}^*$ defines a morphism $S : A \to A^{opp}$

We write then $A = C(G) = C^*(\Gamma)$, with G compact quantum group, and Γ discrete quantum group [Woronowicz 87].

Examples. Compact Lie groups, discrete group duals (NC tori), liberations and half-liberations, product operations..

Tools. Comultiplication, counit and antipode Δ, ε, S, in analogy with multiplication, unit and inverse m, u, i.

Representations 1/4

Definition. A corepresentation of a Woronowicz algebra A is a biunitary matrix $v \in M_n(\mathcal{A})$ satisfying

– $\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$
– $\varepsilon(v_{ij}) = \delta_{ij}$
– $S(v_{ij}) = v_{ji}^*$

where $\mathcal{A} \subset A$ is the dense $*$-subalgebra of "smooth elements".

Examples. 1 (trivial), u (fundamental), $\bar{u}$ (conjugate).

Idea. The corepresentations of $A = C(G)$ can be thought of as corresponding to the representations of G.

Representations 2/4

Theorem. Given a closed subgroup $G \subset U_N$, the corepresentations of $C(G)$ are in one-to-one correspondence, given by

$$\pi(g) = \begin{pmatrix} v_{11}(g) & \ldots & v_{1n}(g) \\ \vdots & & \vdots \\ v_{n1}(g) & \ldots & v_{nn}(g) \end{pmatrix}$$

with the finite dimensional unitary smooth representations of G.

Proof. Same computations as when proving that $A = C(G)$ is a Woronowicz algebra, which was already done.

Representations 3/4

Theorem. The corepresentations of a given Woronowicz algebra A are subject to the following operations:

(1) Making sums, $v + w = diag(v, w)$.

(2) Making tensor products, $(v \otimes w)_{ia,jb} = v_{ij} w_{ab}$.

(3) Taking conjugates, $(\bar{v})_{ij} = v_{ij}^*$.

(4) Spinning, $w = UvU^*$, with $U \in U_n$.

Proof. All this is elementary, coming from definitions.

Representations 4/4

Theorem. Given a discrete group $\Gamma = < g_1, \ldots, g_N >$, the corepresentations of $A = C^*(\Gamma)$ are as follows:

(1) Any group element $h \in \Gamma$ is a 1D corepresentation of A, and the operations are the usual ones on group elements.

(2) Any diagonal matrix of type $v = diag(h_1, \ldots, h_n)$, with $n \in \mathbb{N}$, and with $h_1, \ldots, h_n \in \Gamma$, is a corepresentation of A.

(3) More generally, any matrix $w = Udiag(h_1, \ldots, h_n)U^*$ with $h_1, \ldots, h_n \in \Gamma$ and with $U \in U_n$, is a corepresentation of A.

Proof. Follows from $\Delta(h) = h \otimes h$, $\varepsilon(h) = 1$, $S(h) = h^{-1}$.

Comment. We'll see later that (3) gives all corepresentations.

Theory 1/6

Definition. Given corepresentations $v \in M_n(A), w \in M_m(A)$, we set

$$Hom(v,w) = \left\{ T \in M_{m\times n}(\mathbb{C}) \middle| Tv = wT \right\}$$

and we use the following conventions:

(1) $Fix(v) = Hom(1, v)$ and $End(v) = Hom(v, v)$.

(2) $v \sim w$ when $Hom(v, w)$ contains an invertible element.

(3) v is called irreducible, $v \in Irr(G)$, when $End(v) = \mathbb{C}1$.

Theory 2/6

Theorem. We have the following results:

$$T \in Hom(u,v), S \in Hom(v,w) \implies ST \in Hom(u,w)$$

$$S \in Hom(p,q), T \in Hom(v,w) \implies S \otimes T \in Hom(p \otimes v, q \otimes w)$$

$$T \in Hom(v,w) \implies T^* \in Hom(w,v)$$

In other words, the Hom spaces form a tensor $*$-category.

Proof. All this is elementary, coming from definitions.

Comment. We'll be back to this later (Tannakian duality).

Theory 3/6

Theorem. Let $B \subset M_N(\mathbb{C})$ be a C^*-algebra.

(1) We have $1 = p_1 + \ldots + p_k$, with $p_i \in B$ minimal projections.

(2) The spaces $B_i = p_i B p_i$ are non-unital $*$-subalgebras of B.

(3) We have a non-unital $*$-algebra sum $B = B_1 \oplus \ldots \oplus B_k$.

(4) Unital $*$-algebra isomorphisms $B_i \simeq M_{N_i}(\mathbb{C})$, $N_i = rank(p_i)$.

(5) Thus, we can decompose $B \simeq M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$.

(6) This holds in fact for any finite dimensional C^*-algebra.

Proof. This is something that we already know from lecture 1, the idea being (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5) $\implies$ (6).

Theory 4/6

Theorem (PW1). Any corepresentation $v \in M_n(A)$ decomposes as a direct sum of irreducible corepresentations

$$v = v_1 + \ldots + v_k$$

with each v_i being obtained by restricting v to $Im(p_i)$, where $1 = p_1 + \ldots + p_k$ is the partition of unity for $B = End(v)$.

Proof. (1) Let $\Phi : \mathbb{C}^n \to A \otimes \mathbb{C}^n$, $\Phi(e_i) = \sum_j v_{ij} \otimes e_j$. If $V \subset \mathbb{C}^n$ is invariant, $\Phi(V) \subset A \otimes V$, then $\Phi_{|V} : V \to A \otimes V$ is a coaction too, which must come from a subcorepresentation $w \subset v$.

(2) Given $p \in End(v)$, $V = Im(p)$ must be invariant, coming from $w \subset v$, and $p \to w$ maps subprojections to subcorepresentations, and minimal projections to irreducible corepresentations.

(3) With these preliminaries in hand, the result follows.

Theory 5/6

Definition. We denote by $u^{\otimes k}$, with $k = \circ\bullet\bullet\circ\ldots$ being a colored integer, the various tensor products between $u, \bar{u}$, with the rules

$$u^{\otimes\emptyset} = 1 \quad , \quad u^{\otimes\circ} = u \quad , \quad u^{\otimes\bullet} = \bar{u}$$

along with multiplicativity condition

$$u^{\otimes kl} = u^{\otimes k} \otimes u^{\otimes l}$$

and call them Peter-Weyl corepresentations.

Remarks. In the real case, $u = \bar{u}$, we can assume $k \in \mathbb{N}$. In the classical case, we can assume, up to equivalence, $k \in \mathbb{N} \times \mathbb{N}$.

Theory 6/6

Theorem (PW2). Each irreducible corepresentation of A appears inside a Peter-Weyl corepresentation $u^{\otimes k}$.

Proof. Given a corepresentation $v \in M_n(A)$, consider its space of coefficients, $C(v) = span(v_{ij})$. Then $v \to C(v)$ is functorial, mapping subcorepresentations into subspaces. We have:

$$\mathcal{A} = \sum_{k \in \mathbb{N} * \mathbb{N}} C(u^{\otimes k})$$

We have $C(v) \subset \mathcal{A}$, and so, for certain exponents $k_1, \ldots, k_p$:

$$C(v) \subset C(u^{\otimes k_1} \oplus \ldots \oplus u^{\otimes k_p})$$

Thus $v \subset u^{\otimes k_1} \oplus \ldots \oplus u^{\otimes k_p}$, and PW1 gives the result.

Summary

We are interested in the FD unitary smooth representations of G. These come from the biunitary matrices $v \in M_n(\mathcal{A})$ satisfying:

- $\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$
- $\varepsilon(v_{ij}) = \delta_{ij}$
- $S(v_{ij}) = v_{ji}^*$

As basic examples we have $1, u, \bar{u}$, and more generally the PW corepresentations $u^{\otimes k}$, with k colored integer.

The corepresentations decompose into irreducibles (PW1) and the irreducibles can be obtained by splitting the $u^{\otimes k}$ (PW2).

Haar measure 1/8

Theorem. The algebra $A = C(G)$ with $G \subset U_N$, has a unique faithful positive unital linear form $\int_G : A \to \mathbb{C}$ satisfying:

$$\int_G f(xy)dx = \int_G f(yx)dx = \int_G f(x)dx$$

This can be constructed by starting with any faithful positive unital form $\varphi \in A^*$, and taking the Cesàro limit

$$\int_G = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi^{*k}$$

where the convolution operation is $\phi * \psi = (\phi \otimes \psi)\Delta$.

Proof. Well-known, and we'll reprove it anyway.

Haar measure 2/8

Definition. Given a Woronowicz algebra $A = C(G)$, a positive unital tracial state $\int_G : A \to \mathbb{C}$ subject to the conditions

$$\left(\int_G \otimes id\right)\Delta = \left(id \otimes \int_G\right)\Delta = \int_G (.)1$$

called left/right invariance, is called Haar integration over G.

Remark. In the classical case, $G \subset U_N$, we know that $\int_G$ exists, is unique, and can be constructed via a Cesàro limit.

Haar measure 3/8

Theorem. Given a discrete group $\Gamma = < g_1, \ldots, g_N >$, the algebra $A = C^*(\Gamma)$ has a Haar functional, given on group elements by:

$$\int_{\widehat{\Gamma}} g = \delta_{g,1}$$

This functional is faithful on the image on $C^*(\Gamma)$ in the regular representation. In the abelian case, this is the counit of $C(\widehat{\Gamma})$.

