The magic of random matrices

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Abstract. This is an introduction to random matrices, going from the basics up to advanced aspects, written by using standard calculus and combinatorics.

Preface

There has been debate over the last decades on whether advanced branches of physics, such as statistical or quantum mechanics, are about usual matrices $A \in M_N(\mathbb{C})$, or about random matrices $Z \in M_N(L^\infty(X))$, or about general operators $T \in B(H)$:

(1) Generally speaking, the usual matrices $A \in M_N(\mathbb{C})$ can encode a lot of subtle information, with famous matrices such as the Pauli ones, or the Dirac ones, witnessing for that. There are even more complicated matrices, such as the Gell-Mann or the Cabibbo-Kobayashi-Maskawa ones, worth a lifetime study. Also nice, in relation with usual matrices, is that so much theory has been developed for them, that regardless of the fact that your problem is relevant or not, you will certainly get results about it.

(2) With the general operators $T \in B(H)$, you are quite safe too. These generalize the usual matrices $A \in M_N(\mathbb{C})$, which correspond to the case where the Hilbert space H is finite dimensional, $H = \mathbb{C}^N$. They are widely believed to be the relevant objects of study, in connection with general questions in quantum mechanics. However, and here comes the drawback, their formalism is perhaps a bit too general, and you can feel this even at the purely foundational level, whose understanding is quite painful.

(3) As for the random matrices $Z \in M_N(L^\infty(X))$, these are yet another type of beast, lying somewhere between matrices and operators. On one hand, they generalize the usual matrices, which can be obtained with $X = \{.\}$. On the other hand, any random matrix can be regarded as an operator, over the Hilbert space $H = \mathbb{C}^N \otimes L^2(X)$. So, nice theory in between, and at the level of both available tools and connections with physics, once again, what we have is a beautiful compromise between matrices and operators.

But you might probably say at this point, too vague all this and what am I doing here, I'm not looking for a Nobel Prize in physics, just want to learn some random matrices, and make some money out of that, with applications to basic engineering, or finance. Good point, and with due excuses, yes indeed, I forgot to mention an important point: the random matrices $Z \in M_N(L^\infty(X))$ generalize the usual random variables $f \in L^\infty(X)$, which correspond to the case $N = 1$, and quite often in real life, when usual probability theory is not enough, random matrices provide the answer. So, stay with us.

The present book is an introduction to random matrices, going from the basics up to more advanced aspects, and meant to be as elementary as possible, while keeping things relatively complete. It is for the most written by using primitive technology, from the Stone Age, namely basic calculus and combinatorics. Some chapters at the end will have to borrow some tools from the Bronze Age, but we will attempt to make this learning as pleasant as possible. And in any case, do not worry, we will never get to fearsome tools like sharp blades and axes, not to talk firearms or atomic bombs, from the Iron Age.

The book is organized in four parts, as follows:

– Part I contains all needed preliminaries, and then the basic theory of the Gaussian, Wigner and Wishart matrices, done via the moment method.

– Part II discusses more advanced aspects of these same matrices, in relation with universality, fluctuations, and asymptotic freeness.

– Part III goes into wild combinatorics, by performing all sorts of tricky modifications on our basic classes of matrices, namely Gaussian, Wigner and Wishart.

– Part IV discusses the love and rivalry between operator algebras and random matrices, via a number of carefully chosen topics.

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World, hold on
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CHAPTER 1

Linear algebra

1a. Laws of matrices

Welcome to random matrices. These are matrices $Z\in M_N(L^\infty(X))$, having as entries random variables $Z_{ij}\in L^\infty(X)$. Equivalently, if you prefer, these are beasts of type $Z\in L^\infty(M_N(\mathbb C))$, meaning families of matrices $Z^x\in M_N(\mathbb C)$, indexed by points of our probability space $x\in X$. Either way, what we have is obviously a "random matrix".

The claim is that random matrices can be useful in connection with all sorts of purposes, ranging from reasonable things like engineering, chemistry and biology, up to abstract and dangerous things like modern mathematics and physics. Not to forget finance, and whether that counts as reasonable or dangerous, I will leave it up to you.

The aim of this book is to provide an introduction to random matrices. As prerequisites, we will assume some familiarity with calculus, linear algebra and probability theory. A bit later, we will also need some basic group theory, and basic functional analysis. All these can be learned from many places, and for a compact starting package, talking a bit about everything that will be needed here, you can check my book [7].

One more thing, you probably noticed that I used in the above the mathematicians' notation $L^\infty(X)$ for the algebra of random variables $f:X\to\mathbb C$ over a probability space X. More notations of this type, of rather mathematical flavor, to follow, and please, be not scared by this. Personally I'm something in between a mathematician and a physicist, but if there are two things that I really love, these are chemistry and engineering. In the hope that so are you, sort of scientist at large, and so let's agree to be pragmatic on everything. And in what concerns notations for mathematics, just trust me, this is one point where mathematics shines, so we will mostly use mathematical notations.

Getting started now, here is our first claim, which is something non-trivial:

Claim 1.1. *The usual matrices $A\in M_N(\mathbb C)$ have probability distributions, a bit as the random variables $f\in L^\infty(X)$ do. In fact, we can talk about the probability distribution of a random matrix $Z\in M_N(L^\infty(X))$, generalizing both these situations.*

Obviously, many things going on here, and untangling all this mess will take us some time, and more precisely, the whole present chapter. Then, with this understood, we will go ahead in chapters 2-4 and afterwards with the study of the distributions of various interesting random matrices, with computations and everything, no worries for that.

Getting started for real now, we need to talk about probability distributions of three types, namely for random variables $f \in L^\infty(X)$, for matrices $A \in M_N(\mathbb{C})$, and then for random matrices $Z \in M_N(L^\infty(X))$. Regarding the random variables $f \in L^\infty(X)$, you are surely familiar with all this, so just some quick reminders, as follows:

Definition 1.2. *Let X be a probability space, that is, a space with a probability measure, and with the corresponding integration denoted $\mathbb{E}$, and called expectation.*

1. *The random variables are the real functions $f \in L^\infty(X)$.*
2. *The moments of such a variable are the numbers $M_k(f) = \mathbb{E}(f^k)$.*
3. *The law of such a variable is the measure given by $M_k(f) = \int_\mathbb{R} x^k d\mu_f(x)$.*

Here, and in what follows, we use the term "law" for "probability distribution", which means exactly the same thing, and is more convenient. Regarding now the fact that the law μ_f exists indeed, this is true, but not exactly trivial. By linearity, we would like to have a probability measure making hold the following formula, for any $P \in \mathbb{C}[X]$:

$$\mathbb{E}(P(f)) = \int_\mathbb{R} P(x) d\mu_f(x)$$

By using a standard continuity argument, it is enough to have this formula for the characteristic functions χ_I of the arbitrary measurable sets of real numbers $I \subset \mathbb{R}$:

$$\mathbb{E}(\chi_I(f)) = \int_\mathbb{R} \chi_I(x) d\mu_f(x)$$

But this latter formula, which reads $\mathbb{P}(f \in I) = \mu_f(I)$, can serve as a definition for μ_f, and we are done. Alternatively, assuming some familiarity with measure theory, μ_f is the push-forward of the probability measure on X, via the function $f : X \to \mathbb{R}$.

Let us summarize this discussion in the form of a theorem, as follows:

Theorem 1.3. *The law μ_f of a random variable f exists indeed, and we have*

$$\mathbb{E}(\varphi(f)) = \int_\mathbb{R} \varphi(x) d\mu_f(x)$$

for any integrable function $\varphi : \mathbb{R} \to \mathbb{C}$.

Proof. This follows from the above discussion, and with the precise assumption on $\varphi : \mathbb{R} \to \mathbb{C}$, which is its integrability, in the abstract mathematical sense, being in fact something that we will not really need, in what follows. In fact, for most purposes we will get away with polynomials $\varphi \in \mathbb{C}[X]$, and by linearity this means that we can get away with monomials $\varphi(x) = x^k$, which brings us back to Definition 1.2 (3), as stated. $\square$

Getting now to the case of the matrices $A\in M_N(\mathbb C)$, here it is quite tricky to figure out what the law of A should mean, based on intuition only. So, in the lack of a bright idea, let us just reproduce Definition 1.2, with a few modifications, as follows:

Definition 1.4. *Let $N\in\mathbb N$, and consider the algebra $M_N(\mathbb C)$ of complex $N\times N$ matrices, with its normalized trace $tr:M_N(\mathbb C)\to\mathbb C$, given by $tr(A)=Tr(A)/N$.*

1. *We call random variables the self-adjoint matrices $A\in M_N(\mathbb C)$.*
2. *The moments of such a variable are the numbers $M_k(A)=tr(A^k)$.*
3. *The law of such a variable is the measure given by $M_k(A)=\int_{\mathbb R}x^kd\mu_A(x)$.*

Here we have normalized the trace, as to have $tr(1)=1$, in analogy with the formula $\mathbb E(1)=1$ from usual probability. By the way, as a piece of advice here, many confusions appear from messing up tr and Tr, and it is better ot forget about Tr, and always use tr. With the drawback that if you're a physicist, tr might get messed up in quick handwriting with the reduced Planck constant $\hbar=h/2\pi$. However, shall you ever face this problem, I have an advice here too, namely forgetting about h, and using h instead of $\hbar$.

Another comment is that we assumed in (1) that our matrix is self-adjoint, $A=A^*$, with the adjoint matrix being given, as usual, by the formula $(A^*)_{ij}=\bar{A}_{ji}$. Why this, because for instance at $N=1$ we would like our matrix, which in the case $N=1$ is a number, to be real, and so we must assume $A=A^*$. Of course there is still some discussion here, for instance because you might argue that why not assuming instead that the entries of A are real. But let us leave this for later, and in the meantime, just trust me. Or perhaps, let us both trust Heisenberg, who was the first intensive user of complex matrices, and who declared that such matrices must be self-adjoint. More later.

Back to work now, what we have in Definition 1.4 looks quite reasonable, but as before with the usual random variables $f\in L^\infty(X)$, some discussion is needed, in order to understand if the law μ_A exists indeed, and by which mechanism. And, good news here, in the case of the simplest matrices, the real diagonal ones, we have:

Proposition 1.5. *For any diagonal matrix $A\in M_N(\mathbb R)$ we have the formula*

$$tr(P(A))=\frac{1}{N}(P(\lambda_1)+\ldots+P(\lambda_N))$$

where $\lambda_1,\ldots,\lambda_N\in\mathbb R$ are the diagonal entries of A. Thus the measure

$$\mu_A=\frac{1}{N}(\delta_{\lambda_1}+\ldots+\delta_{\lambda_N})$$

is the law of A, in the abstract sense of Definition 1.4.

PROOF. Assume indeed that we have a real diagonal matrix, as follows, with the convention that the matrix entries which are missing are by definition 0 entries:

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

The powers of A are then diagonal too, given by the following formula:

$$A^k = \begin{pmatrix} \lambda_1^k & & \\ & \ddots & \\ & & \lambda_N^k \end{pmatrix}$$

In fact, given any polynomial $P \in \mathbb{C}[X]$, we have the following formula:

$$P(A) = \begin{pmatrix} P(\lambda_1) & & \\ & \ddots & \\ & & P(\lambda_N) \end{pmatrix}$$

Thus, the first formula in the statement holds indeed. In particular, we conclude that the moments of A are given by the following formula:

$$M_k(A) = tr(A^k) = \frac{1}{N}\sum_i \lambda_i^k$$

On the other hand, with $\mu_A = \frac{1}{N}(\delta_{\lambda_1} + \ldots + \delta_{\lambda_N})$ as in the statement, we have:

$$\int_{\mathbb{R}} x^k d\mu_A(x) = \frac{1}{N}\sum_i \int_{\mathbb{R}} x^k d\delta_{\lambda_i}(x) = \frac{1}{N}\sum_i \lambda_i^k$$

Thus that the law of A exists indeed, and is the measure μ_A, as claimed. □

In order to discuss now the general case, that of the arbitrary self-adjoint matrices, we will need a well-known result from linear algebra, namely:

THEOREM 1.6. *Any matrix $A \in M_N(\mathbb{C})$ which is self-adjoint, $A = A^*$, is diagonalizable, with the diagonalization being of the following type,*

$$A = UDU^*$$

with $U \in U_N$, and with $D \in M_N(\mathbb{R})$ diagonal. The converse holds too.

