RIGIDITY QUESTIONS FOR REAL HALF-CLASSICAL MANIFOLDS

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Abstract. We study the quantum isometry groups of the half-classical manifolds $X \subset S^{N-1}_{\mathbb{R},*}$, and in particular, of the half-classical affine homogeneous spaces. We formulate some rigidity questions for such manifolds, at the affine level, and at the metric level too, by using the orthogonal filtration formalism that we introduced with Skalski.

Introduction

According to the general operator algebra philosophy, one reasonable definition for the noncommutative analogues of the real compact algebraic manifolds comes by considering the abstract spectra X of the universal C^*-algebras of the following type:

$$C(X) = C^*\left(x_1,\ldots,x_N\Big|x_i=x_i^*,P_\alpha(x_1,\ldots,x_N)=0\right)$$

To be more precise, given noncommutative polynomials $P_\alpha \in \mathbb{R} < X_1,\ldots,X_N >$ which are such that the maximal C^*-norm on the associated universal $*$-algebra is bounded, the above C^*-algebra exists indeed, and we can call it $C(X)$. According to a well-known theorem of Gelfand, when we divide $C(X)$ by its commutator ideal, we obtain the algebra $C(X_{class})$ of continuous functions on the following real algebraic manifold:

$$X_{class} = \left\{x\in\mathbb{R}^N\Big|P_\alpha(x_1,\ldots,x_N)=0\right\}$$

In particular when $C(X)$ happens to be commutative, we have $X = X_{class}$, and so our formalism covers all the real compact algebraic manifolds. In general, X can be thought of as being a "liberation" of X_{class}, and the question of studying it appears.

All this is of course quite abstract. Motivated by some quantum group work in [1], [5], [10], [11], we will be interested here in the case $X \subset S^{N-1}_{\mathbb{R},*}$, where:

$$C(S^{N-1}_{\mathbb{R},*}) = C^*\left(x_1,\ldots,x_N\Big|x_i=x_i^*,\, x_ix_jx_k=x_kx_jx_i,\,\sum_i x_i^2=1\right)$$

The relations $abc = cba$ are called half-commutation relations, and $S^{N-1}_{\mathbb{R},*}$ itself is called half-classical sphere. The point with all this comes from the fact that the submanifolds $X \subset S^{N-1}_{\mathbb{R},*}$ can be completely classified, as shown in [10], and under a mild assumption,

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they come via a crossed product type construction from the submanifolds $Y\subset S^{N-1}_\mathbb{C}$. In short, the half-commutation relations $abc=cba$ bring us into something very concrete, and there is definitely some work to be done.

We will study here the quantum isometry groups of the submanifolds $X\subset S^{N-1}_{\mathbb{R},*}$. We will formulate some rigidity questions for such manifolds, at the affine level, and at the metric level too, by using the orthogonal filtration formalism [7], [18]. Our main tools will be the maximal tori, following some previous work from [3], [6]. We will pay particular attention to the case of the half-classical affine homogeneous spaces [2].

The present work is of course inspired by Goswami's theory of quantum isometry groups [15], [16], and in particular, by the Goswami-Joardar rigidity result [17]. Our main results will always state that, under some suitable assumptions, of algebraic or metric nature, "only half-classical quantum groups can act on half-classical manifolds". We should mention, however, that in regards with the various rigidity results available in the classical case [9], [12], [17], our statements here are quite modest. For going further we would need much more detailed information about the Riemannian structure of $X\subset S^{N-1}_{\mathbb{R},*}$, in the case where the base manifold $Y\subset S^{N-1}_\mathbb{C}$ happens to be Riemannian.

There are as well many questions regarding the potential extensions of the theory of half-liberation, and of the present work in particular. One of them regards the extension via twisting methods, in the spirit of [14]. Another one regards complex extensions, using [4]. This latter question is of particular interest in the framework of the noncommutative geometry à la Connes [13], due to the occurrence of the half-classical versions of U_N in the study of the associated Standard Model quantum gauge group [8].

The paper is organized as follows: 1 is a preliminary section, in 2 we discuss affine quantum isometries, in 3 we discuss Riemannian aspects, and in 4 we restrict the attention to the affine homogeneous spaces, and we formulate a conjectural rigidity statement.