Proof. Consider indeed the left regular representation:

$$\pi : C^*(\Gamma) \to B(l^2(\Gamma)) \quad , \quad \pi(g)(h) = gh$$

The composition $\int_{\widehat{\Gamma}}$ of π with $T \to < T1, 1 >$ is given by:

$$\int_{\widehat{\Gamma}} g = < g1, 1 > = \delta_{g,1}$$

But this gives all the assertions, the last one being clear too.

Haar measure 4/8

Theorem. Given an arbitrary unital linear form $\varphi \in A^*$, the limit

$$\int_\varphi a = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi^{*k}(a)$$

exists, and for a corepresentation coefficient $a = (\tau \otimes id)v$, we have

$$\int_\varphi a = \tau(P)$$

where P is the projection onto the 1-eigenspace of $(id \otimes \varphi)v$.

Proof. This is linear algebra, on the space of coefficients of v.

Haar measure 5/8

Theorem. Given a faithful unital linear form $\varphi \in A^*$, the limit

$$\int_\varphi a = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \varphi^{*k}(a)$$

exists, and is independent of φ, given on coefficients by

$$\left(id \otimes \int_\varphi\right) v = P$$

where P is the projection onto $Fix(v) = \{\xi \in \mathbb{C}^n | v\xi = \xi\}$.

Proof. With $M = (id \otimes \varphi)v$ we must prove that $M\xi = \xi$ implies $v\xi = \xi$. But this follows via a standard positivity trick.

Haar measure 6/8

Assume indeed $M\xi = \xi$, and consider the following element:

$$a = \sum_i \left(\sum_j v_{ij}\xi_j - \xi_i \right) \left(\sum_k v_{ik}\xi_k - \xi_i \right)^*$$

We must prove that $a = 0$. Since v is biunitary, we have:

$$\begin{aligned} a &= \sum_i \left(\sum_j \left(v_{ij}\xi_j - \frac{1}{N}\xi_i \right) \right) \left(\sum_k \left(v_{ik}^*\bar{\xi}_k - \frac{1}{N}\bar{\xi}_i \right) \right) \\ &= 2(||\xi||^2 - Re(< v\xi, \xi >)) \end{aligned}$$

By using now $M\xi = \xi$, we obtain from this $\varphi(a) = 0$. Now since φ is faithful, this gives $a = 0$, and so $v\xi = \xi$, as desired.

Haar measure 7/8

Theorem. Any Woronowicz algebra has a unique Haar integration functional, which can be constructed by starting with any faithful positive unital state $\varphi \in A^*$, and setting

$$\int_G = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \varphi^{*k}$$

where $\phi * \psi = (\phi \otimes \psi)\Delta$. Moreover, for any corepresentation v,

$$\left(id \otimes \int_G \right) v = P$$

where P is the projection onto $Fix(v) = \{\xi \in \mathbb{C}^n | v\xi = \xi\}$.

Proof. We already know all this, modulo a few extra minor things.

Haar measure 8/8

Theorem. We have the following results:

(1) For a product $G \times H$, we have $\int_{G\times H} = \int_G \otimes \int_H$.

(2) For a dual free product $G \,\hat{*}\, H$, we have $\int_{G \,\hat{*}\, H} = \int_G * \int_H$.

(3) For a quotient $G \to H$, we have $\int_H = \left(\int_G\right)_{|C(H)}$.

(4) For a projective version $G \to PG$, we have $\int_{PG} = \left(\int_G\right)_{|C(PG)}$.

Proof. All these results follow from uniqueness.

Theory 1/4

Theorem. We have a Frobenius type isomorphism

$$Hom(v,w)\simeq Fix(v\otimes\bar{w})$$

valid for any two corepresentations v,w.

Proof. We have the following equivalences:

$$T\in Hom(v,w)\iff Tv=wT\iff\sum_j T_{aj}v_{ji}=\sum_b w_{ab}T_{bi}$$

$$T\in Fix(v\otimes\bar{w})\iff(v\otimes\bar{w})T=\xi\iff\sum_{jb}v_{ij}w_{ab}^*T_{bj}=T_{ai}$$

With this, both inclusions follow from the biunitarity of v,w.

Theory 2/4

Theorem (PW3). The dense subalgebra $\mathcal{A} \subset A$ decomposes as

$$\mathcal{A} = \bigoplus_{v \in Irr(A)} M_{\dim(v)}(\mathbb{C})$$

isomorphism of $*$-coalgebras, with the summands being pairwise orthogonal with respect to $< a, b >= \int_G ab^*$.

Proof. We must prove that for $v, w \in Irr(A)$ we have:

$$v \not\sim w \implies C(v) \perp C(w)$$

The matrix P given by $P_{ia,jb} = \int_G v_{ij} w_{ab}^*$ is the projection onto:

$$Fix(v \otimes \bar{w}) \simeq Hom(v, w) = \{0\}$$

Thus we have $P = 0$, and this gives the result.

Theory 3/4

Theorem. The characters of the corepresentations, given by

$$\chi_v = \sum_i v_{ii}$$

behave as follows, in respect to the various operations:

$$\chi_{v+w} = \chi_v + \chi_w \quad , \quad \chi_{v \otimes w} = \chi_v \chi_w \quad , \quad \chi_{\bar{v}} = \chi_v^*$$

In addition, assuming $v \sim w$, we have $\chi_v = \chi_w$.

Proof. All this is clear, coming from definitions.

Theory 4/4

Theorem (PW4). The characters of irreducible corepresentations belong to the algebra of "smooth central functions"

$$\mathcal{A}_{central} = \left\{ a \in \mathcal{A} \middle| \Sigma\Delta(a) = \Delta(a) \right\}$$

and form an orthonormal basis of it.

Proof. The only tricky assertion is the norm 1 one. But:

$$\int_G \chi_v \chi_v^* = \sum_{ij} \int_G v_{ii} v_{jj}^* = \sum_i \frac{1}{N} = 1$$

Here we have used the fact that the integrals $\int_G v_{ij} v_{kl}^*$ form the orthogonal projection onto $Fix(v \otimes \bar{v}) \simeq End(v) = \mathbb{C}1$.

Examples 1/2

Theorem. Let $\Gamma =< g_1, \ldots, g_N >$ be a discrete group.

(1) The 1D corepresentations of $C^*(\Gamma)$ are the elements $g \in \Gamma$.

(2) The corepresentations of $C^*(\Gamma)$ are sums of group elements.

Theorem. The cocommutative Woronowicz algebras appear as

$$C^*(\Gamma) \to A \to C^*_{red}(\Gamma)$$

with Γ being a discrete group, $A = C^*_\pi(\Gamma)$ with $\pi \otimes \pi \subset \pi$.

Proofs. All this is clear from the Peter-Weyl theory.

Examples 2/2

Theorem. We have the following results:

(1) The irreps of a product $G \times H$ are the tensor products of the form $\pi \otimes \nu$, with π, ν being irreps of G, H.

(2) The irreps of a dual free product $G \,\hat{*}\, H$ appear as alternating tensor products of irreps of G, H.

(3) The irreps of a quotient $G \to H$ are the irreps of G whose coefficients belong to $C(H)$.

(4) The irreps of $G \to PG$ are the irreps of G which appear by decomposing the tensor powers of $ad(\pi) = \pi \otimes \bar{\pi}$.

Proofs. Once again, all this is clear from the Peter-Weyl theory.

Amenability 1/3

Theorem. Let A_{full} be the enveloping C^*-algebra of $\mathcal{A}$, and let A_{red} be the quotient of A by the null ideal of the Haar integration. The following are then equivalent:

(1) The Haar functional of A_{full} is faithful.

(2) The projection map $A_{full} \to A_{red}$ is an isomorphism.

(3) The counit map $\varepsilon : A_{full} \to \mathbb{C}$ factorizes through A_{red}.

(4) We have $N \in \sigma(Re(\chi_u))$, the spectrum being taken inside A_{red}.

If this is the case, we say that G is coamenable, and Γ is amenable.

Amenability 2/3

(1) $\iff$ (2) This follows from the fact that the GNS construction for the algebra A_{full} produces the algebra A_{red}.

(2) $\iff$ (3) Here $\implies$ is trivial. Conversely, the comultiplication of $\mathcal{A}$ can be extended into a map $\Phi : A_{red} \to A_{red} \otimes A_{full}$, and the composition $(\varepsilon \otimes id)\Phi$ is then our desired isomorphism.

(3) $\iff$ (4) The implication $\implies$ is clear, because from $\varepsilon(u_{ii}) = 1$ for any i, we obtain the following formula:

$$\varepsilon(N - Re(\chi(u))) = 0$$

Thus $N - Re(\chi(u))$ is not invertible in A_{red}, as claimed.

Amenability 3/3

(4) $\implies$ (3) With $v = u \oplus \bar{u}$, our assumption reads:

$$\dim v \in \sigma(\chi_v)$$

By functional calculus the same holds for $w = v + 1$, and in fact for any tensor power $w_k = w^{\otimes k}$. Now choose for each $k \in \mathbb{N}$ a state $\varepsilon_k \in A_{red}^*$ having the following property:

$$\varepsilon_k(w_k) = \dim w_k$$

By Peter-Weyl we must have $\varepsilon_k(v) = \dim v$, for any $v \leq w_k$, and since each irreducible corepresentation of A appears in this way, the sequence ε_k converges to a counit map $\varepsilon : A_{red} \to \mathbb{C}$, as desired.