PROOF. As a first remark, the converse trivially holds, because if we take a matrix of the form $A = UDU^*$, with U unitary and D diagonal and real, then we have:

$$A^* = (UDU^*)^* = UD^*U^* = UDU^* = A$$

In the other sense now, assume that A is self-adjoint, $A = A^*$. Our first claim is that the eigenvalues are real. Indeed, assuming $Av = \lambda v$, we have:

$$\begin{aligned}\lambda < v, v > &= < \lambda v, v > \\ &= < Av, v > \\ &= < v, Av > \\ &= < v, \lambda v > \\ &= \bar{\lambda} < v, v >\end{aligned}$$

Thus we obtain $\lambda \in \mathbb{R}$, as claimed. Our next claim now is that the eigenspaces corresponding to different eigenvalues are pairwise orthogonal. Assume indeed that:

$$Av = \lambda v \quad , \quad Aw = \mu w$$

We have then the following computation, using $\lambda, \mu \in \mathbb{R}$:

$$\begin{aligned}\lambda < v, w > &= < \lambda v, w > \\ &= < Av, w > \\ &= < v, Aw > \\ &= < v, \mu w > \\ &= \mu < v, w >\end{aligned}$$

Thus $\lambda \neq \mu$ implies $v \perp w$, as claimed. In order now to finish, it remains to prove that the eigenspaces span $\mathbb{C}^N$. For this purpose, we will use a recurrence method. Let us pick an eigenvector, $Av = \lambda v$. Assuming $v \perp w$, we have:

$$\begin{aligned}< Aw, v > &= < w, Av > \\ &= < w, \lambda v > \\ &= \lambda < w, v > \\ &= 0\end{aligned}$$

Thus, if v is an eigenvector, then the vector space $v^\perp$ is invariant under A. In order to do the recurrence, it still remains to prove that the restriction of A to the vector space $v^\perp$ is self-adjoint. But this comes from the fact that an arbitary square matrix A is self-adjoint precisely when the following happens, for any vector v:

$$< Av, v > \in \mathbb{R}$$

Indeed, it is clear from this self-adjointness criterion that the restriction of A to any invariant subspace, and in particular to the subspace $v^\perp$, is self-adjoint. Thus, we can proceed by recurrence, as explained above, and we obtain the result. □

With the above result in hand, we can go back to our probability questions, and we have the following generalization of Proposition 1.5, dealing with the general case:

Theorem 1.7. *For a self-adjoint matrix $A\in M_N(\mathbb C)$ we have the formula*

$$tr(P(A))=\frac{1}{N}(P(\lambda_1)+\ldots+P(\lambda_N))$$

where $\lambda_1,\ldots,\lambda_N\in\mathbb R$ are the eigenvalues of A. Thus the measure

$$\mu_A=\frac{1}{N}(\delta_{\lambda_1}+\ldots+\delta_{\lambda_N})$$

is the law of A, in the abstract sense of Definition 1.4.

Proof. We already know, from Proposition 1.5, that the result holds indeed for the diagonal matrices. In the general case now, that of an arbitrary self-adjoint matrix, we know from Theorem 1.6 that our matrix is diagonalizable, as follows:

$$A=UDU^*$$

Now observe that the moments of A are given by the following formula:

$$\begin{aligned}tr(A^k)&=tr(UDU^*\cdot UDU^*\ldots UDU^*)\\&=tr(UD^kU^*)\\&=tr(D^k)\end{aligned}$$

We conclude, by looking carefully at Definition 1.4 and reasoning by linearity, that the matrices A,D have the same law, $\mu_A=\mu_D$, and this gives all the assertions. □

1b. Normal matrices

The above theory, both for random variables $f\in L^\infty(X)$ and for scalar matrices $A\in M_N(\mathbb C)$, is not the end of the story, because we can talk about complex random variables, $f:X\to\mathbb C$, and about non-self-adjoint matrices too, $A\neq A^*$. We will see that, with a bit of know-how, we can have some law technology going on, for both.

Let us start with the complex variables $f\in L^\infty(X)$. The main difference with respect to the real case comes from the fact that we have now a pair of variables instead of one, namely $f:X\to\mathbb C$ itself, and its conjugate $\bar f:X\to\mathbb C$. Thus, we are led to:

Definition 1.8. *The moments a complex variable $f\in L^\infty(X)$ are the numbers*

$$M_k(f)=\mathbb E(f^k)$$

depending on colored integers $k=\circ\bullet\bullet\circ\ldots$, with the conventions

$$f^\emptyset=1\quad,\quad f^\circ=f\quad,\quad f^\bullet=\bar f$$

and multiplicativity, in order to define the colored powers f^k.

Observe that, since $f, \bar{f}$ commute, we can permute terms, and restrict the attention to exponents of type $k = \ldots \circ\circ\circ\bullet\bullet\bullet\bullet\ldots$, if we want to. However, our various results below will look better without doing this, so we will use Definition 1.8 as stated.

Regarding now the notion of law, this extends too, the result being as follows:

THEOREM 1.9. *Each complex variable $f \in L^\infty(X)$ has a law, which is by definition a complex probability measure μ_f making the following formula hold,*

$$M_k(f) = \int_{\mathbb{C}} z^k d\mu_f(z)$$

for any colored integer k. Moreover, we have in fact the formula

$$\mathbb{E}(\varphi(f)) = \int_{\mathbb{C}} \varphi(x) d\mu_f(x)$$

valid for any integrable function $\varphi : \mathbb{C} \to \mathbb{C}$.

PROOF. The first assertion follows exactly as in the real case, and with z^k being defined exactly as f^k, namely by the following formulae, and multiplicativity:

$$z^\emptyset = 1 \quad , \quad z^\circ = z \quad , \quad z^\bullet = \bar{z}$$

As for the second assertion, this basically follows from this by linearity and continuity, by using standard measure theory, again as in the real case. □

Moving ahead towards matrices, all this leads to a mixture of easy and complicated problems. First, Definition 1.8 has the following straightforward analogue:

DEFINITION 1.10. *The moments a matrix $A \in M_N(\mathbb{C})$ are the numbers*

$$M_k(A) = tr(A^k)$$

depending on colored integers $k = \circ\bullet\bullet\circ\ldots$, with the usual conventions

$$A^\emptyset = 1 \quad , \quad A^\circ = A \quad , \quad A^\bullet = A^*$$

and multiplicativity, in order to define the colored powers A^k.

As a first observation, unless the matrix is normal, $AA^* = A^*A$, we cannot switch to exponents of type $k = \ldots \circ\circ\circ\bullet\bullet\bullet\bullet\ldots$, as it was theoretically possible for the complex variables $f \in L^\infty(X)$. Here is an explicit counterexample for this:

PROPOSITION 1.11. *The following matrix, which is not normal,*

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

has the property $tr(JJ^*JJ^*) \neq tr(JJJ^*J^*)$.

PROOF. The adjoint of the matrix in the statement is given by:

$$J^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

We have the following product formulae, which show indeed that J is not normal:

$$JJ^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$J^*J = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Let us compute now the quantities in the statement. We first have:

$$tr(JJ^*JJ^*) = tr((JJ^*)^2) = tr \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}$$

On the other hand, we have as well the following formula:

$$tr(JJJ^*J^*) = tr \left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) = tr \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

Thus, we are led to the conclusion in the statement. □

The above counterexample makes it quite clear that things will be complicated, when attempting to talk about the law of an arbitrary matrix $A \in M_N(\mathbb{C})$. But, there is solution to everything. By being a bit smart, we can formulate things as follows:

DEFINITION 1.12. *The law of a complex matrix $A \in M_N(\mathbb{C})$ is the following functional, on the algebra of polynomials in two noncommuting variables X, X^*:*

$$\mu_A : \mathbb{C} < X, X^* > \to \mathbb{C} \quad , \quad P \to tr(P(A))$$

In the case where we have a complex probability measure $\mu_A \in \mathcal{P}(\mathbb{C})$ such that

$$tr(P(A)) = \int_{\mathbb{C}} P(x)\, d\mu_A(x)$$

we identify this complex measure with the law of A.

As mentioned above, this is something smart, that will take us some time to understand. As a first observation, knowing the law is the same as knowing the moments, because if we write our polynomial as $P = \sum_k c_k X^k$, then we have:

$$tr(P(A)) = tr\left(\sum_k c_k A^k\right) = \sum_k c_k M_k(A)$$

Let us try now to compute some matrix laws, and see what we get. We already did some computations in the real case, and then for the basic 2×2 Jordan block J too, and based on all this, we can formulate the following result, with mixed conclusions:

Theorem 1.13. *The following happen:*

(1) *If $A = A^*$ then $\mu_A = \frac{1}{N}(\lambda_1 + \ldots + \lambda_N)$, with $\lambda_i \in \mathbb{R}$ being the eigenvalues.*
(2) *If A is diagonal, $\mu_A = \frac{1}{N}(\lambda_1 + \ldots + \lambda_N)$, with $\lambda_i \in \mathbb{C}$ being the eigenvalues.*
(3) *For the basic Jordan block J, the law μ_J is not a complex measure.*
(4) *In fact, assuming $AA^* \neq A^*A$, the law μ_A is not a complex measure.*

Proof. This basically follows from what we have, the idea being as follows:

(1) This is clear, because when looking at Definition 1.12, with our matrix being now assumed to be self-adjoint, $A = A^*$, what we have there is the "old" law, as previously axiomatized in Definition 1.4, and so the result follows from Theorem 1.7.

(2) This is again clear, by doing exactly the same trivial computation as in the real case, from the proof of Proposition 1.5.

(3) This follows from the formula $tr(JJ^*JJ^*) \neq tr(JJJ^*J^*)$ from Proposition 1.11. Indeed, since these numbers are different, we cannot obtain them both by integrating $|z|^2$ with respect to a measure $\mu_J \in \mathcal{P}(\mathbb{C})$, and this leads to the conclusion.

(4) This is something which generalizes (3). Assuming $AA^* \neq A^*A$, in order to show that μ_A does not exist as a measure, we can use a positivity trick, as follows:

$$\begin{aligned}
AA^* - A^*A \neq 0 &\implies (AA^* - A^*A)^2 > 0 \\
&\implies AA^*AA^* - AA^*A^*A - A^*AAA^* + A^*AA^*A > 0 \\
&\implies tr(AA^*AA^* - AA^*A^*A - A^*AAA^* + A^*AA^*A) > 0 \\
&\implies tr(AA^*AA^* + A^*AA^*A) > tr(AA^*A^*A + A^*AAA^*) \\
&\implies tr(AA^*AA^*) > tr(AAA^*A^*)
\end{aligned}$$

Thus, we can conclude as in the proof of (3), the point being that we cannot obtain both the above numbers by integrating $|z|^2$ with respect to a measure $\mu_A \in \mathcal{P}(\mathbb{C})$. □

All the above might look a bit confusing, so what to do? Relax, of course. A careful examination of Theorem 1.13 shows that the situation is in fact not that desperate, because all the results there allow formulating the following nice conjecture:

Conjecture 1.14. *The law μ_A is a measure precisely when $AA^* = A^*A$.*

We will prove in what follows that this conjecture holds indeed. For this purpose, the idea will be that of extending our main diagonalization result, Theorem 1.6, from the case of the self-adjoint matrices, $A = A^*$, to the case of the normal matrices, $AA^* = A^*A$.

This will be something a bit technical, called Spectral Theorem, in its most general form. First, we can complement Theorem 1.6 with the following result, regarding the unitary matrices, $U^* = U^{-1}$, which are normal too, but for rather different reasons:

THEOREM 1.15. *Any matrix $U \in M_N(\mathbb{C})$ which is unitary, $U^* = U^{-1}$, is diagonalizable, with the eigenvalues being on the unit circle $\mathbb{T}$. More precisely we have*

$$U = VDV^*$$

with $V \in U_N$, and with $D \in M_N(\mathbb{T})$ diagonal. The converse holds too.

PROOF. As a first remark, the converse holds indeed, because given a matrix of type $U = VDV^*$, with $V \in U_N$, and with $D \in M_N(\mathbb{T})$ being diagonal, we have:

$$U^* = VD^*V^* = VD^{-1}V^{-1} = (VDV^*)^{-1} = U^{-1}$$

Let us prove now the first assertion, stating that the eigenvalues of a unitary matrix $U \in U_N$ belong to the unit circle $\mathbb{T}$. Indeed, by assuming $Ux = \lambda x$, we have:

$$\begin{aligned} <x,x> &= <U^*Ux,x> \\ &= <Ux,Ux> \\ &= <\lambda x,\lambda x> \\ &= |\lambda|^2 <x,x> \end{aligned}$$

Thus we obtain $\lambda \in \mathbb{T}$, as desired. Our next claim now is that the eigenspaces corresponding to different eigenvalues are pairwise orthogonal. Assume indeed that:

$$Ux = \lambda x \quad , \quad Uy = \mu y$$

We have then the following computation, by using $U^* = U^{-1}$ and $\lambda, \mu \in \mathbb{T}$:

$$\begin{aligned} \lambda <x,y> &= <Ux,y> \\ &= <x,U^*y> \\ &= <x,U^{-1}y> \\ &= <x,\mu^{-1}y> \\ &= \mu <x,y> \end{aligned}$$

Thus $\lambda \neq \mu$ implies $x \perp y$, as claimed. In order now to finish, it remains to prove that the eigenspaces span the whole $\mathbb{C}^N$. For this purpose, we will use a recurrence method. Let us pick an eigenvector, $Ux = \lambda x$. Assuming $x \perp y$, we have:

$$\begin{aligned} <Uy,x> &= <y,U^*x> \\ &= <y,U^{-1}x> \\ &= <y,\lambda^{-1}x> \\ &= \lambda <y,x> \\ &= 0 \end{aligned}$$

Thus, if x is an eigenvector of U, then the vector space $x^\perp$ is invariant under U. Now since U is an isometry, so is its restriction to this space $x^\perp$. Thus this restriction is a unitary, and so we can proceed by recurrence, and we obtain the result. □

We can now, eventually, diagonalize the general normal matrices, as follows:

THEOREM 1.16. *Any matrix $A\in M_N(\mathbb{C})$ which is normal, $AA^*=A^*A$, is diagonalizable, with the diagonalization being of the following type,*

$$A=UDU^*$$

with $U\in U_N$, and with $D\in M_N(\mathbb{C})$ diagonal. The converse holds too.