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1. Half-classical manifolds

We use Woronowicz's quantum group formalism [20], under the assumption $S^2=id$. We are particularly interested in Wang's free quantum groups [19].

The real half-liberation theory was developed in [1], [10], [11]. We will need in particular the following key result, due to Bichon and Dubois-Violette [11]:

Proposition 1.1. *Given a conjugation-stable closed subgroup $H\subset U_N$, consider the algebra $C([H])\subset M_2(C(H))$ generated by the following variables:*

$$u_{ij}=\begin{pmatrix}0&v_{ij}\\ \bar{v}_{ij}&0\end{pmatrix}$$

Then $[H]$ is a compact quantum group, we have $[H]\subset O_N^*$, and any non-classical subgroup $G\subset O_N^*$ appears in this way, with $G=O_N^*$ itself appearing from $H=U_N$.

Proof. The 2×2 matrices in the statement are self-adjoint, half-commute, and the $N\times N$ matrix $u=(u_{ij})$ that they form is orthogonal, so we have an embedding $[H]\subset O_N^*$. The quantum group property of $[H]$ is also elementary to check, by using an alternative, equivalent construction, with a quantum group embedding $C([H])\subset C(H)\rtimes\mathbb{Z}_2$.

The surjectivity part is non-trivial, and we refer here to [11]. □

We will need as well the following result, also from [11]:

Proposition 1.2. *We have a bijection $Irr([H])\simeq Irr_0(H)\coprod Irr_1(H)$, where*

$$Irr_k(H)=\left\{r\in Irr(H)\Big|\exists l\in\mathbb{N},r\in u^{\otimes k}\otimes(u\otimes\bar{u})^{\otimes l}\right\}$$

induced by the canonical identification $Irr(H\rtimes\mathbb{Z}_2)\simeq Irr(H)\coprod Irr(H)$.

Proof. We have an equality of projective versions $P[H]=PH$, and so an inclusion $Irr_0(H)=Irr(PH)\subset Irr([H])$. The remaining irreducible representations of $[H]$ must come from an inclusion $Irr_1(H)\subset Irr([H])$, appearing as above. See [11]. □

Regarding now the manifolds, given a subspace $Y\subset S^{N-1}_\mathbb{C}$, with standard coordinates $z_1,\ldots,z_N$, we can construct a subspace $[Y]\subset S^{N-1}_{\mathbb{R},*}$, with coordinates as follows:

$$x_i=\begin{pmatrix}0&z_i\\ \bar{z}_i&0\end{pmatrix}$$

Observe that these matrices are indeed self-adjoint, half-commute, and their squares sum up to 1. Let us also call $X\subset S^{N-1}_{\mathbb{R},*}$ symmetric when it has an automorphism given by $x\to -x$. With these conventions, we have the following result, due to Bichon [10]:

Theorem 1.3. *The symmetric closed subspaces $X\subset S^{N-1}_{\mathbb{R},*}$ appear as follows:*

1. *We have $X=[Y]$, for a certain conjugation-invariant subspace $Y\subset S^{N-1}_\mathbb{C}$.*
2. *$PX=P[Y]$, and $X\subset S^{N-1}_{\mathbb{R},*}$ is maximal with this property.*
3. *$X\subset S^{N-1}_{\mathbb{R},*}$ is algebraic if and only if $Y\subset S^{N-1}_\mathbb{C}$ is algebraic.*

Proof. The key observation is that for any closed subspace $X\subset S^{N-1}_{\mathbb{R},*}$, the projective version PX is classical. Thus we have $PX\subset P^{N-1}_\mathbb{C}$, and one can proceed as follows:

(1) The idea here is to construct $Y\subset S^{N-1}_\mathbb{C}$ as being the affine lift of $PX\subset P^{N-1}_\mathbb{C}$, and then to prove that we have indeed $X=[Y]$. For the details, see [10].

(2) With the above proof of (1), the condition $PX=PY$ is automatic, and the fact that X is maximal with this property is clear too. We refer here to [1].