Conclusion

We have a fully working Haar integration theory and Peter-Weyl theory, the applications of all this being, so far:

(1) Representations of group duals, and of various products.

(2) A fully satisfactory notion of amenability/coamenability.

(3) In particular, a Kesten amenability criterion, $N \in \sigma(Re(\chi_u))$.

(4) Suggesting that computing $law(\chi_u)$ is the "main problem".

$\Longrightarrow$ next lecture: Tannakian duality, easiness

Tannakian duality, diagrams and easiness

Teo Banica

"Introduction to quantum groups", 4/6

06/20

Plan

(1) Tensor categories

(2) Tannakian duality

(3) Diagrams, easiness

(4) Free quantum groups

$\Longrightarrow$ next lecture: quantum permutations

Representations

(1) A corepresentation of a Woronowicz algebra A is a biunitary matrix $v \in M_n(\mathcal{A})$ satisfying:

– $\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj}$
– $\varepsilon(v_{ij}) = \delta_{ij}$
– $S(v_{ij}) = v_{ji}^*$

(2) Basic example: the fundamental corepresentation u. In fact, the axioms state that u must be a faithful corepresentation.

(3) With $A = C(G)$, the corepresentations of A correspond to the FD unitary smooth representations of the quantum group G.

(4) We have a full Peter-Weyl theory for them, the main result stating that $\mathcal{A}$ decomposes as an orthogonal direct sum.

Categories 1/6

Definition. The Tannakian category of a Woronowicz algebra (A, u) is the collection $C = (C(k,l))$ of vector spaces

$$C(k,l) = Hom(u^{\otimes k}, u^{\otimes l})$$

where the corepresentations $u^{\otimes k}$ with $k = \circ \bullet \bullet \circ \ldots$ colored integer are defined by $u^{\otimes \circ} = u$, $u^{\otimes \bullet} = \bar{u}$ and multiplicativity.

Remark 1. We already know that C is a tensor $*$-category, the verification of all conditions being elementary.

Remark 2. In fact, C appears by definition as subcategory of the tensor $*$-category $E(k,l) = \mathcal{L}(H^{\otimes k}, H^{\otimes l})$, where $H = \mathbb{C}^N$.

Categories 2/6

Our purpose will be that of reconstructing (A,u) in terms of $C=(C(k,l))$. Here is a useful preliminary result:

Theorem. Given a morphism $\pi:(A,u)\to(B,v)$ we have

$$Hom(u^{\otimes k},u^{\otimes l})\subset Hom(v^{\otimes k},v^{\otimes l})$$

and if these inclusions are all equalities, π is an isomorphism.

Proof. Follows from Peter-Weyl, by contradiction, because each irreducible corepresentation is contained in some $u^{\otimes k}$.

Categories 3/6

In order to exploit the fact that u is biunitary, we can use:

Theorem. An matrix $u\in M_N(A)$ is biunitary if and only if

$$R\in Hom(1,u\otimes\bar{u})\quad,\quad R\in Hom(1,\bar{u}\otimes u)$$

$$R^*\in Hom(u\otimes\bar{u},1)\quad,\quad R^*\in Hom(\bar{u}\otimes u,1)$$

where $R:\mathbb{C}\to\mathbb{C}^N\otimes\mathbb{C}^N$ is given by $R(1)=\sum_i e_i\otimes e_i$.

Proof. This follows from some elementary computations.

Definition. Let H be a finite dimensional Hilbert space. A tensor category over H is a collection $C=(C(k,l))$ of subspaces

$$C(k,l)\subset\mathcal{L}(H^{\otimes k},H^{\otimes l})$$

satisfying the following conditions:

(1) $S,T\in C$ implies $S\otimes T\in C$.

(2) If $S,T\in C$ are composable, then $ST\in C$.

(3) $T\in C$ implies $T^*\in C$.

(4) Each $C(k,k)$ contains the identity operator.

(5) $C(\emptyset,\circ\bullet)$ and $C(\emptyset,\bullet\circ)$ contain the map $R:1\to\sum_i e_i\otimes e_i$.

Categories 5/6

Theorem. Let (A, u) be a Woronowicz algebra, with fundamental corepresentation $u \in M_N(A)$. The associated Tannakian category

$$C(k,l) = Hom(u^{\otimes k}, u^{\otimes l})$$

is then a tensor category over the Hilbert space $H = \mathbb{C}^N$.

Proof. We already know that axioms (1-4) hold indeed, this being elementary, and (5) is something that we just did, clear too.

Categories 6/6

Theorem. Given a tensor category $C = (C(k,l))$, the following algebra is a Woronowicz algebra:

$$A_C = C(U_N^+) \Big/ \left\langle T \in Hom(u^{\otimes k}, u^{\otimes l}) \Big| \forall k, l, \forall T \in C(k,l) \right\rangle$$

In the case where C comes from a Woronowicz algebra (A, v), we have a quotient map $A_C \to A$.

Proof. We have indeed a Woronowicz algebra, because the relations $T \in Hom(u^{\otimes k}, u^{\otimes l})$ are of "Hopf type", i.e. Δ, ε, S factorize.

The fact that we have a quotient map $A_C \to A$ is clear, because the relations defining A_C are satisfied inside A.

Summary

We have so far:

(1) Axioms for A: $N \times N$ Woronowicz algebra

(2) Axioms for C: tensor category over $\mathbb{C}^N$

(3) Correspondence $A \to C$: set $C_A = (Hom(u^{\otimes k}, u^{\otimes l}))_{kl}$

(4) Correspondence $C \to A$: set $A_C = C(U_N^+)/ < C \subset C_A >$

$\implies$ we want to prove that we have a bijection $A \leftrightarrow C$

Step 1

Theorem. Consider the following conditions:

(1) $C = C_{A_C}$, for any Tannakian category C.

(2) $A = A_{C_A}$, for any Woronowicz algebra (A, u).

We have then (1) $\implies$ (2). Also, $C \subset C_{A_C}$ is automatic.

Proof. We know that we have an arrow as follows:

$$A_{C_A} \to A$$

On the other hand, assuming (1), with $C = C_A$ we get:

$$C_A = C_{A_{C_A}}$$

Thus, we can use our quotient map criterion from before, and we get $A_{C_A} = A$, as desired. Finally, the last assertion is clear.

Step 2

Definition. Given a tensor category C over H, we set:

$$E_C = \bigoplus_{k,l} C(k,l) \subset \bigoplus_{k,l} B(H^{\otimes k}, H^{\otimes l}) \subset B\left(\bigoplus_k H^{\otimes k}\right)$$

Also, for any $s \in \mathbb{N}$, we consider the following truncation:

$$E_C^{(s)} = \bigoplus_{|k|,|l| \leq s} C(k,l) \subset \bigoplus_{|k|,|l| \leq s} B(H^{\otimes k}, H^{\otimes l}) = B\left(\bigoplus_{|k| \leq s} H^{\otimes k}\right)$$

Remark. We obtain in this way certain $*$-algebras.

Step 3

Theorem. For any C^*-algebra $B \subset M_n(\mathbb{C})$ we have

$$B = B''$$

where prime denotes the commutant, taken inside $M_n(\mathbb{C})$.

Proof. Let us decompose B as a direct sum of matrix algebras:

$$B = M_{r_1}(\mathbb{C}) \oplus \ldots \oplus M_{r_k}(\mathbb{C})$$

The commutant of this algebra is then as follows:

$$B' = \mathbb{C} \oplus \ldots \oplus \mathbb{C}$$

By taking once again the commutant we obtain B itself.

(This is a particular case of von Neumann's bicommutant theorem)

Step 4

Theorem. Given a category C, the following are equivalent:

(1) $C = C_{A_C}$.

(2) $E_C = E_{C_{A_C}}$.

(3) $E_C^{(s)} = E_{C_{A_C}}^{(s)}$, for any $s \in \mathbb{N}$.

(4) $E_C^{(s)'} = E_{C_{A_C}}^{(s)'}$, for any $s \in \mathbb{N}$.

In addition, $\subset$, $\subset$, $\subset$, $\supset$ respectively are automatically satisfied.

Proof. Here (1) $\iff$ (2) is clear from definitions, (2) $\iff$ (3) is clear from definitions as well, and (3) $\iff$ (4) comes from the bicommutant theorem. As for the last assertion, we have indeed $C \subset C_{A_C}$, and the other inclusions follow from this.

Step 5

Theorem. Given a Woronowicz algebra (A, u), we have

$$E_{C_A}^{(s)} = End\left(\bigoplus_{|k|\leq s} u^{\otimes k}\right)$$

as subalgebras of $B(\oplus_{|k|\leq s} H^{\otimes k})$.

Proof. The algebra $E_{C_A}^{(s)}$ appears by definition as follows:

$$E_{C_A}^{(s)} = \bigoplus_{|k|,|l|\leq s} Hom(u^{\otimes k}, u^{\otimes l}) \subset B\left(\bigoplus_{|k|\leq s} H^{\otimes k}\right)$$

But this is precisely the algebra of intertwiners of $\bigoplus_{|k|\leq s} u^{\otimes k}$.