PROOF. As a first remark, the converse holds indeed, because if we take a matrix of the form $A=UDU^*$, with U unitary and with D diagonal, then we have:

$$AA^*=UDD^*U^*=UD^*DU^*=A^*A$$

In the other sense now, this is something more technical. Our first claim is that a matrix A is normal precisely when the following is satisfied, for any vector x:

$$||Ax||=||A^*x||$$

Indeed, this equality can be written in the following way, which gives $AA^*=A^*A$:

$$<AA^*x,x>=<A^*Ax,x>$$

Our claim now is that A,A^* have the same eigenvectors, with conjugate eigenvalues:

$$Ax=\lambda x\implies A^*x=\bar{\lambda}x$$

Indeed, this follows from the following computation, and from the trivial fact that if A is normal, then so is any matrix of type $A-\lambda 1_N$, with $\lambda\in\mathbb{C}$:

$$\begin{aligned}||(A^*-\bar{\lambda}1_N)x|| &= ||(A-\lambda 1_N)^*x||\\ &= ||(A-\lambda 1_N)x||\\ &= 0\end{aligned}$$

Let us prove now, by using this fact, that the eigenspaces of A are pairwise orthogonal. Assuming $Ax=\lambda x$ and $Ay=\mu y$ with $\lambda\neq\mu$, we have:

$$\begin{aligned}\lambda<x,y> &= <Ax,y>\\ &= <x,A^*y>\\ &= <x,\bar{\mu}y>\\ &= \mu<x,y>\end{aligned}$$

Thus $\lambda\neq\mu$ implies $x\perp y$, as desired. In order to finish now the proof, it remains to prove that the eigenspaces of A span the whole $\mathbb{C}^N$. This is something quite tricky, and our plan here will be that of proving that the eigenspaces of AA^* are eigenspaces of A. In order to do so, let us pick two eigenvectors x,y of the matrix AA^*, corresponding to different eigenvalues, $\lambda\neq\mu$. The eigenvalue equations are then as follows:

$$AA^*x=\lambda x\quad,\quad AA^*y=\mu y$$

We have the following computation, by using the normality condition $AA^* = A^*A$, and the fact that the eigenvalues of AA^*, and in particular μ, are real:

$$\begin{aligned}\lambda < Ax, y > &= < A\lambda x, y > \\ &= < AAA^*x, y > \\ &= < AA^*Ax, y > \\ &= < Ax, AA^*y > \\ &= < Ax, \mu y > \\ &= \mu < Ax, y >\end{aligned}$$

We conclude that we have $< Ax, y >= 0$. But this reformulates as follows:

$$\lambda \neq \mu \implies A(E_\lambda) \perp E_\mu$$

Now since the eigenspaces of AA^* are pairwise orthogonal, and span the whole $\mathbb{C}^N$, we deduce that these eigenspaces are invariant under A:

$$A(E_\lambda) \subset E_\lambda$$

But with this result in hand, we can now finish the proof. Indeed, we can decompose the diagonalization problem, and the matrix A itself, following these eigenspaces of AA^*, which in practice amounts in saying that we can assume that we only have 1 eigenspace. By rescaling, this is the same as assuming that we have $AA^* = 1$, and so we are now into the unitary case, that we know how to solve, as explained in Theorem 1.15. □

Getting back now to probability, we can formulate a final result, as follows:

THEOREM 1.17. *Given a matrix $A \in M_N(\mathbb{C})$ which is normal, $AA^* = A^*A$, we have the following formula, valid for any polynomial $P \in \mathbb{C} < X, X^* >$,*

$$tr(P(A)) = \frac{1}{N}(P(\lambda_1) + \ldots + P(\lambda_N))$$

where $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$ are the eigenvalues of A. Thus the complex measure

$$\mu_A = \frac{1}{N}(\delta_{\lambda_1} + \ldots + \delta_{\lambda_N})$$

is the law of A. In the non-normal case, the law μ_A is not a measure.

PROOF. According to Theorem 1.16, our matrix A is diagonalizable, and in fact A, A^* are jointly diagonalizable. To be more precise, let us write, as in Theorem 1.16:

$$A = UDU^*$$

Here $U \in U_N$, and $D \in M_N(\mathbb{C})$ is diagonal. The adjoint matrix is then given by:

$$A^* = UD^*U$$

As before in the diagonal matrix case, since our matrix is normal, $AA^* = A^*A$, knowing its law in the abstract sense of Definition 1.12 is the same as knowing the restriction of this abstract distribution to the usual polynomials in two variables:

$$\mu_A : \mathbb{C}[X, X^*] \to \mathbb{C} \quad , \quad P \to tr(P(A))$$

In order now to compute this functional, we can change the basis via the above unitary matrix $U \in U_N$, which in practice means that we can simply assume $U = 1$. Thus if we denote by $\lambda_1, \ldots, \lambda_N$ the diagonal entries of D, which are the eigenvalues of A, the law that we are looking for is the following functional:

$$\mu_A : \mathbb{C}[X, X^*] \to \mathbb{C} \quad , \quad P \to \frac{1}{N}(P(\lambda_1) + \ldots + P(\lambda_N))$$

But this functional corresponds to integrating P with respect to the following complex measure, that we agree to still denote by μ_A, and call distribution of A:

$$\mu_A = \frac{1}{N}(\delta_{\lambda_1} + \ldots + \delta_{\lambda_N})$$

Thus, we are led to the conclusion in the statement. □

The above result does of course not close the discussion, because there are non-normal matrices such as the basic Jordan block J whose laws are waiting to be further investigated. However, in what follows, the above result will be what we will need.

1c. Spectral theory

In order to reach now to random matrices, we must unify what we know about the random variables $f \in L^\infty(X)$, and the scalar matrices $A \in M_N(\mathbb{C})$. But this can indeed be done, via a bit of spectral theory and operator algebras. Let us start with:

THEOREM 1.18. *Given a Hilbert space H, the linear operators $T : H \to H$ which are bounded, in the sense that the quantity*

$$||T|| = \sup_{||x|| \leq 1} ||Tx||$$

is finite, form a complex algebra $B(H)$, having the following properties:

(1) *$B(H)$ is complete with respect to $||.||$, and so we have a Banach algebra.*
(2) *$B(H)$ has an involution $T \to T^*$, given by $< Tx, y >=< x, T^*y >$.*

In addition, the norm and the involution are related by the formula $||TT^*|| = ||T||^2$.

PROOF. The fact that we have indeed an algebra follows from:

$$||S + T|| \leq ||S|| + ||T|| \quad , \quad ||\lambda T|| = |\lambda| \cdot ||T|| \quad , \quad ||ST|| \leq ||S|| \cdot ||T||$$

(1) Assuming that $\{T_k\} \subset B(H)$ is a Cauchy sequence, the sequence $\{T_k x\}$ is Cauchy for any $x \in H$, so we can define the limit $T = \lim_{k\to\infty} T_k$ by setting:

$$Tx = \lim_{k\to\infty} T_k x$$

It is routine then to check that this formula defines indeed an operator $T \in B(H)$, and that we have $T_k \to T$ in norm, and this gives the result.

(2) The existence of T^* comes from the fact that $\psi(x) = < Tx, y >$ being a linear map $H \to \mathbb{C}$, we must have a formula as follows, for a certain vector $T^*y \in H$:

$$\psi(x) = < x, T^*y >$$

Moreover, since this vector T^*y is unique, T^* is unique too, and we have as well:

$$(S+T)^* = S^* + T^* \quad , \quad (\lambda T)^* = \bar{\lambda} T^*$$

$$(ST)^* = T^*S^* \quad , \quad (T^*)^* = T$$

Observe also that we have indeed $T^* \in B(H)$, due to the following equality:

$$\begin{aligned} ||T|| &= \sup_{||x||=1} \sup_{||y||=1} < Tx, y > \\ &= \sup_{||y||=1} \sup_{||x||=1} < x, T^*y > \\ &= ||T^*|| \end{aligned}$$

(3) Regarding now the last assertion, observe first that we have:

$$||TT^*|| \leq ||T|| \cdot ||T^*|| = ||T||^2$$

On the other hand, we have as well the following estimate:

$$\begin{aligned} ||T||^2 &= \sup_{||x||=1} | < Tx, Tx > | \\ &= \sup_{||x||=1} | < x, T^*Tx > | \\ &\leq ||T^*T|| \end{aligned}$$

Now by replacing in this formula $T \to T^*$ we obtain $||T||^2 \leq ||TT^*||$. Thus, we have proved both the needed inequalities, and we are done. □

The above is a bit abstract, but more concretely now, we have:

THEOREM 1.19. *When H comes with an orthonormal basis $\{e_i\}_{i \in I}$, the bounded operators $T \in B(H)$ can be identified with matrices $M \in M_I(\mathbb{C})$ via*

$$Tx = Mx \quad , \quad M_{ij} = < Te_j, e_i >$$

and we obtain in this way an embedding as follows, which is multiplicative:

$$B(H) \subset M_I(\mathbb{C})$$

In the case $H = \mathbb{C}^N$ this embedding is the usual isomorphism $B(H) \simeq M_N(\mathbb{C})$. In the separable case, where $I \simeq \mathbb{N}$, we obtain a proper embedding $B(H) \subset M_\infty(\mathbb{C})$.

PROOF. We have several assertions to be proved, the idea being as follows:

(1) Regarding the first assertion, given a bounded operator $T : H \to H$, let us associate to it a matrix $M \in M_I(\mathbb{C})$ as in the statement, by the following formula:

$$M_{ij} = < Te_j, e_i >$$

It is clear that this correspondence $T \to M$ is linear, and also that its kernel is $\{0\}$. Thus, we have an embedding of linear spaces $B(H) \subset M_I(\mathbb{C})$.

(2) Our claim now is that this embedding is multiplicative. But this is clear too, because if we denote by $T \to M_T$ our correspondence, we have:

$$\begin{aligned}(M_{ST})_{ij} &= \sum_k < Se_k, e_i >< Te_j, e_k > \\ &= \sum_k (M_S)_{ik}(M_T)_{kj} \\ &= (M_S M_T)_{ij}\end{aligned}$$

(3) Finally, we must prove that the original operator $T : H \to H$ can be recovered from its matrix $M \in M_I(\mathbb{C})$ via the formula in the statement, namely $Tx = Mx$. But this latter formula holds for the vectors of the basis, $x = e_j$, because we have:

$$(Te_j)_i = < Te_j, e_i > = M_{ij} = (Me_j)_i$$

Now by linearity we obtain from this that the formula $Tx = Mx$ holds everywhere, on any vector $x \in H$, and this finishes the proof of the first assertion.

(4) In finite dimensions we obtain an isomorphism, because any matrix $M \in M_N(\mathbb{C})$ determines an operator $T : \mathbb{C}^N \to \mathbb{C}^N$, according to the formula $< Te_j, e_i > = M_{ij}$. However, over $H = l^2(\mathbb{N})$, the following matrix does not define an operator:

$$M = \begin{pmatrix} 1 & 1 & \ldots \\ 1 & 1 & \ldots \\ \vdots & \vdots \end{pmatrix}$$

Indeed, $T(e_1)$ should be the all-one vector, which is not square-summable. □

We will be interested here in the algebras of operators, rather than in the operators themselves. The axioms here, coming from Theorem 1.18, are as follows:

DEFINITION 1.20. *A C^*-algebra is a complex algebra with unit A, having:*

1. *A norm $a \to ||a||$, making it a Banach algebra (the Cauchy sequences converge).*
2. *An involution $a \to a^*$, which satisfies $||aa^*|| = ||a||^2$, for any $a \in A$.*

According to Theorem 1.18, the full operator algebra $B(H)$ is a C^*-algebra, and it follows that any closed $*$-subalgebra $A \subset B(H)$ is a C^*-algebra too. It is possible to prove that any C^*-algebra appears in this way, but we will not need this deep result here.

As basic examples now, we have the usual matrix algebras $M_N(\mathbb{C})$, with the norm and the involution being the usual matrix norm and involution, given by:

$$||A||=\sup_{||x||=1}||Ax||\quad,\quad (A^*)_{ij}=\overline{A}_{ji}$$

Some other basic examples are the algebras $L^\infty(X)$ of essentially bounded functions $f:X\to\mathbb{C}$ on a measured space X, with the usual norm and involution, namely:

$$||f||=\sup_{x\in X}|f(x)|\quad,\quad f^*(x)=\overline{f(x)}$$

We can put these two basic classes of examples together, as follows:

Proposition 1.21. *The random matrix algebras $A=M_N(L^\infty(X))$ are C^*-algebras, with their usual norm and involution, given by:*

$$||Z||=\sup_{x\in X}||Z_x||\quad,\quad (Z^*)_{ij}=\overline{Z}_{ij}$$

These algebras generalize both the algebras $M_N(\mathbb{C})$, and the algebras $L^\infty(X)$.