(3) This is just an abstract consequence of (1), or of (2), see [1], [10]. □

2. Quantum isometries

We discuss here the computation of affine quantum isometry groups. We recall that the free complex sphere $S^{N-1}_{\mathbb{C},+}$ is the noncommutative compact space whose coordinates $z_1,\ldots,z_N$ are subject to the relations $\sum_i z_iz_i^*=\sum_i z_i^*z_i=1$. See [1].

We use the following formalism of affine quantum isometry groups:

Definition 2.1. *A closed quantum subgroup $G\subset U_N^+$ is said to be acting affinely on an algebraic submanifold $X\subset S^{N-1}_{\mathbb{C},+}$ when we have a morphism of algebras, as follows:*

$$\Phi:C(X)\to C(G)\otimes C(X)\quad,\quad z_i\to\sum_j u_{ij}\otimes z_j$$

The biggest closed quantum group $G\subset U_N^+$ acting affinely on X is called affine quantum isometry group of X, and is denoted $G^+(X)$.

Here the fact that $G^+(X)$ exists indeed, and is unique, follows by dividing the algebra $C(U_N^+)$ by a suitable ideal, coming from the existence of Φ. See [1], [16].

We will be interested in what follows in the affine quantum isometry groups of the submanifolds $X\subset S^{N-1}_{\mathbb{R},*}\subset S^{N-1}_{\mathbb{C},+}$, with the embedding on the right being the standard one, given by $x_i\to z_i$. We first have the following functoriality result:

Proposition 2.2. *Given a closed subgroup $H\subset U_N$, $H\curvearrowright Y$ implies $[H]\curvearrowright[Y]$.*

Proof. We use Proposition 1.1 and Theorem 1.3 above, in their crossed product picture. Let us recall indeed from [10], [11] that we have inclusions $C([H])\subset C(H)\rtimes\mathbb{Z}_2$ and $C([Y])\subset C(Y)\rtimes\mathbb{Z}_2$ constructed as follows, where τ is the standard generator of $\mathbb{Z}_2$:

$$u_{ij}=v_{ij}\otimes\tau\quad,\quad x_i=z_i\otimes\tau$$

The idea now will be, starting with an affine coaction $\Phi:C(Y)\to C(H)\otimes C(Y)$, to constuct a certain morphism $\widetilde{\Phi}:[C(Y)\rtimes\mathbb{Z}_2]\to[C(H)\rtimes\mathbb{Z}_2]\otimes[C(Y)\rtimes\mathbb{Z}_2]$, and then to obtain by restriction an affine coaction $\Psi:C([Y])\to C([H])\otimes C([Y])$, as desired.

To be more precise, starting from an affine action Φ, we can define a morphism $\widetilde{\Phi}$ as above by the following formulae, using the standard leg-numbering convention:

$$f\otimes 1\to\Phi(f)_{13}\quad,\quad \tau\to(\tau\otimes\tau)_{24}$$

Our claim now is that we have a factorization diagram, as follows:

$$\begin{matrix}C(Y\rtimes\mathbb{Z}_2)&\to&[C(H)\rtimes\mathbb{Z}_2]\otimes[C(Y)\rtimes\mathbb{Z}_2]\\ \cup&&\cup\\ C([Y])&\to&C([H])\otimes C([Y])\end{matrix}$$

Indeed, in order to construct the map on the bottom, it is enough to show that the standard generators $x_i\in C([Y])$ map into $C([H])\otimes C([Y])$. But this is indeed the case, because via the above identifications from [10], [11], these generators map as follows:

$$\begin{array}{ccc} z_i\otimes\tau & \to & \sum_j(v_{ij}\otimes\tau)\otimes(z_j\otimes\tau)\\ \uparrow & & \uparrow\\ x_i & \to & \sum_j v_{ij}\otimes x_j \end{array}$$

Thus we have constructed a morphism $\Psi : C([Y]) \to C([H]) \otimes C([Y])$, which is by definition an affine coaction in the sense of Definition 2.1, as claimed. □

The problem now is that of deciding whether the above actions $[H] \curvearrowright [Y]$ can be universal or not. There are several computations available here, from [1] and a number of other papers, which tell us that in general, this is not the case. However, under suitable rigidity assumptions on Y, we have a rigidity result for $[Y]$, as follows:

Theorem 2.3. *Let $X \subset S^{N-1}_{\mathbb{R},*}$ be symmetric, write $X = [Y]$ with $Y \subset S^{N-1}_{\mathbb{C}}$ as above, and assume that $Y \subset S^{N-1}_{\mathbb{C}}$ is "projectively rigid", in the sense that:*

1. *The quantum isometry group $H = G^+(Y)$ is classical, $H \subset U_N$.*
2. *Moreover, PH is the projective quantum isometry group of PY.*

Then X itself follows to be rigid, in the sense that we have $G^+(X) \subset O^*_N$.