Step 6

<u>Theorem</u>. For any corepresentation $v \in M_n(A)$, the map

$$\pi_v : A^* \to M_n(\mathbb{C}) \quad , \quad \varphi \to (\varphi(v_{ij}))_{ij}$$

is a representation, having as image $Im(\pi_v) = End(v)'$.

<u>Proof</u>. The first assertion is clear, coming from:

$$\begin{aligned}
(\pi_v(\varphi * \psi))_{ij} &= (\varphi \otimes \psi)\Delta(v_{ij}) \\
&= \sum_k \varphi(v_{ik})\psi(v_{kj}) \\
&= \sum_k (\pi_v(\varphi))_{ik}(\pi_v(\psi))_{kj} \\
&= (\pi_v(\varphi)\pi_v(\psi))_{ij}
\end{aligned}$$

As for the second assertion, this comes by double inclusion.

Conclusion

$\Longrightarrow$ We want to prove Tannakian duality, $A \leftrightarrow C$. Passed a few trivialities, this amounts in proving that:

$$C_{A_C} \subset C$$

$\Longrightarrow$ By using the $C \to E_C$ construction, truncated at $s \in \mathbb{N}$, and then a bicommutant trick, this is the same as proving that:

$$E_C^{(s)'} \subset E_{C_{A_C}}^{(s)'}$$

$\Longrightarrow$ We know that for any A we have $E_{C_A}^{(s)'} = Im(\pi_v)$, where

$$v = \bigoplus_{|k| \leq s} u^{\otimes k}$$

and where $\pi_v : A^* \to M_n(\mathbb{C})$ is given by $\varphi \to (\varphi(v_{ij}))_{ij}$.

Modelling 1/4

In order to model A_C, and to fine-tune the results that we have, consider the following pair of dual vector spaces:

$$F = \bigoplus_k B\left(H^{\otimes k}\right) \quad , \quad F^* = \bigoplus_k B\left(H^{\otimes k}\right)^*$$

Let $f_{ij}, f_{ij}^* \in F^*$ be the standard generators of $B(H)^*, B(\bar{H})^*$.

(1) F^* is a $*$-algebra, with multiplication $\otimes$ and involution $f_{ij} \leftrightarrow f_{ij}^*$.

(2) F^* is a $*$-bialgebra, with $\Delta(f_{ij}) = \sum_k f_{ik} \otimes f_{kj}$ and $\varepsilon(f_{ij}) = \delta_{ij}$.

(3) We have a $*$-bialgebra isomorphism $< u_{ij} > \simeq F^*$, $u_{ij} \to f_{ij}$.

Modelling 2/4

Theorem. The smooth part of the algebra A_C is given by

$$\mathcal{A}_C \simeq F^*/J$$

where $J \subset F^*$ is the ideal coming from the following relations,

$$\sum_{p_1,\ldots,p_k} T_{i_1\ldots i_l,p_1\ldots p_k} f_{p_1 j_1} \otimes \ldots \otimes f_{p_k j_k}$$

$$= \sum_{q_1,\ldots,q_l} T_{q_1\ldots q_l,j_1\ldots j_k} f_{i_1 q_1} \otimes \ldots \otimes f_{i_l q_l} \quad , \quad \forall i,j$$

one for each pair of colored integers k, l, and each $T \in C(k,l)$.

Proof. This is indeed clear from definitions.

Modelling 3/4

Theorem. The linear space $\mathcal{A}_C^*$ is given by the formula

$$\mathcal{A}_C^* = \left\{ a \in F \middle| T a_k = a_l T, \forall T \in C(k,l) \right\}$$

and its representation constructed before, namely

$$\pi_v : \mathcal{A}_C^* \to B(\oplus_{|k| \leq s} H^{\otimes k})$$

appears diagonally, by truncating, $\pi_v : a \to (a_k)_{kk}$.

Proof. Once again, this an elementary computation.

Modelling 4/4

In order to conclude, consider the following spaces:

$$F_s = \bigoplus_{|k|\leq s} B\left(H^{\otimes k}\right) \quad , \quad F_s^* = \bigoplus_{|k|\leq s} B\left(H^{\otimes k}\right)^*$$

We denote by $a \to a_s$ the truncation $F \to F_s$. We have:

(1) $E_C^{(s)'} \subset F_s$.

(2) $E_C' \subset F$.

(3) $\mathcal{A}_C^* = E_C'$.

(4) $Im(\pi_v) = (E_C')_s$.

Indeed, all this follows from the above interpretation of $\mathcal{A}_C^*$.

Duality

Theorem. We have a Tannakian duality correspondence

$$A \leftrightarrow C$$

between Woronowicz algebras and tensor categories, given by

$$C_A = (Hom(u^{\otimes k}, u^{\otimes l}))_{kl}$$

in one sense, from algebras to categories, and by

$$A_C = C(U_N^+)/ < C \subset C_A >$$

in the other sense, from categories to algebras.

Proof 1/2

We have to prove that, for any category C, and any $s \in \mathbb{N}$:

$$E_C^{(s)'} = (E'_C)_s$$

By taking duals, this is the same as proving that:

$$\left\{ f \in F_s^* \middle| f_{|(E'_C)_s} = 0 \right\} = \left\{ f \in F_s^* \middle| f_{|E_C^{(s)'}} = 0 \right\}$$

We use $\mathcal{A}_C^* = E'_C$. Since we have $\mathcal{A}_C = F^*/J$, we conclude that the ideal $J \subset F^*$ previously constructed is given by:

$$J = \left\{ f \in F^* \middle| f_{|E'_C} = 0 \right\}$$

Proof 2/2

The point now is that we have, for any $s \in \mathbb{N}$:

$$J \cap F_s^* = \left\{ f \in F_s^* \middle| f_{|E_C^{(s)'}} = 0 \right\}$$

On the other hand, we have as well:

$$\begin{aligned} J \cap F_s^* &= \left\{ f \in F^* \middle| f_{|E_C'} = 0 \right\} \cap F_s^* \\ &= \left\{ f \in F_s^* \middle| f_{|E_C'} = 0 \right\} \\ &= \left\{ f \in F_s^* \middle| f_{|(E_C')_s} = 0 \right\} \end{aligned}$$

Thus, we are led to the equality that we wanted to prove.

Applications

Many applications, and to begin with, we have as plan:

(1) The biggest quantum group, namely U_N^+, must correspond to the smallest tensor category, namely $< R >$.

(2) It is well-known that $R : 1 \to \sum_i e_i \otimes e_i$ can be pictured as a semicircle $\cap$, so we have to get into diagrams.

(3) We will reach in this way to a notion of "easy quantum group", covering O_N, O_N^+, U_N, U_N^+, and many other examples.

(4) As a main application, we will solve the problem of computing the law of the main character for O_N, O_N^+, U_N, U_N^+.

Easiness 1/3

Let $P(k,l)$ be the set of partitions between an upper colored integer k, and a lower colored integer l.

Definition. A collection of subsets $D(k,l) \subset P(k,l)$ is called a category of partitions when it satisfies:

(1) Stability under the horizontal concatenation, $(\pi,\sigma) \to [\pi\sigma]$.

(2) Stability under vertical concatenation $(\pi,\sigma) \to [^{\sigma}_{\pi}]$ (matching).

(3) Stability under the upside-down turning $*$, with $\circ \leftrightarrow \bullet$.

(4) Each $P(k,k)$ contains the identity partition $|| \ldots ||$.

(5) Both $P(\emptyset, \circ\bullet)$ and $P(\emptyset, \bullet\circ)$ contain the semicircle $\cap$.

Easiness 2/3

Definition. A closed subgroup $G \subset U_N^+$ is called easy when

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

for a certain category of partitions $D \subset P$, where

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with $\delta_\pi \in \{0,1\}$ depending on whether the indices fit or not.

Easiness 3/3

Theorem. The basic unitary quantum groups, namely

$$\begin{array}{ccc} U_N & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & O_N^+ \end{array}$$

are all easy, coming from the following categories of pairings:

$$\begin{array}{ccc} \mathcal{P}_2 & \longleftarrow & \mathcal{NC}_2 \\ \downarrow & & \downarrow \\ P_2 & \longleftarrow & NC_2 \end{array}$$

Proof. This comes from Tannaka (classical case: Brauer).

Applications 1/3

Theorem. We have the following free complexification formula,

$$\widetilde{O}_N^+ = U_N^+$$

and for projective versions we have the following isomorphism,

$$PO_N^+ = PU_N^+$$

by identifying as usual the full and reduced versions.

Proof. We know that we have $\widetilde{O}_N^+ \subset U_N^+$, and since the Tannakian categories coincide, this is an isomorphism.

Applications 2/3

<u>Theorem.</u> The moments of the main characters for

$$\begin{array}{ccc} U_N & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & O_N^+ \end{array}$$

are, in the $N \to \infty$ limit, as follows:

(1) On the bottom, with $k = 2l$, we have $(2l)!!$ and $\frac{1}{l+1}\binom{2l}{l}$.

(2) On top we have similar numbers, with k being now colored.

<u>Proof</u>. This follows by counting the pairings, with $N \to \infty$ being needed as for $\{T_\pi\}$ to be linearly independent.