Proof. The fact that the C^*-algebra axioms are satisfied is clear from definitions. As for the last assertion, this follows by taking $X=\{.\}$ and $N=1$, respectively. □

Our purpose in what follows is to develop the spectral theory of the C^*-algebras, and in particular that of the random matrix algebras $A=M_N(L^\infty(X))$ that we are interested in, one of our objectives being that of talking about spectral measures, in the normal case, in analogy with what we know about the usual matrices. Let us start with:

Definition 1.22. *The spectrum of an element $a\in A$ is the set*

$$\sigma(a)=\left\{\lambda\in\mathbb{C}\Big|a-\lambda\notin A^{-1}\right\}$$

where $A^{-1}\subset A$ is the set of invertible elements.

Given an element $a\in A$, and a rational function $f=P/Q$ having poles outside $\sigma(a)$, we can construct the element $f(a)=P(a)Q(a)^{-1}$. For simplicity, we write:

$$f(a)=\frac{P(a)}{Q(a)}$$

With this convention, we have the following result:

Proposition 1.23. *We have the "rational functional calculus" formula*

$$\sigma(f(a))=f(\sigma(a))$$

valid for any rational function $f\in\mathbb{C}(X)$ having poles outside $\sigma(a)$.

PROOF. We can prove this result in two steps, as follows:

(1) Assume first that we are in the usual polynomial case, $f \in \mathbb{C}[X]$. We pick a number $\lambda \in \mathbb{C}$, and we decompose the polynomial $f - \lambda$:

$$f(X) - \lambda = c(X - p_1) \ldots (X - p_n)$$

We have then, as desired, the following computation:

$$\begin{aligned}
\lambda \notin \sigma(f(a)) &\iff f(a) - \lambda \in A^{-1} \\
&\iff c(a - p_1) \ldots (a - p_n) \in A^{-1} \\
&\iff a - p_1, \ldots, a - p_n \in A^{-1} \\
&\iff p_1, \ldots, p_n \notin \sigma(a) \\
&\iff \lambda \notin f(\sigma(a))
\end{aligned}$$

(2) In the general case now, $f \in \mathbb{C}(X)$, we pick $\lambda \in \mathbb{C}$, we write $f = P/Q$, and we set $R = P - \lambda Q$. By using (1) above, we obtain:

$$\begin{aligned}
\lambda \in \sigma(f(a)) &\iff R(a) \notin A^{-1} \\
&\iff 0 \in \sigma(R(a)) \\
&\iff 0 \in R(\sigma(a)) \\
&\iff \exists \mu \in \sigma(a), R(\mu) = 0 \\
&\iff \lambda \in f(\sigma(a))
\end{aligned}$$

Thus, we have obtained the formula in the statement. □

Given an element $a \in A$, its spectral radius $\rho(a)$ is the radius of the smallest disk centered at 0 containing $\sigma(a)$. With this convention, we have the following key result:

THEOREM 1.24. *Let A be a C^*-algebra.*

1. *The spectrum of a norm one element is in the unit disk.*
2. *The spectrum of a unitary element ($a^* = a^{-1}$) is on the unit circle.*
3. *The spectrum of a self-adjoint element ($a = a^*$) consists of real numbers.*
4. *The spectral radius of a normal element ($aa^* = a^*a$) is equal to its norm.*

PROOF. We use the various results established above, as follows:

(1) This comes from the following basic formula, valid when $||a|| < 1$:

$$\frac{1}{1-a} = 1 + a + a^2 + \ldots$$

(2) Assuming $a^* = a^{-1}$, we have the following computations:

$$||a|| = \sqrt{||aa^*||} = \sqrt{1} = 1$$

$$||a^{-1}|| = ||a^*|| = ||a|| = 1$$

If we denote by D the unit disk, we obtain from this, by using (1):

$$\sigma(a) \subset D \quad , \quad \sigma(a^{-1}) \subset D$$

On the other hand, by using the function $f(z) = z^{-1}$, we have:

$$\sigma(a^{-1}) \subset D \implies \sigma(a) \subset D^{-1}$$

Thus we have $\sigma(a) \subset D \cap D^{-1} = \mathbb{T}$, as desired.

(3) This follows by using the result (2), just established above, and Proposition 1.23, with the following rational function, depending on a parameter $t \in \mathbb{R}$:

$$f(z) = \frac{z + it}{z - it}$$

Indeed, for $t >> 0$ the element $f(a)$ is well-defined, and we have:

$$\left(\frac{a + it}{a - it} \right)^* = \frac{a - it}{a + it} = \left(\frac{a + it}{a - it} \right)^{-1}$$

Thus the element $f(a)$ is a unitary, and by using (2) above its spectrum is contained in $\mathbb{T}$. We conclude that we have an inclusion as follows:

$$f(\sigma(a)) = \sigma(f(a)) \subset \mathbb{T}$$

Thus, we obtain an inclusion $\sigma(a) \subset f^{-1}(\mathbb{T}) = \mathbb{R}$, and we are done.

(4) We already know from (1) that we have the following inequality:

$$\rho(a) \leq ||a||$$

For the converse, we fix an arbitrary number $\rho > \rho(a)$. We have then:

$$\int_{|z|=\rho} \frac{z^n}{z - a}\, dz = \sum_{k=0}^{\infty} \left(\int_{|z|=\rho} z^{n-k-1} dz \right) a^k = a^{n-1}$$

By applying the norm and taking n-th roots we obtain from this:

$$\rho \geq \lim_{n \to \infty} ||a^n||^{1/n}$$

In the case $a = a^*$ we have $||a^n|| = ||a||^n$ for any exponent of the form $n = 2^k$, and by taking n-th roots we get $\rho \geq ||a||$. But this gives the missing inequality, namely:

$$\rho(a) \geq ||a||$$

In the general case $aa^* = a^*a$ we have $a^n(a^n)^* = (aa^*)^n$. Thus $\rho(a)^2 = \rho(aa^*)$, and since the element aa^* is self-adjoint, we obtain $\rho(aa^*) = ||a||^2$, and we are done. □

We are now in position of proving a key result, due to Gelfand, as follows:

THEOREM 1.25. *Any commutative C^*-algebra is the form*

$$A = C(X)$$

with its "spectrum" $X = Spec(A)$ appearing as the space of characters $\chi : A \to \mathbb{C}$.

PROOF. Given a commutative C^*-algebra A, we can define X to be the set of characters $\chi : A \to \mathbb{C}$, with topology making continuous all evaluation maps $ev_a : \chi \to \chi(a)$. Then X is a compact space, and $a \to ev_a$ is a morphism of algebras, as follows:

$$ev : A \to C(X)$$

(1) We first prove that ev is involutive. For this purpose we use the following formula, which is similar to the $z = Re(z) + iIm(z)$ formula for usual complex numbers:

$$a = \frac{a+a^*}{2} + i \cdot \frac{a-a^*}{2i}$$

Thus it is enough to prove the equality $ev_{a^*} = ev_a^*$ for self-adjoint elements a. But this is the same as proving that $a = a^*$ implies that ev_a is a real function, which is in turn true, because $ev_a(\chi) = \chi(a)$ is an element of the spectrum $\sigma(a)$, contained in $\mathbb{R}$.

(2) Since A is commutative, each element is normal, so ev is isometric, due to:

$$||ev_a|| = \rho(a) = ||a||$$

(3) It remains to prove that ev is surjective. But this follows from the Stone-Weierstrass theorem, because $ev(A)$ is a closed subalgebra of $C(X)$, which separates the points. □

As a main consequence of the Gelfand theorem, we have:

PROPOSITION 1.26. *For any $a \in A$ normal we have an identification as follows:*

$$< a >= C(\sigma(a))$$

In addition, given a function $f \in C(\sigma(a))$, we can apply it to a, and we have

$$\sigma(f(a)) = f(\sigma(a))$$

which generalizes the previous rational calculus formula, in the normal case.

PROOF. Since a is normal, the C^*-algebra $< a >$ that is generates is commutative, so if we denote by X the space of the characters $\chi :< a >\to \mathbb{C}$, we have:

$$< a >= C(X)$$

Now since the map $X \to \sigma(a)$ given by evaluation at a is bijective, we obtain:

$$< a >= C(\sigma(a))$$

Thus, we are dealing here with usual functions, and this gives all the assertions. □

In order to get now towards probability, we first have to develop the theory of positive elements, and linear forms. First, we have the following result:

PROPOSITION 1.27. *For an element $a \in A$, the following are equivalent:*

(1) *a is positive, in the sense that $\sigma(a) \subset [0, \infty)$.*
(2) *$a = b^2$, for some $b \in A$ satisfying $b = b^*$.*
(3) *$a = cc^*$, for some $c \in A$.*

PROOF. This is something very standard, as follows:

(1) $\implies$ (2) Observe first that $\sigma(a) \subset \mathbb{R}$ implies $a = a^*$. Thus the algebra $< a >$ is commutative, and by using Theorem 1.26, we can set $b = \sqrt{a}$.

(2) $\implies$ (3) This is trivial, because we can simply set $c = b$.

(2) $\implies$ (1) This is clear too, because we have:

$$\sigma(a) = \sigma(b^2) = \sigma(b)^2 \subset \mathbb{R}^2 = [0, \infty)$$

(3) $\implies$ (1) We proceed by contradiction. By multiplying c by a suitable element of $< cc^* >$, we are led to the existence of an element $d \neq 0$ satisfying:

$$-dd^* \geq 0$$

By writing now $d = x + iy$ with $x = x^*, y = y^*$ we have:

$$dd^* + d^*d = 2(x^2 + y^2) \geq 0$$

Thus $d^*d \geq 0$, which is easily seen to contradict the condition $-dd^* \geq 0$. $\square$

We can talk as well about positive linear forms, as follows:

DEFINITION 1.28. *Consider a linear map $\varphi : A \to \mathbb{C}$.*

(1) *φ is called positive when $a \geq 0 \implies \varphi(a) \geq 0$.*
(2) *φ is called faithful and positive when $a \geq 0, a \neq 0 \implies \varphi(a) > 0$.*

In the commutative case, $A = C(X)$, the positive linear forms appear as follows, with μ being positive, and strictly positive if we want φ to be faithful and positive:

$$\varphi(f) = \int_X f(x) d\mu(x)$$

In general, the positive forms can be thought of as being integration functionals with respect to some underlying "positive measures". Based on this, we can formulate:

DEFINITION 1.29. *Let A be a C^*-algebra, given with a positive trace $tr : A \to \mathbb{C}$.*

(1) *The elements $a \in A$ are called random variables.*
(2) *The moments of such a variable are the numbers $M_k(a) = tr(a^k)$.*
(3) *The law of such a variable is the functional $\mu_a : P \to tr(P(a))$.*

Here the exponent $k = \circ \bullet \bullet \circ \ldots$ is as usual a colored integer, and the powers a^k are defined by the following formulae, and multiplicativity:

$$a^{\emptyset} = 1 \quad , \quad a^{\circ} = a \quad , \quad a^{\bullet} = a^*$$

As for the polynomial P, this is a noncommuting $*$-polynomial in one variable:

$$P \in \mathbb{C} < X, X^* >$$

Observe that the law is uniquely determined by the moments, because we have:

$$P(X) = \sum_k \lambda_k X^k \implies \mu_a(P) = \sum_k \lambda_k M_k(a)$$

At the level of the general theory, we have the following key result, extending the various results that we have, regarding the self-adjoint and normal matrices:

THEOREM 1.30. *Let A be a C^*-algebra, with a trace $tr : A \to \mathbb{C}$, and consider an element $a \in A$ which is normal, in the sense that $aa^* = a^*a$.*

1. *μ_a is a complex probability measure, satisfying $supp(\mu_a) \subset \sigma(a)$.*
2. *In the self-adjoint case, $a = a^*$, this measure μ_a is real.*
3. *Assuming that tr is faithful, we have $supp(\mu_a) = \sigma(a)$.*

Moreover, in the non-normal case, $aa^* \neq a^*a$, the law of a is not a measure.

PROOF. This is something very standard, that we already know for the usual complex matrices, and whose proof in general is quite similar, as follows:

(1) In the normal case, $aa^* = a^*a$, the Gelfand theorem, or rather the subsequent continuous functional calculus result, tells us that we have:

$$< a >= C(\sigma(a))$$

Thus the functional $f(a) \to tr(f(a))$ can be regarded as an integration functional on the algebra $C(\sigma(a))$, and by the Riesz theorem, this latter functional must come from a probability measure μ on the spectrum $\sigma(a)$, in the sense that we must have:

$$tr(f(a)) = \int_{\sigma(a)} f(z) d\mu(z)$$

We are therefore led to the conclusions in the statement, with the uniqueness assertion coming from the fact that the elements a^k, taken as usual with respect to colored integer exponents, $k = \circ \bullet \bullet \circ \ldots$, generate the whole C^*-algebra $C(\sigma(a))$.

(2) This is something which is clear from definitions.