Proof. Our claim, which will prove the result, is that we have $G^+(X) = [H]$. In one sense, this follows from Proposition 2.2 above, because we have:

$$H \curvearrowright Y \implies [H] \curvearrowright [Y] = X \implies [H] \subset G^+(X)$$

Conversely now, using the projective quantum isometry group formalism from [4], along with the identifications in Theorem 1.3, and our assumptions, we have:

$$\begin{aligned} L \subset G^+(X) &\implies L \curvearrowright X\\ &\implies PL \curvearrowright PX = P[X] = PY\\ &\implies PL \subset PH \end{aligned}$$

Thus PL is classical, and by using one more time Theorem 1.3, in its quantum group version now, we obtain by lifting $L \subset [H]$, which finishes the proof. □

The above result is of course quite theoretical, and simply transfers the affine rigidity questions for the half-classical manifolds $X \subset S^{N-1}_{\mathbb{R},*}$ into projective rigidity questions for the classical manifolds $Y \subset S^{N-1}_{\mathbb{C}}$. Regarding these latter questions, only a few technical results are available here so far, and for details we refer to [1], [4].

3. Riemannian aspects

One interesting question regarding the half-classical manifolds $X\subset S^{N-1}_{\mathbb R,*}$ concerns their possible Riemannian structure. As explained some time ago in [5], this question is open for the sphere $S^{N-1}_{\mathbb R,*}$ itself. To be more precise, it is not difficult to guess what the eigenspaces of the Laplacian should be, but the problem is with the eigenvalues. Besides this eigenvalue problem, there is a second question, which looks difficult as well, which is that of converting the Laplacian into a Dirac operator à la Connes [13].

This being said, as observed by Goswami in [15], and heavily used in a number of subsequent papers, including [5], knowing the eigenspaces of the Laplacian suffices for talking about quantum isometry groups. So, we will take advantage of this fact.

For our purposes, best is to use directly the notion of orthogonal filtration, introduced in [7], and further studied in [18]. The definition here is as follows:

Definition 3.1. *Given a unital C^*-algebra $A=C(X)$, having a faithful positive unital trace $\int_X:C(X)\to\mathbb C$, an orthogonal filtration for it is a decomposition of type*

$$C(X)=\overline{\bigoplus_i}E_i$$

with the bar sign standing for the norm closure, and with the summands E_i being finite dimensional, and pairwise orthogonal with respect to the trace.

As a main example, assuming that X is a Riemannian manifold, we can equip $C(X)$ with the Riemannian integration functional $\int_X$, and construct as well the corresponding Hodge Laplacian Δ, and then consider its eigenspaces E_i. We obtain in this way an orthogonal filtration for $C(X)$, in the above sense. In general, there are many other examples, and an orthogonal filtration is best thought of as being a kind of "Laplacian without eigenvalues" for the space X. For more on this material, see [3], [7], [18].

The main result in [7], heavily inspired from [15], states that in the setting of Definition 3.1 above, we can talk about the corresponding quantum isometry group of X:

Definition 3.2. *The biggest compact quantum group acting on X by preserving the filtration is called "metric" quantum isometry group of X, and is denoted $\mathcal G^+(X)$.*

Here the notation $\mathcal G^+(X)$ stands for distinguishing this quantum group from the affine quantum isometry one $G^+(X)$, in case where X happens to be a submanifold $X\subset S^{N-1}_{\mathbb C,+}$, because these quantum groups are in general different. Understanding when the equality $\mathcal G^+(X)=G^+(X)$ holds is actually a very interesting problem.