Applications 3/3

Theorem. The asymptotic laws of the main characters for

$$\begin{array}{ccc} U_N & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & O_N^+ \end{array}$$

are the basic measures in probability and free probability:

$$\begin{array}{ccc} \textit{Complex Gaussian} & \longrightarrow & \textit{Voiculescu circular} \\ \uparrow & & \uparrow \\ \textit{Real Gaussian} & \longrightarrow & \textit{Wigner semicircular} \end{array}$$

Proof. Calculus if we guess the answer, Stieltjes inversion otherwise.

Conclusion

We have a theory of easy quantum groups, featuring:

(1) Simple axioms: "C must come from partitions".

(2) The quantum groups O_N, O_N^+, U_N, U_N^+ as main examples.

(3) Many other potential examples, e.g. coming from P, NC.

(4) Interesting connections with probability/free probability.

$\Longrightarrow$ next lecture: quantum permutations

Quantum permutations and quantum reflections

Teo Banica

"Introduction to quantum groups", 5/6

07/20

Tannaka

Theorem. We have a Tannakian duality correspondence

$$A \longleftrightarrow C$$

between Woronowicz algebras and tensor categories, given by

$$C_A = (Hom(u^{\otimes k}, u^{\otimes l}))_{kl}$$

in one sense, from algebras to categories, and by

$$A_C = C(U_N^+)/ < C \subset C_A >$$

in the other sense, from categories to algebras.

Easiness

Theorem. Any category of partitions $D = (D(k,l))$ produces a family of quantum groups $G = (G_N)$ via the formula

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

where the linear maps T_π associated to partitions are given by

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with $\{e_i\}$ being the basis of $\mathbb{C}^N$, and $\delta_\pi \in \{0,1\}$ being Kronecker symbols. These quantum groups G_N are called easy.

Plan

(1) Quantum permutation groups

(2) Easiness: algebra and analysis

(3) Quantum reflection groups

(4) Transitivity, planar algebras

$\Longrightarrow$ next lecture: tori, models

Quantum permutations

The coordinates of $S_N \subset O_N$, permutation matrices, are:

$$u_{ij} = \chi\left(\sigma \in S_N \middle| \sigma(j) = i\right)$$

A quick study of u suggests the following definition:

<u>Definition</u>. The quantum permutation group S_N^+ is defined via

$$C(S_N^+) = C^*\left((u_{ij}) \middle| u = N \times N \text{ magic}\right)$$

where "magic" = made of projections, sum 1 on rows/columns.

[the verification of the CQG axioms is routine: Wang 98]

Alternative definition

Theorem. S_N^+ is the biggest quantum group acting on

$$X = \{1, \ldots, N\}$$

by keeping the counting measure invariant.

Proof. In order to have a quantum group action

$$G \times X \to X \quad , \quad (\sigma, i) \to \sigma(i)$$

we need a coaction map $\Phi : C(X) \to C(G) \otimes C(X)$. With

$$\Phi(\delta_i) = \sum_j u_{ij} \otimes \delta_j$$

the matrix $u = (u_{ij})$ must be magic. Thus $G_{max} = S_N^+$.

Basic properties 1/4

We have a quotient map $C(S_N^+) \to C(S_N)$, given by:

$$u_{ij} \to \chi\left(\sigma \in S_N \middle| \sigma(j) = i\right)$$

Thus we have an embedding $S_N \subset S_N^+$. Study:

$\underline{N = 2}$: We have $S_2^+ = S_2$, because the 2×2 magics are

$$u = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

and their entries commute. Thus $C(S_2^+)$ is commutative.

$\underline{N = 3}$: We have $S_3^+ = S_3$, by similar arguments.

Basic properties 2/4

We know $S_N \subset S_N^+$ isomorphism at $N = 2, 3$. Continuation:

$\underline{N = 4}$: Here S_4^+ is non-classical and infinite, because

$$u = \begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

with $p, q \in B(H)$ shows that $C(S_4^+)$ is NC and ∞D.

$\underline{N \geq 5}$: Here S_N^+ stays non-classical and infinite (clear).

Basic properties 3/4

$\Longrightarrow$ Can we understand better <u>why $S_4^+ \neq S_4$</u>?

Recall that given $\Gamma =< g_1, \ldots, g_N >$ discrete group, $A = C^*(\Gamma)$ is a Woronowicz algebra, written $A = C(\widehat{\Gamma})$, with:

$$u = diag(g_1, \ldots, g_N)$$

Now observe that we have, trivially by Fourier transform:

$$\widehat{\mathbb{Z}_2} = \mathbb{Z}_2 = S_2 = S_2^+$$

Thus <u>our concatenation trick at $N = 4$</u> amounts in saying that:

$$\widehat{D_\infty} = \widehat{\mathbb{Z}_2 * \mathbb{Z}_2} \subset S_4^+$$

Even better, we have $\widehat{D_\infty} \subset G^+(\square)$. More on this later.

Basic properties 4/4

$\Longrightarrow$ Can we understand what this S_4^+ beast is?

Algebra $C(SO_3^{-1})$, with orthogonal coordinates a_{ij}, satisfying:

– $a_{ij}a_{kl} = \pm a_{kl}a_{ij}$, with $+$ if $i \neq k, j \neq l$, and $-$ otherwise

– twisted determinant condition: $\sum_{\sigma \in S_3} a_{1\sigma(1)}a_{2\sigma(2)}a_{3\sigma(3)} = 1$

The point is that the following matrix must be magic:

$$\frac{1}{4}\begin{pmatrix}1&1&1&1\\1&-1&-1&1\\1&-1&1&-1\\1&1&-1&-1\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&a_{11}&a_{12}&a_{13}\\0&a_{21}&a_{22}&a_{23}\\0&a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}1&1&1&1\\1&-1&-1&1\\1&-1&1&-1\\1&1&-1&-1\end{pmatrix}$$

Thus $S_4^+ = SO_3^{-1}$, via the Fourier transform over $K = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Representations 1/4

Theorem. The Tannakian category of S_N is given by

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in P(k,l)\right)$$

where the linear maps associated to partitions are:

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1,\ldots,j_l} \delta_\pi \begin{pmatrix} i_1 \ldots i_k \\ j_1 \ldots j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

Regarding now S_N^+, the situation is quite similar:

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in NC(k,l)\right)$$

In other words, S_N, S_N^+ are easy, coming from P, NC.

Representations 2/4

Proof for S_N. Consider the one-block partition $\mu \in P(2,1)$. We have $T_\mu(e_i \otimes e_j) = \delta_{ij} e_i$, and a computation gives:

$$T_\mu \in Hom(u^{\otimes 2}, u) \iff u_{ij} u_{ik} = \delta_{jk} u_{ij}, \forall i, j, k$$

On the right we have the magic condition. We conclude that:

$$C(S_N) = C(O_N) \Big/ \Big\langle T_\mu \in Hom(u^{\otimes 2}, u) \Big\rangle$$

Now since P is generated by $\mu \in P(2,1)$, we are done.

Proof for S_N^+. Similar, by using the Brauer theorem for O_N^+.

Representations 3/4

Theorem. The fusion rules for S_N^+ are the same as for SO_3,

$$r_k \otimes r_l = r_{|k-l|} + r_{|k-l|+1} + \ldots + r_{k+l}$$

with $\dim(r_k) = \frac{q^{k+1}-q^{-k}}{q-1}$, where $q^2 - (N-2)q + 1 = 0$.

Proof. We know from easiness that we have:

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in NC(k,l)\right)$$

Thus, the main character χ is squared-semicircular:

$$\int_{S_N^+} \chi^p = |NC(0,p)| = \frac{1}{p+1}\binom{2p}{p}$$

But this gives the result, using $S^3_{\mathbb{R}} \simeq SU_2 \to SO_3$.

Representations 4/4

Comment: the above proof is valid in fact only with $N >> 0$, where the maps $\{T_\pi\}$ are linearly independent.

However, things are in fact fine as long as $N \geq 4$.

Why 4? Because this is a "Jones index". We have indeed

$$NC(0,p) \simeq NC_2(0,2p) \simeq NC_2(p,p) = \{basis\ of\ \ TL(p)\}$$

and according to Jones, we must have $N \geq 4$ for things to work.

$\Longrightarrow$ note that all this is simpler than for S_N (!)

Analysis 1/4

Let $S_N \subset O_N$ as usual. The main character is then:

$$\chi(\sigma) = \sum_i u_{ii}(\sigma) = \sum_i \delta_{\sigma(i)i} = \#\left\{i \middle| \sigma(i) = i\right\}$$

By using the inclusion-exclusion principle, we obtain:

$$\mathbb{P}(\chi = 0) = 1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + (-1)^{N-1}\frac{1}{(N-1)!} + (-1)^N \frac{1}{N!}$$

Thus, we have the following asymptotic formula:

$$\lim_{N\to\infty} \mathbb{P}(\chi = 0) = \frac{1}{e}$$

In fact, the character χ becomes Poisson with $N \to \infty$.

Analysis 2/4

Theorem. If G is easy, coming from a category of partitions D,

$$\int_G u_{i_1j_1}\ldots u_{i_kj_k}=\sum_{\pi,\sigma\in D(k)}\delta_\pi(i)\delta_\sigma(j)W_{kN}(\pi,\sigma)$$

where $W_{kN}=G_{kN}^{-1}$ is the inverse of $G_{kN}(\pi,\sigma)=N^{|\pi\vee\sigma|}$.