(3) Once again, this is something which is clear from definitions.

(4) Finally, the last assertion follows by using the same trick as in the matrix case. $\square$

1d. Random matrices

With the above general theory in hand, we can now talk about the laws of random matrices, which appear as elements $Z \in A$, with the C^*-algebra A being as follows:

$$A = M_N(L^\infty(X))$$

To be more precise, for this particular C^*-algebra A, the main result that we have so far, namely Theorem 1.30, takes the following very concrete and useful form:

THEOREM 1.31. *Given a random matrix $Z \in M_N(L^\infty(X))$ which is normal,*

$$ZZ^* = Z^*Z$$

its law, which is by definition the following abstract functional,

$$\mu_Z : \mathbb{C} < X, X^* > \to \mathbb{C} \quad , \quad P \to \frac{1}{N}\int_X tr(P(Z))$$

when restricted to the usual polynomials in two variables,

$$\mu_Z : \mathbb{C}[X, X^*] \to \mathbb{C} \quad , \quad P \to \frac{1}{N}\int_X tr(P(Z))$$

must come from a probability measure on the spectrum $\sigma(Z) \subset \mathbb{C}$, as follows:

$$\mu_Z(P) = \int_{\sigma(T)} P(x) d\mu_Z(x)$$

In the non-normal case, the law μ_Z is not a complex probability measure.

PROOF. This follows indeed from what we know from Theorem 1.30, applied to the normal element $a = Z$, belonging to the C^*-algebra $A = M_N(L^\infty(X))$. □

So long for theory, and in the hope that you survived, and are still with us. We will see explicit examples for all this in the next chapters.

1e. Exercises

Exercises.

CHAPTER 2

Gaussian matrices

2a. Normal laws

Welcome to random matrices, again. What we have learned so far is useful, but a bit theoretical. In what follows, starting from the present chapter, we will go into the real thing, namely computations, for various interesting classes of random matrices.

Generally speaking, if our random matrix $Z \in M_N(L^\infty(X))$ is really "random", this should mean that its entries $Z_{ij} \in L^\infty(X)$ should be normal variables:

$$Z = \begin{pmatrix} Z_{11} & \ldots & Z_{1N} \\ \vdots & & \vdots \\ Z_{N1} & \ldots & Z_{NN} \end{pmatrix} \quad , \quad Z_{ij} = \text{normal}$$

But, what is normal? This is something quite tricky, and in order to discuss this, we first need to talk about independence. This can be done as follows:

DEFINITION 2.1. *Two variables $f, g \in L^\infty(X)$ are called independent when*

$$\mathbb{E}(f^k g^l) = \mathbb{E}(f^k)\,\mathbb{E}(g^l)$$

happens, for any $k, l \in \mathbb{N}$.

As usual, this definition hides some non-trivial things. Indeed, by linearity, we would like to have a formula as follows, valid for any polynomials $P, Q \in \mathbb{C}[X]$:

$$\mathbb{E}[P(f)Q(g)] = \mathbb{E}[P(f)]\,\mathbb{E}[Q(g)]$$

By using a continuity argument, it is enough to have this formula for the characteristic functions χ_I, χ_J of the measurable sets of real numbers $I, J \subset \mathbb{R}$:

$$\mathbb{E}[\chi_I(f)\chi_J(g)] = \mathbb{E}[\chi_I(f)]\,\mathbb{E}[\chi_J(g)]$$

Thus, we are led to the usual definition of independence, namely:

$$\mathbb{P}(f \in I, g \in J) = \mathbb{P}(f \in I)\,\mathbb{P}(g \in J)$$

All this might seem a bit abstract, but in practice, the idea is of course that f, g must be independent, in an intuitive, real-life sense. As a first result now, we have:

Proposition 2.2. *Assuming that $f,g\in L^\infty(X)$ are independent, we have*

$$\mu_{f+g}=\mu_f*\mu_g$$

where $*$ is the convolution of real probability measures.

Proof. We have the following computation, using the independence of f,g:

$$\begin{aligned}M_k(f+g)&=\mathbb{E}((f+g)^k)\\&=\sum_r\binom{k}{r}\mathbb{E}(f^rg^{k-r})\\&=\sum_r\binom{k}{r}M_r(f)M_{k-r}(g)\end{aligned}$$

On the other hand, by using the Fubini theorem, we have as well:

$$\begin{aligned}\int_\mathbb{R}x^kd(\mu_f*\mu_g)(x)&=\int_{\mathbb{R}\times\mathbb{R}}(x+y)^kd\mu_f(x)d\mu_g(y)\\&=\sum_r\binom{k}{r}\int_\mathbb{R}x^rd\mu_f(x)\int_\mathbb{R}y^{k-r}d\mu_g(y)\\&=\sum_r\binom{k}{r}M_r(f)M_{k-r}(g)\end{aligned}$$

Thus μ_{f+g} and $\mu_f*\mu_g$ have the same moments, so they coincide. □

Here is now a second result on independence, which is more advanced:

Theorem 2.3. *Assuming that $f,g\in L^\infty(X)$ are independent, we have*

$$F_{f+g}=F_fF_g$$

where $F_f(x)=\mathbb{E}(e^{ixf})$ is the Fourier transform.

Proof. We have indeed, by using Proposition 2.2 and Fubini:

$$\begin{aligned}F_{f+g}(x)&=\int_\mathbb{R}e^{ixz}d\mu_{f+g}(z)\\&=\int_\mathbb{R}e^{ixz}d(\mu_f*\mu_g)(z)\\&=\int_{\mathbb{R}\times\mathbb{R}}e^{ix(z+t)}d\mu_f(z)d\mu_g(t)\\&=\int_\mathbb{R}e^{ixz}d\mu_f(z)\int_\mathbb{R}e^{ixt}d\mu_g(t)\\&=F_f(x)F_g(x)\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Let us go back now to our problem, regarding what normality means. Intuitively, we are in need of a formula for what comes out of various "normal" measurements, such as measuring the temperature of the room, over some time, or the pressure of a tyre, or why not, recording what comes out by grading a calculus exam. And here, intuition suggests that we should get some kind of bell-shaped curve, with most students doing average, and then with this average dropping on both sides, towards good and bad.

Now let us think a bit, on how these students actually produce the bell-shaped curve. Since students' contributions to the various exercises, and so to this curve, are rather independent, barring of course any cheating, we are led to the following conclusion:

Conclusion 2.4. *The normal law is the bell-shaped curve coming out of a "central limiting" procedure, consisting in adding i.i.d. variables.*

Summarizing, we are in need of a "central limiting theorem", telling us what the normal law is. However, doing this with bare hands is a bit complicated, so we will do instead some reverse engineering. And with the comment that, as calculus teachers, we are of course entitled to cheat a bit. Following Gauss, we first have:

$$\begin{aligned}
\int_{\mathbb{R}} e^{-x^2} dx &= \sqrt{\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-x^2-y^2} dxdy} \\
&= \sqrt{\int_0^{2\pi}\int_0^\infty e^{-r^2} r\, drdt} \\
&= \sqrt{2\pi\int_0^\infty \left(-\frac{e^{-r^2}}{2}\right)' dr} \\
&= \sqrt{\pi}
\end{aligned}$$

We can now introduce candidates for the normal distributions, as follows:

Definition 2.5. *The normal law of parameter* 1 *is the following measure:*

$$g_1 = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

More generally, the normal law of parameter $t > 0$ is the following measure:

$$g_t = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$$

These are also called Gaussian distributions, with "g" standing for Gauss.

These laws are usually denoted $\mathcal{N}(0,1)$ and $\mathcal{N}(0,t)$, but here we will use the above simplified notations. Observe that our laws have indeed mass 1, as shown by:

$$\int_{\mathbb{R}} e^{-x^2/2t} dx = \int_{\mathbb{R}} e^{-y^2} \sqrt{2t}\, dy = \sqrt{2\pi t}$$

Generally speaking, the normal laws appear as bit everywhere, in the real life. The reasons for this come from the Central Limit Theorem (CLT), that we will explain in a moment. As a first result now, regarding the mean and variance, we have:

PROPOSITION 2.6. *We have the mean and variance formulae*

$$M(g_t)=0\quad,\quad V(g_t)=t$$

valid for any $t>0$.

PROOF. The first moment, or mean, is 0, because our normal law g_t is centered. Now observe that the second moment can be computed as follows:

$$\begin{aligned}M_2(g_t)&=\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}x^2e^{-x^2/2t}dx\\&=\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}(tx)\left(-e^{-x^2/2t}\right)'dx\\&=\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}te^{-x^2/2t}dx\\&=t\end{aligned}$$

We conclude from this that the variance is $V(g_t)=M_2(g_t)=t$. □

Regarding now the Fourier transforms, the result here is as follows:

PROPOSITION 2.7. *The Fourier transforms of the normal laws are given by:*

$$F_{g_t}(x)=e^{-tx^2/2}$$

In particular, the normal laws satisfy $g_s*g_t=g_{s+t}$, for any $s,t>0$.

PROOF. The Fourier transform formula can be established as follows:

$$\begin{aligned}F_{g_t}(x)&=\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}e^{-z^2/2t+ixz}dz\\&=\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}e^{-(z/\sqrt{2t}-\sqrt{t/2}\,iz)^2-tx^2/2}dz\\&=\frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}}e^{-y^2-tx^2/2}\sqrt{2t}\,dy\\&=\frac{1}{\sqrt{\pi}}e^{-tx^2/2}\int_{\mathbb{R}}e^{-y^2}dy\\&=e^{-tx^2/2}\end{aligned}$$

As for $g_s*g_t=g_{s+t}$, this comes from Theorem 2.3, since $\log F_{g_t}$ is linear in t. □

We are now ready to state and prove the CLT, as follows:

THEOREM 2.8 (CLT). *Given real variables $f_1,f_2,f_3,\ldots\in L^\infty(X)$ which are i.i.d., centered, and with common variance $t>0$, we have*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n f_i\sim g_t$$

with $n\to\infty$, in moments.

PROOF. In terms of moments, the Fourier transform is given by:

$$\begin{aligned}F_f(x)&=\mathbb{E}\left(\sum_{r=0}^\infty\frac{(ixf)^r}{r!}\right)\\&=\sum_{r=0}^\infty\frac{(ix)^r\mathbb{E}(f^r)}{r!}\\&=\sum_{r=0}^\infty\frac{i^rM_r(f)}{r!}\,x^r\end{aligned}$$

Thus, the Fourier transform of the variable in the statement is:

$$\begin{aligned}F(x)&=\left[F_f\left(\frac{x}{\sqrt{n}}\right)\right]^n\\&=\left[1-\frac{tx^2}{2n}+O(n^{-2})\right]^n\\&\simeq e^{-tx^2/2}\end{aligned}$$

But this function being the Fourier transform of g_t, we obtain the result. $\square$

All this is very nice, but we are not done yet with probability theory, because we will need as well the complex normal laws. These are defined as follows:

DEFINITION 2.9. *The complex normal law of parameter $t>0$ is*

$$G_t=law\left(\frac{1}{\sqrt{2}}(a+ib)\right)$$

where a,b are independent, each following the law g_t.

Generally speaking, the basic theory of these laws can be developed by adapting the results from the real case. As a first illustration for this principle, we have:

PROPOSITION 2.10. *The complex Gaussian laws have the property*

$$G_s*G_t=G_{s+t}$$

for any $s,t>0$, and so they form a convolution semigroup.

PROOF. This follows indeed from the real result, namely $g_s * g_t = g_{s+t}$, established in Proposition 2.7, simply by taking real and imaginary parts. □

We have as well the following complex analogue of the CLT:

THEOREM 2.11 (CCLT). *Given complex variables $f_1, f_2, f_3, \ldots \in L^\infty(X)$ which are i.i.d., centered, and with common variance $t > 0$, we have*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n f_i \sim G_t$$

with $n \to \infty$, in moments.

PROOF. This follows indeed from the real CLT, established in Theorem 2.8, simply by taking the real and imaginary parts of all variables involved. □

So long for basic probability, and normal laws. All the above was of course quite quick, and for a more detailed introduction, you can check Feller [43], or Durrett [40].

2b. Wick formula

As a last topic regarding the real and complex normal laws, we will need formulae for their moments, and joint moments. We first have the following result:

PROPOSITION 2.12. *The even moments of the normal law are the numbers*

$$M_k(g_t) = t^{k/2} \times k!!$$

where $k!! = (k-1)(k-3)(k-5)\ldots$, and the odd moments vanish.

PROOF. We have the following computation, valid for any integer $k \in \mathbb{N}$:

$$\begin{aligned}
M_k(g_t) &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} y^k e^{-y^2/2t}dy \\
&= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} (ty^{k-1})\left(-e^{-y^2/2t}\right)' dy \\
&= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} t(k-1)y^{k-2}e^{-y^2/2t}dy \\
&= t(k-1) \times \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} y^{k-2}e^{-y^2/2t}dy \\
&= t(k-1)M_{k-2}(g_t)
\end{aligned}$$

Thus by recurrence, we are led to the formula in the statement. □

We have the following alternative formulation of the above result:

PROPOSITION 2.13. *The moments of the normal law are the numbers*

$$M_k(g_t) = t^{k/2}|P_2(k)|$$

where $P_2(k)$ is the set of pairings of $\{1,\ldots,k\}$.