Now back to our questions regarding the submanifolds $X\subset S^{N-1}_{\mathbb R,*}$, we have two problems to be solved, namely the construction of the filtration, and then the study of the corresponding quantum isometry group $\mathcal G^+(X)$. Regarding the first question, for the sphere itself this was discussed in [5]. In general the answer is similar, as follows:

Proposition 3.3. *Let $X\subset S^{N-1}_{\mathbb{R},*}$ be symmetric, write $X=[Y]$, and assume that $Y\subset S^{N-1}_{\mathbb{C}}$ is Riemannian. The algebra $C(X)$ has then an orthogonal filtration, as follows:*

1. *The trace $tr: C([Y])\subset C(Y)\rtimes\mathbb{Z}_2\to\mathbb{C}$ is the canonical one, coming from the Riemannian integration functional $tr: C(Y)\to\mathbb{C}$.*
2. *The filtration itself appears by decomposing $C([Y])\simeq C(PY)\oplus C(PY)^\perp$, and by using the Laplacian filtration on $C(Y)=C(PY)\oplus C(PY)^\perp$.*

Proof. The functional tr in the statement is indeed positive, unital and tracial. Moreover, it follows from the identifications in Theorem 1.3 that this trace is simply the restriction to $C([Y])$ of the usual matrix trace on $M_2(C(Y))$, via the canonical embedding:

$$\begin{matrix} C(Y)\rtimes\mathbb{Z}_2 & \subset & M_2(C(Y))\\ \cup & & \downarrow\\ C([Y]) & \to & \mathbb{C}\end{matrix}$$

Now since the usual trace on $M_2(C(Y))$ is faithful, so is its restriction tr.

Regarding now the filtration, we use here the fact, coming from Theorem 1.3 above, that when performing the passage $Y\to[Y]=X$, the projective algebra part $C(PY)$ is left unchanged, and the linear space $C(PY)^\perp$ gets replaced by a space which is isomorphic to it. Thus, we obtain in this way a direct sum decomposition as in Definition 3.1, and the orthogonality property follows from the above description of the trace. □

We recall now that for a connected connected Riemannian manifold X, the eigenfunctions of the Laplacian have the domain property, in the sense that $f,g\neq 0$ implies $fg\neq 0$. This is for instance because the set of zeros of each nonzero eigenfunction of the Laplacian is known to have Hausdorff dimension $\dim X-1$, and hence measure zero.

Based on this fact, we have the following result, from [8]:

Proposition 3.4. *A compact connected Riemannian manifold X cannot have an isometric action of a non-classical group dual.*

Proof. Assume that we have a group dual coaction $\Phi: C(X)\to C^*(\Gamma)\otimes C(X)$. Let $E=E_1\oplus E_2$ be the direct sum of two eigenspaces of L. Pick a basis $\{x_i\}$ such that the corepresentation on E becomes diagonal, i.e. $\Phi(x_i)=g_i\otimes x_i$ with $g_i\in\Gamma$. The formula $\Phi(x_ix_j)=\Phi(x_jx_i)$ reads $g_ig_j\otimes x_ix_j=g_jg_i\otimes x_ix_j$, and by using the domain property we obtain $g_ig_j=g_jg_i$. Also, the formula $\Phi(x_i\bar{x}_j)=\Phi(\bar{x}_jx_i)$ reads $g_ig_j^{-1}\otimes x_i\bar{x}_j=g_j^{-1}g_i\otimes x_i\bar{x}_j$, and by using the domain property again, we obtain $g_ig_j^{-1}=g_j^{-1}g_i$. Thus the elements $\{g_i,g_i^{-1}\}$ mutually commute, and with E varying, this shows that Γ is abelian. See [3]. □

The above result is of course obsolete, in view of the general rigidity result obtained by Goswami and Joardar in [17]. However, and here comes our point, the same idea can be used as well in the half-classical setting, where the technology from [17] does not seem to apply, and where we can therefore obtain some non-trivial applications.