Proof. This is the Weingarten formula, coming from the fact that the above integrals form the projection onto $Fix(u^{\otimes k})$.

In the unitary case we must use colored integers.
Works too in the symplectic case, and other cases.

Analysis 3/4

<u>Theorem</u>. The main character $\chi = \sum_{i=1}^N u_{ii}$ for the quantum groups S_N, S_N^+ follows with $N \to \infty$ the laws

$$p_1 = \frac{1}{e} \sum_k \frac{\delta_k}{k!}$$

$$\pi_1 = \frac{1}{2\pi} \sqrt{4x^{-1} - 1} dx$$

called Poisson and Marchenko-Pastur (or free Poisson) of parameter 1, and appearing via the PLT and FPLT.

<u>Proof</u>. Here we do not really need Weingarten, because:

$$\int_G \chi^k \simeq |D(k)|$$

By using standard calculus (e.g. cumulants) we can conclude.

Analysis 4/4

<u>Theorem</u>. The truncated characters $\chi_t = \sum_{i=1}^{[tN]} u_{ii}$ for the quantum groups S_N, S_N^+ follow with $N \to \infty$ the laws

$$p_t = e^{-t} \sum_k \frac{t^k}{k!} \delta_k$$

$$\pi_t = \max(1-t, 0)\delta_0 + \frac{\sqrt{4t - (x-1-t)^2}}{2\pi x} \, dx$$

called Poisson and Marchenko-Pastur (or free Poisson) of parameter t, and appearing via the PLT and FPLT.

<u>Proof</u>. Here, by using the Weingarten formula, we have:

$$\int_G \chi_t^k \simeq \sum_{\pi \in D(k)} t^{|\pi|}$$

By using standard calculus (e.g. cumulants) we can conclude.

Summary

(1) The analogy between S_N, S_N^+ is best understood via easiness

$$\begin{array}{ccccc}
S_N^+ & \cdots\cdots\cdots & NC & \cdots\cdots\cdots & \pi_t \\
\updownarrow & & \updownarrow & & \updownarrow \\
S_N & \cdots\cdots\cdots & P & \cdots\cdots\cdots & p_t
\end{array}$$

with N generic, for algebra, and with $N \to \infty$, for analysis.

(2) When N is fixed things collapse for both S_N, S_N^+, the collapsing being worse for S_N in algebra, and worse for S_N^+ in analysis.

(3) All this is just the "tip of the iceberg". Many advanced results, both algebra and analysis (planar algebras, Diaconis type).

Graphs 1/4

Let X be a finite graph, $|X| = N < \infty$, with adjacency matrix $d \in M_N(0,1)$. Its quantum symmetry group is given by:

$$G^+(X) = C(S_N^+)\Big/\Big\langle du = ud\Big\rangle$$

We have then a diagram of inclusions, as follows:

$$\begin{array}{ccc}
G^+(X) & \longrightarrow & S_N^+ \\
\uparrow & & \uparrow \\
G(X) & \longrightarrow & S_N
\end{array}$$

Trivial example: no edges (or complete graph) $\Longrightarrow$ get S_N^+.

Graphs 2/4

Cycle graph C_N. Here generically we have, by algebra,

$$G^+(C_N) = G(C_N) = D_N$$

unless at $N = 4$, where the following thing happens:

$$G^+(C_4) = G^+(\square) = G^+(|\ |) \supset \widehat{\mathbb{Z}_2 * \mathbb{Z}_2} = \widehat{D_\infty}$$

$\implies$ Question: what is $G^+(\square)$?

Looking at hypercube graphs $\square_N$. Here we have:

$$G^+(\square_N) = O_N^{-1}$$

$\implies$ In particular, we obtain $G^+(\square) = O_2^{-1}$.

Graphs 3/4

This is still not ok, because $H_N \to O_N^{-1}$ cannot be a "true liberation", for analytic reasons (same law as for O_N).

$\implies$ Question: what is H_N^+?

Answer. Consider the graph $|| \ldots ||$ consisting of N segments (the $[-1,1]$ segments on the N coordinate axes). Then:

$$G(|| \ldots ||) = \mathbb{Z}_2 \wr S_N = H_N \longleftrightarrow P_{even}$$

We can therefore define H_N^+ as follows, and we are done:

$$G^+(|| \ldots ||) = \mathbb{Z}_2 \wr_* S_N^+ = H_N^+ \longleftrightarrow NC_{even}$$

Graphs 4/4

More generally, for any $s \in \{1, 2, \ldots, \infty\}$ we have:

$$G(\triangle_s \ldots \triangle_s) = \mathbb{Z}_s \wr S_N = H_N^s \longleftrightarrow P^s$$

We can liberate this reflection group as follows:

$$G^+(\triangle_s \ldots \triangle_s) = \mathbb{Z}_s \wr_* S_N^+ = H_N^{s+} \longleftrightarrow NC^s$$

(the "s" at right mean $\# \circ = \# \bullet (s)$, signed, in each block)

– at $s = 1$ we recover S_N, S_N^+
– at $s = 2$ we recover H_N, H_N^+
⋮
– at $s = \infty$ non-QPG, called K_N, K_N^+

Many other interesting results here.

Orbits 1/4

Recall that for $G \subset S_N$ the coordinates via $S_N \subset O_N$ are:

$$u_{ij} = \chi\left(\sigma \in G \middle| \sigma(j) = i\right)$$

<u>Definition</u>. A quantum permutation group $G \subset S_N^+$ is called transitive when $u_{ij} \neq 0$, for any i, j.

As basic examples, all QPG that we met so far:

– we have $G^+(X)$ with X transitive (i.e. with $G(X)$ transitive)

– in particular we have H_N^s, H_N^{s+}, for any $s \in \mathbb{N}$

– also in particular, we have $O_N^{-1} = G^+(\square_N)$

Orbits 2/4

Orbits. Given a closed subgroup $G \subset S_N^+$, let us set:

$$i \sim j \iff u_{ij} \neq 0$$

This is an equivalence relation. Indeed (using positivity):

$$\begin{aligned}
\Delta(u_{ik}) = \sum_j u_{ij} \otimes u_{jk} \quad &\implies \quad [i \sim j, j \sim k \implies i \sim k] \\
\varepsilon(u_{ii}) = 1 \quad &\implies \quad i \sim i \\
S(u_{ij}) = u_{ji} \quad &\implies \quad [i \sim j \implies j \sim i]
\end{aligned}$$

In the classical case, $G \subset S_N$, we recover the usual orbits.

$\implies$ what to do with this notion? (no examples so far)

Orbits 3/4

Consider a quotient group of type $\mathbb{Z}_{N_1} * \ldots * \mathbb{Z}_{N_k} \to \Gamma$, with $N = N_1 + \ldots + N_k$. We have then, by Fourier:

$$\begin{aligned}\widehat{\Gamma} &\subset \mathbb{Z}_{N_1} \widehat{* \ldots *} \mathbb{Z}_{N_k} = \widehat{\mathbb{Z}_{N_1}} \,\hat{*} \ldots \hat{*}\, \widehat{\mathbb{Z}_{N_k}} \\ &\simeq \mathbb{Z}_{N_1} \,\hat{*} \ldots \hat{*}\, \mathbb{Z}_{N_k} \subset S_{N_1} \,\hat{*} \ldots \hat{*}\, S_{N_k} \\ &\subset S_{N_1}^+ \,\hat{*} \ldots \hat{*}\, S_{N_k}^+ \subset S_N^+\end{aligned}$$

Theorem. Any group dual subgroup $\widehat{\Gamma} \subset S_N^+$ appears in this way, for a certain partition $N = N_1 + \ldots + N_k$.

Proof. Orbit decomposition $N = N_1 + \ldots + N_k$.

Orbits 4/4

<u>Orbitals</u>. Let $G \subset S_N^+$, and $k \in \mathbb{N}$. The relation

$$(i_1, \ldots, i_k) \sim (j_1, \ldots, j_k) \iff u_{i_1 j_1} \ldots u_{i_k j_k} \neq 0$$

is then reflexive and symmetric (proof as before, at $k = 1$).

Transitivity <u>holds at $k = 1$</u>. <u>Also at $k = 2$</u>, the trick being:

$$\begin{aligned} & (u_{i_1 j_1} \otimes u_{j_1 l_1}) \Delta(u_{i_1 l_1} u_{i_2 l_2})(u_{i_2 j_2} \otimes u_{j_2 l_2}) \\ = & \sum_{s_1 s_2} u_{i_1 j_1} u_{i_1 s_1} u_{i_2 s_2} u_{i_2 j_2} \otimes u_{j_1 l_1} u_{s_1 l_1} u_{s_2 l_2} u_{j_2 l_2} \\ = & \ u_{i_1 j_1} u_{i_2 j_2} \otimes u_{j_1 l_1} u_{j_2 l_2} \end{aligned}$$

<u>At $k \geq 3$ this fails</u> (but few things still hold), at $k \geq 4$ totally fails.

Algebra 1/4

What can be said about the arbitrary subgroups $G \subset S_N^+$?

(in addition to the orbit/orbital theory explained above)

Theorem. Quantum Cayley fails.

Recall indeed the Cayley theorem, stating that, for classical groups:

$$|G| = N \implies G \subset S_N$$

This does not work for quantum groups. There are finite quantum groups which are not quantum permutation groups (!)