PROOF. Let us count the pairings of $\{1,\ldots,k\}$. In order to have such a pairing, we must pair 1 with one of the numbers $2,\ldots,k$, and then use a pairing of the remaining $k-2$ numbers. Thus, we have the following recurrence formula:

$$|P_2(k)| = (k-1)|P_2(k-2)|$$

As for the initial data, this is $P_1 = 0$, $P_2 = 1$. Thus, we obtain by recurrence:

$$|P_2(k)| = k!!$$

Now by comparing this with Proposition 2.12, we obtain the result. □

We are not done yet, and here is one more improvement of the above, which looks quite conceptual, and which will be our final saying on the subject:

THEOREM 2.14. *The moments of the normal law are the numbers*

$$M_k(g_t) = \sum_{\pi\in P_2(k)} t^{|\pi|}$$

where $P_2(k)$ is the set of pairings of $\{1,\ldots,k\}$, and $|.|$ is the number of blocks.

PROOF. This follows indeed from Proposition 2.13, because the number of blocks of a pairing of $\{1,\ldots,k\}$ is trivially $k/2$, independently of the pairing. □

In the complex case now, things are more tricky, because here we must compute, as explained in chapter 1, moments labeled by colored integers. We first have:

PROPOSITION 2.15. *The moments of the complex normal law are given by*

$$M_k(G_t) = \begin{cases} t^p p! & (k\text{ uniform, of length }2p)\\ 0 & (k\text{ not uniform})\end{cases}$$

where $k = \circ\bullet\bullet\circ\ldots$ is called uniform when it contains the same number of $\circ$ and $\bullet$.

PROOF. We must compute the moments, with respect to colored integer exponents $k = \circ\bullet\bullet\circ\ldots$, of the variable from Definition 2.9, namely:

$$f = \frac{1}{\sqrt{2}}(a+ib)$$

We can assume that we are in the case $t=1$, and the proof here goes as follows:

(1) As a first observation, in the case where our exponent $k = \circ\bullet\bullet\circ\ldots$ is not uniform, a standard rotation argument shows that the corresponding moment of f vanishes. To

be more precise, the variable $f'=wf$ is complex Gaussian too, for any complex number $w\in\mathbb T$, and from $M_k(f)=M_k(f')$ we obtain $M_k(f)=0$, in this case.

(2) In the uniform case now, where the exponent $k=\circ\bullet\bullet\circ\ldots$ consists of p copies of $\circ$ and p copies of $\bullet$, the corresponding moment can be computed as follows:

$$\begin{aligned}M_k(f)&=\frac{1}{2^p}\sum_r\binom{p}{r}\int a^{2r}\int b^{2p-2r}\\&=\frac{1}{2^p}\sum_r\binom{p}{r}(2r)!!(2p-2r)!!\\&=\frac{p!}{4^p}\sum_r\binom{2r}{r}\binom{2p-2r}{p-r}\end{aligned}$$

(3) In order to finish the computation, let us recall that we have the following formula, coming from the generalized binomial formula, or from the Taylor formula:

$$\frac{1}{\sqrt{1+t}}=\sum_{q=0}^\infty\binom{2q}{q}\left(\frac{-t}{4}\right)^q$$

By taking the square of this series, we obtain the following formula:

$$\begin{aligned}\frac{1}{1+t}&=\sum_{qr}\binom{2q}{q}\binom{2r}{r}\left(\frac{-t}{4}\right)^{q+r}\\&=\sum_p\left(\frac{-t}{4}\right)^p\sum_r\binom{2r}{r}\binom{2p-2r}{p-r}\end{aligned}$$

Now by looking at the coefficient of t^p on both sides, we conclude that the sum on the right equals 4^p. Thus, we can finish the moment computation in (2), as follows:

$$M_k(f)=\frac{p!}{4^p}\times 4^p=p!$$

We are therefore led to the conclusion in the statement. □

As before with the real Gaussian laws, a better-looking statement is in terms of partitions. Given a colored integer $k=\circ\bullet\bullet\circ\ldots$, we say that a pairing $\pi\in P_2(k)$ is matching when it pairs $\circ-\bullet$ symbols. With this convention, we have the following result:

Theorem 2.16. *The moments of the complex normal law are the numbers*

$$M_k(G_t)=\sum_{\pi\in\mathcal P_2(k)}t^{|\pi|}$$

where $\mathcal P_2(k)$ are the matching pairings of $\{1,\ldots,k\}$, and $|.|$ is the number of blocks.

PROOF. This is a reformulation of Proposition 2.15. Indeed, we can assume that we are in the case $t = 1$, and here we know from Proposition 2.15 that we have:

$$M_k(G_t) = \begin{cases} (|k|/2)! & (k \text{ uniform}) \\ 0 & (k \text{ not uniform}) \end{cases}$$

On the other hand, the numbers $|\mathcal{P}_2(k)|$ are given by exactly the same formula. Indeed, in order to have a matching pairing of k, our exponent $k = \circ\bullet\bullet\circ\ldots$ must be uniform, consisting of p copies of $\circ$ and p copies of $\bullet$, with $p = |k|/2$. But then the matching pairings of k correspond to the permutations of the $\bullet$ symbols, as to be matched with $\circ$ symbols, and so we have $p!$ such pairings. Thus, we have the same formula as for the moments of f, and we are led to the conclusion in the statement. □

In practice, we also need to know how to compute joint moments of independent normal variables. We have here the following result, to be heavily used later on:

THEOREM 2.17 (Wick formula). *Given independent variables f_i, each following the complex normal law G_t, with $t > 0$ being a fixed parameter, we have the formula*

$$\mathbb{E}\left(f_{i_1}^{k_1}\ldots f_{i_s}^{k_s}\right) = t^{s/2}\#\left\{\pi\in\mathcal{P}_2(k)\Big|\pi\leq\ker i\right\}$$

where $k = k_1\ldots k_s$ and $i = i_1\ldots i_s$, for the joint moments of these variables, where $\pi \leq \ker i$ means that the indices of i must fit into the blocks of π, in the obvious way.

PROOF. This is something well-known, which can be proved as follows:

(1) Let us first discuss the case where we have a single variable f, which amounts in taking $f_i = f$ for any i in the formula in the statement. What we have to compute here are the moments of f, with respect to colored integer exponents $k = \circ\bullet\bullet\circ\ldots$, and the formula in the statement tells us that these moments must be:

$$\mathbb{E}(f^k) = t^{|k|/2}|\mathcal{P}_2(k)|$$

But this is the formula in Theorem 2.16, so we are done with this case.

(2) In general now, when expanding the product $f_{i_1}^{k_1}\ldots f_{i_s}^{k_s}$ and rearranging the terms, we are left with doing a number of computations as in (1), and then making the product of the expectations that we found. But this amounts in counting the partitions in the statement, with the condition $\pi \leq \ker i$ there standing for the fact that we are doing the various type (1) computations independently, and then making the product. □

The above statement is one of the possible formulations of the Wick formula, and there are in fact many other formulations, which are all useful. Here is one alternative such formulation, which is quite popular, and that we will often use in what follows:

THEOREM 2.18 (Wick formula 2). *Given independent variables f_i, each following the complex normal law G_t, with $t > 0$ being a fixed parameter, we have the formula*

$$\mathbb{E}\left(f_{i_1}\ldots f_{i_k}f_{j_1}^*\ldots f_{j_k}^*\right)=t^k\#\left\{\pi\in S_k\Big|i_{\pi(r)}=j_r,\forall r\right\}$$

for the non-vanishing joint moments of these variables.

PROOF. This follows from the usual Wick formula, from Theorem 2.17. With some changes in the indices and notations, the formula there reads:

$$\mathbb{E}\left(f_{I_1}^{K_1}\ldots f_{I_s}^{K_s}\right)=t^{s/2}\#\left\{\sigma\in\mathcal{P}_2(K)\Big|\sigma\leq\ker I\right\}$$

Now observe that we have $\mathcal{P}_2(K)=\emptyset$, unless the colored integer $K=K_1\ldots K_s$ is uniform, in the sense that it contains the same number of $\circ$ and $\bullet$ symbols. Up to permutations, the non-trivial case, where the moment is non-vanishing, is the case where the colored integer $K=K_1\ldots K_s$ is of the following special form:

$$K=\underbrace{\circ\circ\ldots\circ}_k\ \underbrace{\bullet\bullet\ldots\bullet}_k$$

So, let us focus on this case, which is the non-trivial one. Here we have $s=2k$, and we can write the multi-index $I=I_1\ldots I_s$ in the following way:

$$I=i_1\ldots i_k\ j_1\ldots j_k$$

With these changes made, the above usual Wick formula reads:

$$\mathbb{E}\left(f_{i_1}\ldots f_{i_k}f_{j_1}^*\ldots f_{j_k}^*\right)=t^k\#\left\{\sigma\in\mathcal{P}_2(K)\Big|\sigma\leq\ker(ij)\right\}$$

The point now is that the matching pairings $\sigma\in\mathcal{P}_2(K)$, with $K=\circ\ldots\circ\bullet\ldots\bullet$, of length $2k$, as above, correspond to the permutations $\pi\in S_k$, in the obvious way. With this identification made, the above modified usual Wick formula becomes:

$$\mathbb{E}\left(f_{i_1}\ldots f_{i_k}f_{j_1}^*\ldots f_{j_k}^*\right)=t^k\#\left\{\pi\in S_k\Big|i_{\pi(r)}=j_r,\forall r\right\}$$

Thus, we have reached to the formula in the statement, and we are done. □

Finally, here is one more formulation of the Wick formula, which is useful as well:

THEOREM 2.19 (Wick formula 3). *Given independent variables f_i, each following the complex normal law G_t, with $t > 0$ being a fixed parameter, we have the formula*

$$\mathbb{E}\left(f_{i_1}f_{j_1}^*\ldots f_{i_k}f_{j_k}^*\right)=t^k\#\left\{\pi\in S_k\Big|i_{\pi(r)}=j_r,\forall r\right\}$$

for the non-vanishing joint moments of these variables.

PROOF. This follows from our second Wick formula, from Theorem 2.18, simply by permuting the terms, as to have an alternating sequence of plain and conjugate variables. Alternatively, we can start with Theorem 2.17, and then perform the same manipulations as in the proof of Theorem 2.18, but with the exponent being this time as follows:

$$K = \underbrace{\circ\bullet\circ\bullet\ldots\ldots\circ\bullet}_{2k}$$

Thus, we are led to the conclusion in the statement. □

As before with the general theory of normal variables, the above was of course quite quick. For a more detailed introduction, you can check Feller [43], or Durrett [40].

2c. Gaussian matrices

Good news, we have now all needed ingredients for launching some explicit random matrix computations, featuring lots of exciting calclulations. Our goal will be that of computing the asymptotic moments, and then the asymptotic laws, with $N \to \infty$, of the main classes of large random matrices. This will be something that will take us some time, keeping us busy for the remainder of this chapter, and for chapters 3-4 as well.

There are many types of random matrices, and we will start with the "central" ones. These are, and no surprise here, the complex Gaussian matrices, appearing as follows:

DEFINITION 2.20. *A complex Gaussian matrix is a random matrix of type*

$$Z \in M_N(L^\infty(X))$$

which has i.i.d. centered complex normal entries.

Here we use the notion of complex normal variable, introduced and studied before. To be more precise, let us recall that the complex Gaussian law of parameter $t > 0$ is by definition the following law, with a, b being independent, each following the law g_t:

$$G_t = law\left(\frac{1}{\sqrt{2}}(a + ib)\right)$$

With this notion in hand, the assumption in the above definition is that all the matrix entries Z_{ij} are independent, and follow this law G_t, for a fixed value of the parameter $t > 0$. We will see that the above matrices have an interesting, and "central" combinatorics, among all kinds of random matrices, with the study of the other random matrices being usually obtained as a modification of the study of the Gaussian matrices.

Here is now our first result, regarding the Gaussian matrices:

THEOREM 2.21. *Given a sequence of Gaussian random matrices*

$$Z_N\in M_N(L^\infty(X))$$

having independent G_t variables as entries, for some fixed $t>0$, we have

$$M_k\left(\frac{Z_N}{\sqrt{N}}\right)\simeq t^{|k|/2}|\mathcal{NC}_2(k)|$$

for any colored integer $k=\circ\bullet\bullet\circ\ldots$, in the $N\to\infty$ limit.