We first need to understand the notion of connectivity, in the orthogonal filtration framework. For our purposes, the result that we will need is as follows:

Proposition 3.5. *Assume that $X=[Y]\subset S^{N-1}_{\mathbb R,*}$ is as in Proposition 3.3 above, with $Y\subset S^{N-1}_{\mathbb C}$ being connected. Then X is "connected" as well, in the sense that*

$$f_1\in E_{i_1}-\{0\},\ldots,f_n\in E_{i_n}-\{0\}\implies f_1\ldots f_n\neq 0$$

where E_i are the spaces of the orthogonal filtration on $C(X)$.

Proof. Since the manifold Y is Riemannian and connected, the above-mentioned standard results on the eigenfunctions of the Laplacian show that the associated orthogonal filtration on $C(Y)$ has indeed the connectivity property in the statement.

Now since the filtration on $C(X)$ is obtained from the one on $C(Y)$ simply by introducing a $\mathbb Z_2$ twist, this property will hold as well for the filtration on $C(X)$, as claimed. □

We have the following half-classical version of Proposition 3.4 above:

Proposition 3.6. *Assume that $X\subset S^{N-1}_{\mathbb R,*}$ is given with an orthogonal filtration of $C(X)$, and is in addition connected, in the sense of Proposition 3.5. Then, given a group dual isometric action $\widehat{\Gamma}\curvearrowright X$, the group Γ follows to be half-classical, $\widehat{\Gamma}\subset O_N^*$.*

Proof. We use the same method as in the proof of Proposition 3.4. Assume that we have a group dual coaction $\Phi:C(X)\to C^*(\Gamma)\otimes C(X)$. Let $E=E_1\oplus E_2\oplus E_3$ be the direct sum of three spaces of the orthogonal filtration. Pick a basis $\{x_i\}$ such that the corepresentation on E becomes diagonal, i.e. $\Phi(x_i)=g_i\otimes x_i$ with $g_i\in\Gamma$.

By using twice the half-commutation relations between the standard coordinates of X, and then the domain property, as formulated in Proposition 3.5, we obtain:

$$\begin{aligned}\Phi(x_ix_jx_k)=\Phi(x_kx_jx_i)&\implies g_ig_jg_k\otimes x_ix_jx_k=g_kg_jg_i\otimes x_kx_jx_i\\&\implies g_ig_jg_k\otimes x_ix_jx_k=g_kg_jg_i\otimes x_ix_jx_k\\&\implies g_ig_jg_k=g_kg_jg_i\end{aligned}$$

Thus the elements g_i mutually half-commute. On the other hand from $\Phi(x_i)=g_i\otimes x_i$ we obtain by conjugating $\Phi(x_i)=g_i^{-1}\otimes x_i$, so these elements are reflections, $g_i^2=1$. Summing up, we have proved that Γ is generated by half-commuting reflections, and this means that we have an embedding $\widehat{\Gamma}\subset O_N^*$, given by $u_{ij}=\delta_{ij}g_i$, as claimed. □

In order to further advance, let us recall from [3], [6] that associated to any subgroup $G\subset U_N^+$ is a family of group dual subgroups $\{\widehat{\Gamma}_Q|Q\in U_N\}$, constructed as follows:

$$C^*(\Gamma_Q)=C(G)\Big/\Big\langle (QuQ^*)_{ij}=0,\forall i\neq j\Big\rangle$$

Here u is the fundamental corepresentation of G, and the key observation is that the elements $g_i = (QuQ^*)_{ii}$ are group-like in the quotient on the right. These group dual subgroups, taken altogether, play the role of the "maximal torus" for G. See [3], [6].

In the half-classical case, we have the following result, from [6]:

Proposition 3.7. *The group dual subgroups $\widehat{[\Gamma]}_Q \subset [H]$ appear via*

$$[\Gamma]_Q = [\Gamma_Q]$$

from the group dual subgroups $\widehat{\Gamma}_Q \subset H$ associated to $H \subset U_N$.