Algebra 2/4

What can be said (good) about the subgroups $G \subset S_N^+$?

Theorem. The collection of vector spaces

$$P_k = Fix(u^{\otimes k})$$

is a planar algebra in the sense of Jones. More precisely, we have an inclusion as follows, where Q_N is the "spin" planar algebra,

$$P \subset Q_N$$

and any planar subalgebra $P \subset Q_N$ appears in this way.

Proof. Tannakian duality, applied in this setting, "rotated".

Algebra 3/4

Planar algebras, more. The correspondence established above

$$G \subset S_N^+ \longleftrightarrow P \subset Q_N$$

makes correspond the following objects and constructions,

$$\{1\} \longleftrightarrow Q_N$$

$$S_N^+ \longleftrightarrow TL_N$$

$$H_N^+ \longleftrightarrow FC_N$$

$$G^+(X) \longleftrightarrow < \square_X >$$

where $\square_X$ is the Laplacian (adjacency matrix) viewed as 2-box.

$\Longrightarrow$ Bisch-Jones, "Laplacian in the box" philosophy

Algebra 4/4

A difficult conjecture states that $S_N \subset S_N^+$ is maximal, in the sense that there is no object in between. Status:

(1) Trivial: no groups, no group duals.

(2) Elementary: no easy solutions.

(3) Advanced: OK at $N = 4$, cf. ADE classification of the subgroups $G \subset S_4^+ = SO_3^{-1}$.

(4) Difficult: OK at $N = 5$, due to the classification of index 5 subfactors. No known QPG proof.

Conclusion

We have a theory of quantum permutations, featuring:

(1) General theory, orbits, easiness.

(2) $S_N, S_N^+, H_N, H_N^+, K_N, K_N^+$ as main examples.

(3) Many other examples, e.g. coming from graphs.

(4) Interesting connections with probability/free probability.

$\implies$ next lecture: tori, models

Orientability, toral subgroups and matrix models

Teo Banica

"Introduction to quantum groups", 6/6

07/20

Plan

(1) Easiness - review, more

(2) Orientability - questions

(3) Tori - diagonal, spinned

(4) Geometry - axiomatization

(5) Models - general theory

(6) Matrices - Weyl, Fourier

Easiness 1/4

A closed subgroup $G \subset U_N^+$ is called easy when

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

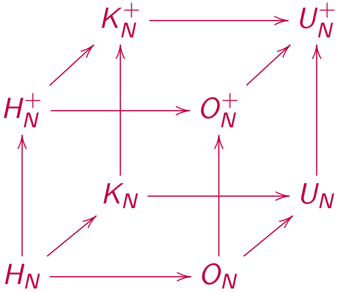
for certain sets of partitions $D(k,l) \subset P(k,l)$, where

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with $\{e_i\}$ = basis of $\mathbb{C}^N$, and δ_π = Kronecker type symbols.

Easiness 2/4

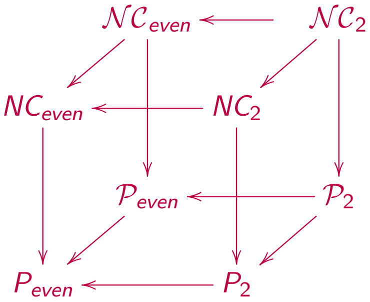
The main examples of easy quantum groups are as follows,



where $H_N = \mathbb{Z}_2 \wr S_N$, $K_N = \mathbb{T} \wr S_N$, $H_N^+ = \mathbb{Z}_2 \wr_* S_N^+$, $K_N = \mathbb{T} \wr_* S_N^+$.

Easiness 3/4

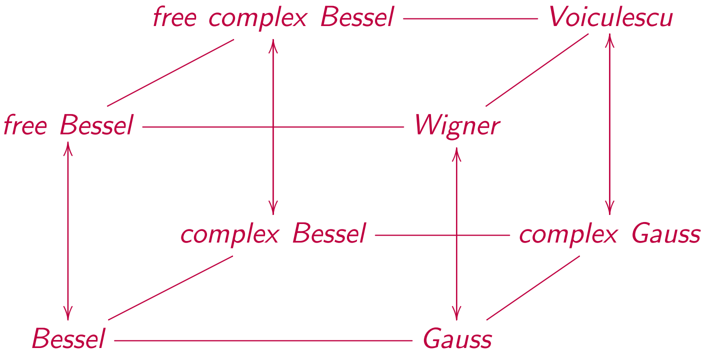
The corresponding categories of partitions are as follows,



with the calligraphic letters standing for "matching".

Easiness 4/4

The asymptotic laws of truncated characters are as follows,



with the vertical arrows standing for the Bercovici-Pata bijection.

Questions

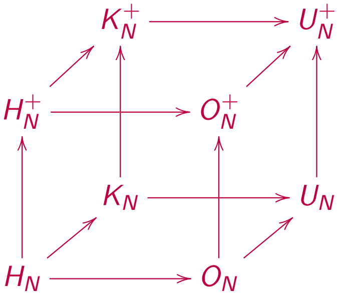
Classify. Compute laws. Find a "contravariant duality" as follows,

$$\begin{array}{ccccccc}
U_N & \longrightarrow & U_N^{(r)} & \longrightarrow & U_N^C & \longrightarrow & U_N^+\\
\vdots & & \vdots & & \vdots & & \vdots\\
H_N^+ & \longleftarrow & H_N^{[r]} & \longleftarrow & H_N^\Gamma & \longleftarrow & H_N
\end{array}$$

between the unitary and real reflection easy quantum groups.

Orientability 1/4

The standard cube is an intersection and generation diagram,



i.e. for any face $P \subset Q, R \subset S$ we have $P = Q \cap R, < Q, R >= S$.

Orientability 2/4

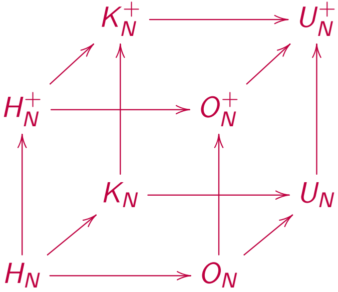
The needed technology here is as follows:

– Intersection $G \cap H$
– Tannaka $C_{G \cap H} =< C_G, C_H >$
– Easy case $D_{G \cap H} =< D_G, D_H >$
$\implies$ and this is OK for our cube problems

– Generation $< G, H >$
– Tannaka $C_{<G,H>} = C_G \cap C_H$
– Easy case $D_{\{G,H\}} = D_G \cap D_H$
– Conjecture $< G, H > \subset \{G, H\}$ iso
$\implies$ ad-hoc methods for 5 faces, 1 face left

Orientability 3/4

Ground Zero: the twistable, easy, uniform, oriented CQG are



where we know what easy means, and:

– uniform means $G = (G_N)$ with $G_{N-1} = G_N \cap U_{N-1}^+$

– twistable means here $H_N \subset G_N$, for any $N \in \mathbb{N}$

– oriented means "not disoriented" with respect to Ox,Oy,Oz

Orientability 4/4

Regarding the oriented CQG, under extra assumptions, mild:

1. classical: O_N, SO_N, U_N^d, H_N^{sd} + bistochastic versions

2. free: the known easy ones, and that's not trivial

3. group duals: abelian + varieties of real reflection groups

Here 1 looks doable, 2 looks hard, 3 is probably the simplest.

Questions

Classification of the "main" closed subgroups $G \subset U_N^+$:

$\Longrightarrow$ use partition methods, and intersection/generation surgery, in 3D or more, in order to "classify" G.

$\Longrightarrow$ the good 3 dimensions are those above, discrete/continuous, real/complex, classical/free.

$\Longrightarrow$ there are 3 more dimensions, "bad", coming from taking the bistochastic version, special version, diagonal torus.

Conjecture: 6-parameter series + exceptional examples.

Tori 1/4

The diagonal torus $T \subset G$ is the group dual given by

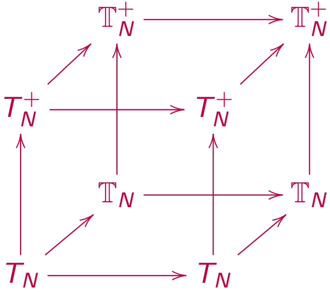
$$C(T) = C(G) \Big/ \left\langle u_{ij} = 0 \middle| \forall i \neq j \right\rangle$$

the generators of $\widehat{T}$ being $g_i = u_{ii}$. Equivalently, we can set

$$T = G \cap \mathbb{T}_N^+$$

where $\mathbb{T}_N^+ \subset U_N^+$ is the abstract dual of the free group F_N.

The diagonal tori of the main quantum groups are as follows,



where $T_N = \mathbb{Z}_2^N$, $\mathbb{T}_N = \mathbb{T}^N$ and $T_N^+ = \widehat{\mathbb{Z}_2^{*N}}$, $\mathbb{T}_N = \widehat{F_N}$.

Tori 3/4

Given $G \subset U_N^+$, consider its diagonal torus $T = G \cap \mathbb{T}_N^+$, and consider as well its reflection subgroup $K = G \cap K_N^+$:

$$T \subset K \subset G$$

Let also $G_{class} = G \cap U_N$. We say that G appears as:

– a soft liberation, when $G = < G_{class}, K >$

– a hard liberation, when $G = < G_{class}, T >$

$\implies$ OK (hard liberation) for O_N^+, U_N^+, and for O_N^*, U_N^* too.
$\implies$ cannot work for S_N^+, or B_N^+, C_N^+, and H_N^+, K_N^+ fail too.