PROOF. This is something standard, which can be done as follows:

(1) We fix $N\in\mathbb N$, and we let $Z=Z_N$. Let us first compute the trace of Z^k. With $k=k_1\ldots k_s$, and with the convention $(ij)^\circ=ij,(ij)^\bullet=ji$, we have:

$$\begin{aligned}Tr(Z^k)&=Tr(Z^{k_1}\ldots Z^{k_s})\\&=\sum_{i_1=1}^N\ldots\sum_{i_s=1}^N(Z^{k_1})_{i_1i_2}(Z^{k_2})_{i_2i_3}\ldots(Z^{k_s})_{i_si_1}\\&=\sum_{i_1=1}^N\ldots\sum_{i_s=1}^N(Z_{(i_1i_2)^{k_1}})^{k_1}(Z_{(i_2i_3)^{k_2}})^{k_2}\ldots(Z_{(i_si_1)^{k_s}})^{k_s}\end{aligned}$$

(2) Next, we rescale our variable Z by a $\sqrt{N}$ factor, as in the statement, and we also replace the usual trace by its normalized version, $tr=Tr/N$. Our formula becomes:

$$tr\left(\left(\frac{Z}{\sqrt{N}}\right)^k\right)=\frac{1}{N^{s/2+1}}\sum_{i_1=1}^N\ldots\sum_{i_s=1}^N(Z_{(i_1i_2)^{k_1}})^{k_1}(Z_{(i_2i_3)^{k_2}})^{k_2}\ldots(Z_{(i_si_1)^{k_s}})^{k_s}$$

Thus, the moment that we are interested in is given by:

$$M_k\left(\frac{Z}{\sqrt{N}}\right)=\frac{1}{N^{s/2+1}}\sum_{i_1=1}^N\ldots\sum_{i_s=1}^N\int_X(Z_{(i_1i_2)^{k_1}})^{k_1}(Z_{(i_2i_3)^{k_2}})^{k_2}\ldots(Z_{(i_si_1)^{k_s}})^{k_s}$$

(3) Let us apply now the Wick formula, from Theorem 2.17. We conclude that the moment that we are interested in is given by the following formula:

$$\begin{aligned}&M_k\left(\frac{Z}{\sqrt{N}}\right)\\=&\ \frac{t^{s/2}}{N^{s/2+1}}\sum_{i_1=1}^N\ldots\sum_{i_s=1}^N\#\left\{\pi\in\mathcal P_2(k)\Big|\pi\leq\ker\left((i_1i_2)^{k_1},(i_2i_3)^{k_2},\ldots,(i_si_1)^{k_s}\right)\right\}\\=&\ t^{s/2}\sum_{\pi\in\mathcal P_2(k)}\frac{1}{N^{s/2+1}}\#\left\{i\in\{1,\ldots,N\}^s\Big|\pi\leq\ker\left((i_1i_2)^{k_1},(i_2i_3)^{k_2},\ldots,(i_si_1)^{k_s}\right)\right\}\end{aligned}$$

(4) Our claim now is that in the $N \to \infty$ limit the combinatorics of the above sum simplifies, with only the noncrossing partitions contributing to the sum, and with each of them contributing precisely with a 1 factor, so that we will have, as desired:

$$\begin{aligned} M_k\left(\frac{Z}{\sqrt{N}}\right) &= t^{s/2}\sum_{\pi\in\mathcal{P}_2(k)}\Big(\delta_{\pi\in NC_2(k)}+O(N^{-1})\Big) \\ &\simeq t^{s/2}\sum_{\pi\in\mathcal{P}_2(k)}\delta_{\pi\in NC_2(k)} \\ &= t^{s/2}|\mathcal{NC}_2(k)| \end{aligned}$$

(5) In order to prove this, the first observation is that when k is not uniform, in the sense that it contains a different number of $\circ,\bullet$ symbols, we have $\mathcal{P}_2(k)=\emptyset$, and so:

$$M_k\left(\frac{Z}{\sqrt{N}}\right) = t^{s/2}|\mathcal{NC}_2(k)| = 0$$

(6) Thus, we are left with the case where k is uniform. Let us examine first the case where k consists of an alternating sequence of $\circ$ and $\bullet$ symbols, as follows:

$$k = \underbrace{\circ\bullet\circ\bullet\ldots\ldots\circ\bullet}_{2p}$$

In this case it is convenient to relabel our multi-index $i=(i_1,\ldots,i_s)$, with $s=2p$, in the form $(j_1,l_1,j_2,l_2,\ldots,j_p,l_p)$. With this done, our moment formula becomes:

$$M_k\left(\frac{Z}{\sqrt{N}}\right) = t^p\sum_{\pi\in\mathcal{P}_2(k)}\frac{1}{N^{p+1}}\#\left\{j,l\in\{1,\ldots,N\}^p\Big|\pi\le\ker(j_1l_1,j_2l_1,j_2l_2,\ldots,j_1l_p)\right\}$$

Now observe that, with k being as above, we have an identification $\mathcal{P}_2(k)\simeq S_p$, obtained in the obvious way. With this done too, our moment formula becomes:

$$M_k\left(\frac{Z}{\sqrt{N}}\right) = t^p\sum_{\pi\in S_p}\frac{1}{N^{p+1}}\#\left\{j,l\in\{1,\ldots,N\}^p\Big|j_r=j_{\pi(r)+1},l_r=l_{\pi(r)},\forall r\right\}$$

(7) We are now ready to do our asymptotic study, and prove the claim in (4). Let indeed $\gamma\in S_p$ be the full cycle, which is by definition the following permutation:

$$\gamma = (1\,2\,\ldots\,p)$$

In terms of γ, the conditions $j_r=j_{\pi(r)+1}$ and $l_r=l_{\pi(r)}$ found above read:

$$\gamma\pi\le\ker j \quad , \quad \pi\le\ker l$$

Counting the number of free parameters in our moment formula, we obtain:

$$\begin{aligned} M_k\left(\frac{Z}{\sqrt{N}}\right) &= \frac{t^p}{N^{p+1}}\sum_{\pi\in S_p} N^{|\pi|+|\gamma\pi|} \\ &= t^p\sum_{\pi\in S_p} N^{|\pi|+|\gamma\pi|-p-1} \end{aligned}$$

(8) The point now is that the last exponent is well-known to be ≤ 0, with equality precisely when the permutation $\pi \in S_p$ is geodesic, which in practice means that π must come from a noncrossing partition. Thus we obtain, in the $N \to \infty$ limit, as desired:

$$M_k\left(\frac{Z}{\sqrt{N}}\right) \simeq t^p|\mathcal{NC}_2(k)|$$

This finishes the proof in the case of the exponents k which are alternating, and the case where k is an arbitrary uniform exponent is similar, by permuting everything. □

2d. Circular law

The problem is now, what to do with what we found in Theorem 2.21. This does not look obvious, so let us begin by giving a name to the asymptotic laws found there:

DEFINITION 2.22. *We call circular law the abstract law Γ_t, in the sense of the abstract laws $\mathbb{C} < X, X^* > \to \mathbb{C}$, whose moments are given by:*

$$M_k(\Gamma_t) = t^{|k|/2}|\mathcal{NC}_2(k)|$$

In other words, Γ_t is an analogue of G_t, with the underlying matching pairings being replaced by matching noncrossing pairings.

This looks quite reasonable, and the only mystery lies in the name "circular" that we chose for designating Γ_t. But it is possible to do all sorts of speculations, justifying our choice. Now with this done, we can reformulate Theorem 2.21, as follows:

THEOREM 2.23. *Given complex Gaussian matrices $Z_N \in M_N(L^\infty(X))$, having independent G_t variables as entries, we have*

$$\frac{Z_N}{\sqrt{N}} \sim \Gamma_t$$

in the $N \to \infty$ limit. That is, the rescalings of Z_N become asymptotically circular.

PROOF. This is indeed a reformulation of Theorem 2.21, using Definition 2.22. □

All this is very nice, aren't we a bit engineers, and our thing works. We will discuss a more conceptual approach to all this, via more mathematics, in the next chapter.

2e. Exercises

Exercises.

CHAPTER 3

Wigner matrices

3a. Wigner matrices

In this chapter and in the next one we investigate some important versions of the complex Gaussian matrices. As a somewhat surprising remark, to start with, using real normal variables in the definition of the Gaussian matrices leads nowhere. The correct real versions of the Gaussian matrices are the Wigner matrices, constructed as follows:

Definition 3.1. *A Wigner matrix is a random matrix of type*

$$Z \in M_N(L^\infty(X))$$

which has i.i.d. centered complex normal entries, up to the constraint $Z = Z^*$.

This definition is something a bit compacted, and to be more precise here, a Wigner matrix is by definition a random matrix as follows, with the diagonal entries being real normal variables, $a_i \sim g_t$, for some $t > 0$, the upper diagonal entries being complex normal variables, $b_{ij} \sim G_t$, the lower diagonal entries being the conjugates of the upper diagonal entries, as indicated, and with all the variables a_i, b_{ij} being independent:

$$Z = \begin{pmatrix} a_1 & b_{12} & \ldots & \ldots & b_{1N} \\ \bar{b}_{12} & a_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_{N-1} & b_{N-1,N} \\ \bar{b}_{1N} & \ldots & \ldots & \bar{b}_{N-1,N} & a_N \end{pmatrix}$$

This might look a bit complicated, but for many concrete applications, the Wigner matrices are in fact the central objects in random matrix theory, and in particular, they are often more important than the Gaussian ones. In fact, these are the random matrices which were first considered and investigated, a long time ago, by Wigner himself [97].

As before with the complex Gaussian matrices, we would like to compute the law of the Wigner matrices, in the $N \to \infty$ limit. But for this purpose, no need to use the Wick formula and do heavy combinatorics again, because we can use the following simple fact, making the connection with our computations for Gaussian matrices from chapter 2:

THEOREM 3.2. *Given a Gaussian matrix Z, with independent entries following the centered complex normal law G_t, with $t > 0$, if we write*

$$Z = \frac{1}{\sqrt{2}}(X + iY)$$

with X, Y being self-adjoint, then both X, Y are Wigner matrices, of parameter t.

PROOF. This is something elementary, which can be done in two steps, as follows:

(1) As a first observation, the result holds at $N = 1$. Indeed, here our Gaussian matrix Z is just a random variable, subject to the condition $Z \sim G_t$. But recall that the law G_t is by definition as follows, with X, Y being independent, each following the law g_t:

$$G_t = law\left(\frac{1}{\sqrt{2}}(X + iY)\right)$$

Thus in this case, $N = 1$, the variables X, Y that we obtain in the statement, as rescaled real and imaginary parts of Z, are subject to the condition $X, Y \sim g_t$, and so are Wigner matrices of size $N = 1$ and parameter $t > 0$, as in Definition 3.1.

(2) In the general case now, $N \in \mathbb{N}$, the proof is similar, by using the basic behavior of the real and complex normal variables with respect to sums. □

By using now Theorem 3.2, and the computations from chapter 2, we obtain:

THEOREM 3.3. *Given a sequence of Wigner random matrices*

$$Z_N \in M_N(L^\infty(X))$$

having independent G_t variables as entries, with $t > 0$, up to $Z_N = Z_N^*$, we have

$$M_k\left(\frac{Z_N}{\sqrt{N}}\right) \simeq t^{k/2}|NC_2(k)|$$

for any integer $k \in \mathbb{N}$, in the $N \to \infty$ limit.

PROOF. This can be deduced from a direct computation based on the Wick formula, similar to that from chapter 2, but the best is to deduce this result from the result in chapter 2 itself. Indeed, we know from there that for a sequence of complex Gaussian matrices $Y_N \in M_N(L^\infty(X))$ we have the following formula, valid for any colored integer $K = \circ \bullet \bullet \circ \ldots$, with $\mathcal{NC}_2$ standing for noncrossing matching pairings:

$$M_K\left(\frac{Y_N}{\sqrt{N}}\right) \simeq t^{|K|/2}|\mathcal{NC}_2(K)|$$

By doing some combinatorics, we deduce from this that we have the following formula for the moments of the matrices $Re(Y_N)$, with respect to usual exponents, $k\in\mathbb N$:

$$\begin{aligned}M_k\left(\frac{Re(Y_N)}{\sqrt N}\right)&=2^{-k}\cdot M_k\left(\frac{Y_N}{\sqrt N}+\frac{Y_N^*}{\sqrt N}\right)\\&=2^{-k}\sum_{|K|=k}M_K\left(\frac{Y_N}{\sqrt N}\right)\\&\simeq 2^{-k}\sum_{|K|=k}t^{k/2}|\mathcal{NC}_2(K)|\\&=2^{-k}\cdot t^{k/2}\cdot 2^{k/2}|\mathcal{NC}_2(k)|\\&=2^{-k/2}\cdot t^{k/2}|NC_2(k)|\end{aligned}$$

Now since the matrices $Z_N=\sqrt2 Re(Y_N)$ are of Wigner type, this gives the result. □

Summarizing, all this brings us into counting noncrossing pairings. So, let us start with some preliminaries here. We first have the following well-known result:

Theorem 3.4. *The Catalan numbers, which count the noncrossing pairings,*

$$C_k=|NC_2(2k)|$$

have the following properties:

1. *They satisfy* $C_{k+1}=\sum_{a+b=k}C_aC_b$.
2. *The series* $f(z)=\sum_{k\geq0}C_kz^k$ *satisfies* $zf^2-f+1=0$.
3. *This series is given by* $f(z)=\frac{1-\sqrt{1-4z}}{2z}$.
4. *We have the formula* $C_k=\frac{1}{k+1}\binom{2k}{k}$.
5. *Numerically,* $\{C_k\}_{k\geq1}$ *is* $1,2,5,14,42,132,429,\ldots$

Proof. All this this well-known, and beautiful, the idea being as follows:

(1) We must count the noncrossing pairings of $\{1,\ldots,2k+2\}$. But such a pairing appears by pairing 1 to an odd number, $2a+1$, and then inserting a noncrossing pairing of $\{2,\ldots,2a\}$, and a noncrossing pairing of $\{2a+2,\ldots,2k+2\}$. We conclude that we have the following recurrence formula for the Catalan numbers, as in the statement:

$$C_{k+1}=\sum_{a+b=k}C_aC_b$$

(2) Consider now generating series of the Catalan numbers, $f(z) = \sum_{k\geq 0} C_k z^k$. In terms of this generating series, the above recurrence gives:

$$\begin{aligned} zf^2 &= \sum_{a,b\geq 0} C_a C_b z^{a+b+1} \\ &= \sum_{k\geq 1} \sum_{a+b=k-1} C_a C_b z^k \\ &= \sum_{k\geq 1} C_k z^k \\ &= f - 1 \end{aligned}$$

Thus f satisfies the equation $zf^2 - f + 1 = 0$, as claimed.