Proof. By using the crossed product picture, the operation $H \to [H]$ constructed in Proposition 1.1 is functorial, and this gives the result. See [6]. □

With all these ingredients in hand, we can now formulate a result in the spirit of those found in [8], but this time in the half-classical case, as follows:

Theorem 3.8. *Let $X \subset S^{N-1}_{\mathbb{R},*}$ be symmetric, write $X = [Y]$, and assume that $Y \subset S^{N-1}_{\mathbb{C}}$ is Riemannian and connected. Endow $C(X)$ with the orthogonal filtration coming from the one on $C(Y)$, and consider the corresponding quantum group $\mathcal{G}^+(X)$. Then:*

1. *Either $\mathcal{G}^+(X)$ is half-classical.*
2. *Or $\mathcal{G}^+(X)$ is "half-strange", in the sense that it is not half-classical, but all its maximal tori are half-classical.*

Proof. We know from Proposition 3.5 that X is connected in the technical sense formulated there, and so Proposition 3.6 applies. It follows that all the maximal tori of $\mathcal{G}^+(X)$ must be half-classical, and so we are led to the dichotomy in the statement. □

Observe the similarity with the main result in [3]. The notion of "half-strangeness" introduced above complements the work in [2], and raises many questions, in the same spirit as those raised there. We believe that a proof of the general conjectures in [6] can help here, but we have no further advances on these questions.

4. Homogeneous spaces

In this section we discuss the homogeneous space case. Following [2], we will be interested in the following special types of homogeneous spaces:

Definition 4.1. *An affine homogeneous space over $G \subset O_N^+$ is a closed subset $X \subset S^{N-1}_{\mathbb{R},+}$, such that there exists an index set $I \subset \{1, \ldots, N\}$ such that*

$$\alpha(x_i) = \frac{1}{\sqrt{|I|}} \sum_{j \in I} u_{ij} \quad , \quad \Phi(x_i) = \sum_j u_{ij} \otimes x_j$$

define morphisms of C^*-algebras, satisfying $(\int_G \otimes id)\Phi = \int_G \alpha(.)1$.

Observe that the condition $(id\otimes\Phi)\Phi=(\Delta\otimes id)\Phi$ is satisfied, and that we have as well $(id\otimes\alpha)\Phi=\Delta\alpha$. Note that we do not assume that α is injective, and this, in order to cover some basic examples, such as the standard action $O_N^+\curvearrowright S^{N-1}_{\mathbb{R},+}$. See [2].

The above definition is quite tricky, coming from a long series of papers, the last of which is [2]. The idea is that this formalism covers all the known examples of real homogeneous spaces for which a Weingarten integration formula is available. See [2].

In order to deal with the half-classical case, we will need:

Proposition 4.2. *For a permutation $\sigma\in S_k$, the following two conditions, involving abstract self-adjoint variables $x_1,\ldots,x_N$ satisfying $\sum_i x_i^2=1$, are equivalent:*

(1) $x_{i_1}\ldots x_{i_k}=x_{i_{\sigma(1)}}\ldots x_{i_{\sigma(k)}}$.
(2) $\sum_{i_1\ldots i_k}x_{i_1}\ldots x_{i_k}x_{i_{\sigma(k)}}\ldots x_{i_{\sigma(1)}}=1$.

Proof. The implication (1) $\implies$ (2) is trivial, coming from the following computation, with our assumption $\sum_i x_i^2=1$ being used k times, in order to conclude:

$$\sum_{i_1\ldots i_k}x_{i_1}\ldots x_{i_k}x_{i_{\sigma(k)}}\ldots x_{i_{\sigma(1)}}=\sum_{i_1\ldots i_k}x_{i_1}\ldots x_{i_k}x_{i_k}\ldots x_{i_1}=1$$

As for (2) $\implies$ (1), this requires a positivity trick. We have indeed:

$$\begin{aligned}
&\sum_{i_1\ldots i_k}(x_{i_1}\ldots x_{i_k}-x_{i_{\sigma(1)}}\ldots x_{i_{\sigma(k)}})(x_{i_1}\ldots x_{i_k}-x_{i_{\sigma(1)}}\ldots x_{i_{\sigma(k)}})^*\\
=&\sum_{i_1\ldots i_k}x_{i_1}\ldots x_{i_k}x_{i_k}\ldots x_{i_1}-x_{i_1}\ldots x_{i_k}x_{i_{\sigma(k)}}\ldots x_{i_{\sigma(1)}}\\
&-x_{i_{\sigma(1)}}\ldots x_{i_{\sigma(k)}}x_{i_k}\ldots x_{i_1}+x_{i_{\sigma(1)}}\ldots x_{i_{\sigma(k)}}x_{i_{\sigma(k)}}\ldots x_{i_{\sigma(1)}}\\
=&\ 1-1-1+1=0
\end{aligned}$$

Here we have used k times our assumption $\sum_i x_i^2=1$, for computing the first and last sums, and the two middle sums were computed by using our assumption (2).