Tori 4/4

Spinned tori, obtained by using the corepresentation $v = QuQ^*$:

$$\{T_Q \subset G | Q \in U_N\}$$

(1) Generation: $G =< (T_Q)_{Q \in U_N} >$.

(2) Weak generation: $G =< G_{class}, (T_Q)_{Q \in U_N} >$.

(3) Fourier liberation: $G =< G_{class}, (T_F)_{F = Fourier} >$.

(4) Hard liberation: $G =< G_{class}, T_1 >$.

No counterexamples to (1,2). It is known that (3) holds, beyond (4), for S_N^+, and for B_N^+, C_N^+ as well. No easy counterexamples.

Questions

The family $\{T_Q | Q \in U_N\}$ is the "maximal torus". Conjectures:

(1) Characters: if G is connected, for any nonzero $P \in C(G)_{central}$ there exists $Q \in U_N$ such that $P \neq 0$ inside $C(T_Q)$.

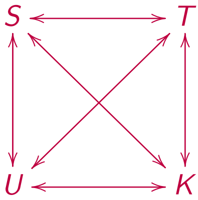
(2) Amenability: G is coamenable if and only if each of the tori T_Q is coamenable, in the usual discrete group sense.

(3) Growth: G has polynomial cogrowth if and only if each T_Q has polynomial cogrowth, in the usual discrete group sense.

$\Longrightarrow$ OK for groups, group duals, main easy cases.

Geometry 1/4

Step 1. Axiomatize and classify the quadruplets



Step 2. Develop the geometries that you found.

Step 3. Integration theory, Riemannian aspects.

Step 4. Work more, reach to "Nash-Connes Geometry".

Geometry 2/4

A first difficulty is with $T\to U$. The axiom here must be:

$$U=<O_N,T>$$

(1) Classical real case: $O_N=<O_N,T_N>$, clear.

(2) Classical complex case: $U_N=<O_N,\mathbb{T}_N>$, true.

(3) Free real case: $O_N^+=<O_N,T_N^+>$. Very technical, by proving first $O_N^+=<O_{N-1}^+,O_N>$, by recurrence on N.

(4) Free complex case: $U_N^+=<O_N,\mathbb{T}_N^+>$. Can be obtained from the free real case formula, by using standard arguments.

Geometry 3/4

A second difficulty is with $T \to K$. We have the following quantum isometry group computations, which are quite surprising:

$$\begin{array}{ccc}
T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\
\uparrow & & \uparrow \\
T_N & \longrightarrow & \mathbb{T}_N
\end{array}
\quad \to \quad
\begin{array}{ccc}
H_N^+ & \longrightarrow & K_N^+ \\
\vdots & & \vdots \\
O_N^{-1} & \longrightarrow & U_N^{-1}
\end{array}$$

The solution is by saying that $T \to K$ appears as follows:

$$K = G^+(T) \cap K_N^+$$

That is, K must be the "quantum reflection group" of T.

Geometry 4/4

An abstract NCG must come from a quadruplet (S, T, U, K),

$$S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{C},+} \quad , \quad T_N \subset T \subset \mathbb{T}_N^+$$

$$O_N \subset U \subset U_N^+ \quad , \quad H_N \subset K \subset K_N^+$$

such that we can pass from each object to all the other objects,

$$\begin{array}{ccccccc} S & = & S_{<O_N,T>} & = & S_U & = & S_{<O_N,K>} \\ S \cap \mathbb{T}_N^+ & = & T & = & U \cap \mathbb{T}_N^+ & = & K \cap \mathbb{T}_N^+ \\ G^+(S) & = & <O_N, T> & = & U & = & <O_N, K> \\ G^+(S) \cap K_N^+ & = & G^+(T) \cap K_N^+ & = & U \cap K_N^+ & = & K \end{array}$$

with all this being up to the "full=reduced" equivalence relation.

Questions

We have 9 main geometries in our sense, which are all easy:

$$
\begin{array}{ccccc}
\mathbb{R}^N_+ & \longrightarrow & \mathbb{T}\mathbb{R}^N_+ & \longrightarrow & \mathbb{C}^N_+ \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}^N_* & \longrightarrow & \mathbb{T}\mathbb{R}^N_* & \longrightarrow & \mathbb{C}^N_* \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}^N & \longrightarrow & \mathbb{T}\mathbb{R}^N & \longrightarrow & \mathbb{C}^N
\end{array}
$$

Under some mild extra axioms (..), these are the only ones.

$\Longrightarrow$ must first "develop" all these geometries
$\Longrightarrow$ then go towards "Nash-Connes Geometry"

Models 1/4

We are interested in random matrix models for our algebras:

$$\pi : C(G) \to M_K(C(T))$$

The Hopf image of π is the smallest quotient Hopf C^*-algebra $C(G) \to C(H)$ producing a factorization of type

$$\pi : C(G) \to C(H) \to M_K(C(T))$$

When $H \subset G$ is an isomorphism, we say that π is inner faithful.

Models 2/4

The inner faithful models $\pi : C(G) \to M_K(C(T))$ are conjectured to exist in general, and remind the quantum group:

(1) The Tannakian category of G is given by the formula

$$C_{kl} = Hom(U^{\otimes k}, U^{\otimes l})$$

where $U_{ij} = \pi(u_{ij})$, with formal intertwiner spaces on the right.

(2) The Haar integration over G is given by the formula

$$\int_G = \lim_{k\to\infty} \frac{1}{k} \sum_{r=1}^{k} \int_G^r$$

where $\int_G^r = (\varphi \circ \pi)^{*r}$, with $\varphi = tr \otimes \int_T$ being the standard trace.

Models 3/4

A model $\pi : C(G) \to M_K(C(T))$ is called stationary when:

$$\int_G = \left(tr \otimes \int_T \right) \pi$$

In this case, the model must be faithful. We have as basic example

$$C(O_N^*) \to M_2(C(U_N)) \quad , \quad u_{ij} \to \begin{pmatrix} 0 & v_{ij} \\ \bar{v}_{ij} & 0 \end{pmatrix}$$

where v is the fundamental corepresentation of $C(U_N)$, as well as

$$C(U_N^*) \to M_2(C(U_N \times U_N)) \quad , \quad u_{ij} \to \begin{pmatrix} 0 & v_{ij} \\ w_{ij} & 0 \end{pmatrix}$$

with v, w corresponding to the two copies of $C(U_N)$.

Models 4/4

As another basic example, we have a stationary matrix model

$$\pi : C(S_4^+) \to M_4(C(SU_2))$$

given on the standard coordinates by the formula

$$\pi(u_{ij}) = [x \to Proj(c_i x c_j)]$$

where $x \in SU_2$, and c_1, c_2, c_3, c_4 are the Pauli matrices.

Questions

(1) Half-liberation.

(2) Weyl matrix models.

(3) Universal flat models.

(4) Sinkhorn and other.

Matrices 1/4

A pair of orthogonal MASA is a pair of maximal abelian subalgebras

$$B, C \subset A$$

which are orthogonal: $tr(bc) = tr(b)tr(c)$, for any $b \in B, c \in C$.

Popa: up to a unitary, the pairs of orthogonal MASA in the simplest von Neumann factor, namely $M_N(\mathbb{C})$, are

$$A = \Delta \quad , \quad B = H\Delta H^*$$

with $\Delta =$ diagonal matrices, and $H \in M_N(\mathbb{C})$ being Hadamard (entries on the unit circle, rows pairwise orthogonal).

Matrices 2/4

(1) Given $H \in M_N(\mathbb{C})$ Hadamard, the associated pair of MASA fit into a "commuting square" in the sense of subfactor theory:

$$\begin{array}{ccc} \Delta & \longrightarrow & M_N(\mathbb{C}) \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & H\Delta H^* \end{array}$$

(2) By "basic construction" we obtain a subfactor $Q \subset R$, whose invariants can be computed using "Ocneanu compactness".

(3) Work of Jones shows that Q must appear as fixed point algebra under some kind of "quantum permutation group" action.

Matrices 3/4

Given an Hadamard matrix $H \in M_N(\mathbb{C})$, the rank 1 projections

$$P_{ij} = Proj\left(\frac{H_i}{H_j}\right)$$

where $H_1, \ldots, H_N \in \mathbb{T}^N$ are the rows of H, form a magic unitary.

$\implies$ We associate to H the quantum permutation group $G \subset S_N^+$ given by the following Hopf image factorization,

$$\begin{array}{ccc} C(S_N^+) & \xrightarrow{\ \ \pi\ \ } & M_N(\mathbb{C}) \\ & \searrow \quad \nearrow & \\ & C(G) & \end{array}$$

where $\pi(u_{ij}) = Proj(H_i/H_j)$ are the above rank 1 projections.

Matrices 4/4

The main results regarding $H \to G$ are as follows:

(1) Fourier: $F_G \to G$. Also $H' \otimes H'' \to G' \times G$".

(2) Various abstract results: Haar, Tannaka, duality.

(3) Relation with commuting squares and subfactors.

(4) Diță deformations of F_G, with various parameters.

(5) Extensions to the partial Hadamard matrix setting.

Questions

Physics!

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