Now since we are in a situation of type $\sum_I A_IA_I^*=0$, by positivity it follows that we have $A_I=0$ for any I, and we recover precisely the condition (1), as desired. □

As a consequence, we have the following result:

Proposition 4.3. *An affine homogeneous space X over a half-classical quantum group $G\subset O_N^*$ must be half-classical itself, $X\subset S^{N-1}_{\mathbb{R},*}$.*

Proof. This is the half-classical analogue of a classical result established in [2], and we can use here the same proof, with Proposition 4.2 above as technical ingredient.

To be more precise, as explained in [2], the ergodicity assumption $(\int_G\otimes id)\Phi=\int_G\alpha(.)1$ made in Definition 4.1, or rather its Tannakian translation, shows that the formula in Proposition 4.2 (2) is satisfied over X, with $\sigma=(321)\in S_3$. Thus the formula in Proposition 4.2 (1) is satisfied as well over X, and this gives $X\subset S^{N-1}_{\mathbb{R},*}$, as desired. □

We recall now from [2] that among the affine homogeneous spaces coming from a pair (G,I) there is a minimal one and a maximal one, $X_{G,I}\subset\widetilde{X}_{G,I}$. The associated quotient map $C(\widetilde{X}_{G,I})\to C(X_{G,I})$ is a bit similar to the quotient maps $C^*(\Gamma)\to C^*_{red}(\Gamma)$, and as in the group dual case, we can use a GNS construction in order to identify $X_{G,I}$ with $\widetilde{X}_{G,I}$ and with the other affine homogeneous spaces coming from (G,I). See [2].

With this identification made, we can now improve the above result, by using the general half-liberation technology from section 1, as follows:

Proposition 4.4. *Assuming that $G\subset O_N^*$ is non-classical, appearing as $G=[H]$ with $H\subset U_N$, we have an identification $X_{G,I}=[X_{H,I}]$.*

Proof. According to the explicit description given in [2] of the minimal affine homogeneous space $X_{G,I}$, and to Proposition 1.1 above, we have inclusions as follows:

$$C(X_{G,I})\subset C(G)\subset M_2(C(H))$$

At the level of the standard generators, these identifications are as follows:

$$x_i=\frac{1}{\sqrt{|I|}}\sum_{j\in I}u_{ij}=\frac{1}{\sqrt{|I|}}\sum_{j\in I}\begin{pmatrix}0&v_{ij}\\ \bar{v}_{ij}&0\end{pmatrix}=\begin{pmatrix}0&z_{ij}\\ \bar{z}_{ij}&0\end{pmatrix}$$

Now since the elements on the right are precisely the standard coordinates on the half-classical space $[X_{H,I}]$ produced via the construction in Theorem 1.3 by the affine homogeneous space $X_{H,I}$, we have an identification $X_{G,I}=[X_{H,I}]$, as claimed. □

Summarizing, the constructions in [2] behave well with respect to the general machinery from section 1 above. Regarding now our rigidity questions, the available results, for spheres and a few related spaces, suggest the following conjecture:

Conjecture 4.5. *Assuming that $G\subset O_N^*$ is non-classical, appearing as $G=[H]$ with $H\subset U_N$ being connected, the quantum isometry group of $X_{G,I}$ is half-classical.*

Here we are of course a bit vague, our more precise claim being that we should have such a statement both in the affine and the metric framework, with the corresponding quantum isometry groups being related to each other. At the affine level, some evidence for the conjecture, or at least the connection with the notion of conectedness, comes from the classification result in Proposition 1.2 above, via the various results and conjectures in [6]. At the metric level, the evidence comes from the results in section 3.

As already mentioned in the introduction, some other interesting questions, having a number of potential applications, concern the extension of the results found above to the twisted setting, and to the unitary setting as well. The common framework for all this would be a twisted unitary setting, but the general theory here is not available yet.

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