(3) By solving the equation $zf^2 - f + 1 = 0$ found above, and choosing the solution which is bounded at $z = 0$, we obtain the following formula, as claimed:

$$f(z) = \frac{1 - \sqrt{1 - 4z}}{2z}$$

(4) In order to compute this function, we use the generalized binomial formula:

$$(1 + t)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k$$

For the exponent $p = 1/2$, the generalized binomial coefficients are:

$$\begin{aligned} \binom{1/2}{k} &= \frac{1/2(-1/2)(-3/2)\ldots(3/2 - k)}{k!} \\ &= (-1)^{k-1} \frac{1 \cdot 3 \cdot 5 \ldots (2k - 3)}{2^k k!} \\ &= (-1)^{k-1} \frac{(2k - 2)!}{2^{k-1}(k - 1)! 2^k k!} \\ &= \frac{(-1)^{k-1}}{2^{2k-1}} \cdot \frac{1}{k} \binom{2k - 2}{k - 1} \\ &= -2 \left(\frac{-1}{4}\right)^k \cdot \frac{1}{k} \binom{2k - 2}{k - 1} \end{aligned}$$

Thus the generalized binomial formula at exponent $p = 1/2$ reads:

$$\sqrt{1 + t} = 1 - 2 \sum_{k=1}^{\infty} \frac{1}{k} \binom{2k - 2}{k - 1} \left(\frac{-t}{4}\right)^k$$

With $t=-4z$ we obtain from this the following formula:

$$\sqrt{1-4z}=1-2\sum_{k=1}^\infty\frac{1}{k}\binom{2k-2}{k-1}z^k$$

Now back to our series f, we obtain the following formula for it:

$$\begin{aligned}f(z)&=\frac{1-\sqrt{1-4z}}{2z}\\&=\sum_{k=1}^\infty\frac{1}{k}\binom{2k-2}{k-1}z^{k-1}\\&=\sum_{k=0}^\infty\frac{1}{k+1}\binom{2k}{k}z^k\end{aligned}$$

Now recall that we have $f(z)=\sum_{k\geq0}C_kz^k$, by definition of our series f. Thus the Catalan numbers are given by the formula the statement, namely:

$$C_k=\frac{1}{k+1}\binom{2k}{k}$$

(5) Numerics left to you, and with the comment that true physicists, engineers and other scientists do not need a computer, for everything ≤ 1000. Are you one? □

Getting back now to the Wigner matrices, we have the following result:

THEOREM 3.5. *Given a sequence of Wigner random matrices*

$$Z_N\in M_N(L^\infty(X))$$

having independent G_t variables as entries, with $t>0$, up to $Z_N=Z_N^*$, we have

$$M_{2k}\left(\frac{Z_N}{\sqrt{N}}\right)\simeq t^kC_k$$

in the $N\to\infty$ limit. As for the asymptotic odd moments, these all vanish.

PROOF. This follows from Theorem 3.3 and Theorem 3.4. Indeed, according to the results there, the asymptotic even moments are given by:

$$M_{2k}\left(\frac{Z_N}{\sqrt{N}}\right)\simeq t^k|NC_2(2k)|=t^kC_k$$

As for the asymptotic odd moments, once again from Theorem 3.3, we know that these all vanish. Thus, we are led to the conclusion in the statement. □

3b. Semicircle law

In order to recapture the asymptotic measure of the Wigner matrices out of the moments, which are the Catalan numbers, there are several methods available, namely:

(1) Stieltjes inversion.

(2) Knowledge of the sphere of space-time, unaltered by Einstein's relativity.

(3) Cheating.

We will follow here the method (1), which looks the most reasonable, and comment on (2,3) later. We will need the Stieltjes inversion formula, which is as follows:

THEOREM 3.6. *The density of a real probability measure μ can be recaptured from the sequence of moments $\{M_k\}_{k\geq 0}$ via the Stieltjes inversion formula*

$$d\mu(x)=\lim_{t\searrow 0}-\frac{1}{\pi}\,Im\left(G(x+it)\right)\cdot dx$$

where the function on the right, given in terms of moments by

$$G(\xi)=\xi^{-1}+M_1\xi^{-2}+M_2\xi^{-3}+\ldots$$

is the Cauchy transform of the measure μ.

PROOF. The Cauchy transform of our measure μ is given by:

$$\begin{aligned}G(\xi) &= \xi^{-1}\sum_{k=0}^\infty M_k\xi^{-k}\\ &= \int_{\mathbb{R}}\frac{\xi^{-1}}{1-\xi^{-1}y}\,d\mu(y)\\ &= \int_{\mathbb{R}}\frac{1}{\xi-y}\,d\mu(y)\end{aligned}$$

Now with $\xi=x+it$, we obtain the following formula:

$$\begin{aligned}Im(G(x+it)) &= \int_{\mathbb{R}}Im\left(\frac{1}{x-y+it}\right)d\mu(y)\\ &= \int_{\mathbb{R}}\frac{1}{2i}\left(\frac{1}{x-y+it}-\frac{1}{x-y-it}\right)d\mu(y)\\ &= -\int_{\mathbb{R}}\frac{t}{(x-y)^2+t^2}\,d\mu(y)\end{aligned}$$

By integrating over $[a,b]$ we obtain, with the change of variables $x=y+tz$:

$$\begin{aligned}\int_a^b Im(G(x+it))dx &= -\int_{\mathbb{R}}\int_a^b \frac{t}{(x-y)^2+t^2}\,dx\,d\mu(y)\\ &= -\int_{\mathbb{R}}\int_{(a-y)/t}^{(b-y)/t}\frac{t}{(tz)^2+t^2}\,t\,dz\,d\mu(y)\\ &= -\int_{\mathbb{R}}\int_{(a-y)/t}^{(b-y)/t}\frac{1}{1+z^2}\,dz\,d\mu(y)\\ &= -\int_{\mathbb{R}}\left(\arctan\frac{b-y}{t}-\arctan\frac{a-y}{t}\right)d\mu(y)\end{aligned}$$

Now observe that with $t\searrow 0$ we have:

$$\lim_{t\searrow 0}\left(\arctan\frac{b-y}{t}-\arctan\frac{a-y}{t}\right)=\begin{cases}\frac{\pi}{2}-\frac{\pi}{2}=0 & (y<a)\\ \frac{\pi}{2}-0=\frac{\pi}{2} & (y=a)\\ \frac{\pi}{2}-(-\frac{\pi}{2})=\pi & (a<y<b)\\ 0-(-\frac{\pi}{2})=\frac{\pi}{2} & (y=b)\\ -\frac{\pi}{2}-(-\frac{\pi}{2})=0 & (y>b)\end{cases}$$

We therefore obtain the following formula:

$$\lim_{t\searrow 0}\int_a^b Im(G(x+it))dx=-\pi\left(\mu(a,b)+\frac{\mu(a)+\mu(b)}{2}\right)$$

Thus, we are led to the conclusion in the statement. □

Before getting further, let us mention that the above result does not fully solve the moment problem, because we still have the question of understanding when a sequence of numbers $M_1,M_2,M_3,\ldots$ can be the moments of a measure μ. We have here:

THEOREM 3.7. *A sequence of numbers $M_0,M_1,M_2,M_3,\ldots\in\mathbb{R}$, with $M_0=1$, is the series of moments of a real probability measure μ precisely when:*

$$\begin{vmatrix}M_0\end{vmatrix}\geq 0\quad,\quad\begin{vmatrix}M_0&M_1\\M_1&M_2\end{vmatrix}\geq 0\quad,\quad\begin{vmatrix}M_0&M_1&M_2\\M_1&M_2&M_3\\M_2&M_3&M_4\end{vmatrix}\geq 0\quad,\quad\ldots$$

That is, the associated Hankel determinants must be all positive.

PROOF. As a first observation, the positivity conditions in the statement tell us that the following associated linear forms must be positive:

$$\sum_{i,j=1}^n c_i\bar{c}_jM_{i+j}\geq 0$$

But this is something very classical, in one sense the result being elementary, coming from the following computation, which shows that we have positivity indeed:

$$\int_{\mathbb{R}}\left|\sum_{i=1}^n c_ix^i\right|^2 d\mu(x)=\int_{\mathbb{R}}\sum_{i,j=1}^n c_i\bar{c}_jx^{i+j}d\mu(x)=\sum_{i,j=1}^n c_i\bar{c}_jM_{i+j}$$

As for the other sense, here the result comes once again from the above formula, this time via some standard functional analysis, as explained for instance in [77]. □

Now back to our questions, as a basic application of the Stieltjes formula, we can solve the moment problem for the Catalan numbers, as follows:

PROPOSITION 3.8. *The real measure having as even moments the Catalan numbers, $C_k=\frac{1}{k+1}\binom{2k}{k}$, and having all odd moments 0 is the measure*

$$\gamma_1=\frac{1}{2\pi}\sqrt{4-x^2}dx$$

called Wigner semicircle law on $[-2,2]$.

PROOF. In order to apply the Stieltjes inversion formula, we need a simple formula for the Cauchy transform. For this purpose, our starting point will be the formula from Theorem 3.4 for the generating series of the Catalan numbers, namely:

$$\sum_{k=0}^\infty C_kz^k=\frac{1-\sqrt{1-4z}}{2z}$$

By using this formula with $z=\xi^{-2}$, we obtain the following formula:

$$\begin{aligned}G(\xi)&=\xi^{-1}\sum_{k=0}^\infty C_k\xi^{-2k}\\&=\xi^{-1}\cdot\frac{1-\sqrt{1-4\xi^{-2}}}{2\xi^{-2}}\\&=\frac{\xi}{2}\left(1-\sqrt{1-4\xi^{-2}}\right)\\&=\frac{\xi}{2}-\frac{1}{2}\sqrt{\xi^2-4}\end{aligned}$$

With this formula in hand, let us apply now the Stieltjes inversion formula, from Theorem 3.6. The study here goes as follows:

(1) According to the general philosophy of the Stieltjes formula, the first term, namely $\xi/2$, which is "trivial", will not contribute to the density.

(2) As for the second term, which is something non-trivial, this will contribute to the density, the rule here being that the square root $\sqrt{\xi^2-4}$ will be replaced by the "dual" square root $\sqrt{4-x^2}\,dx$, and that we have to multiply everything by $-1/\pi$.

(3) As a conclusion, by Stieltjes inversion we obtain the following density:

$$d\mu(x)=-\frac{1}{\pi}\cdot-\frac{1}{2}\sqrt{4-x^2}\,dx=\frac{1}{2\pi}\sqrt{4-x^2}dx$$

Thus, we have obtained the mesure in the statement, and we are done. □

More generally now, we have the following result:

PROPOSITION 3.9. *Given $t>0$, the real measure having as even moments the numbers $M_{2k}=t^kC_k$ and having all odd moments 0 is the measure*

$$\gamma_t=\frac{1}{2\pi t}\sqrt{4t-x^2}dx$$

called Wigner semicircle law on $[-2\sqrt{t},2\sqrt{t}]$.

PROOF. This follows by redoing the Stieltjes inversion computation. We have:

$$\begin{aligned}G(\xi)&=\xi^{-1}\sum_{k=0}^\infty t^kC_k\xi^{-2k}\\&=\xi^{-1}\cdot\frac{1-\sqrt{1-4t\xi^{-2}}}{2t\xi^{-2}}\\&=\frac{\xi}{2t}\left(1-\sqrt{1-4t\xi^{-2}}\right)\\&=\frac{\xi}{2t}-\frac{1}{2t}\sqrt{\xi^2-4t}\end{aligned}$$

Thus, by Stieltjes inversion we obtain the following density:

$$d\mu(x)=\frac{1}{2\pi t}\sqrt{4t-x^2}\,dx$$

But simplest is in fact, perhaps a bit by cheating, simply using the result at $t=1$, from Proposition 3.8, along with a change of variables. □

Talking cheating, another way of recovering Proposition 3.8, without Stieltjes inversion, but by knowing the answer to the question in advance, is as follows:

PROPOSITION 3.10. *The Catalan numbers are the even moments of*

$$\gamma_1=\frac{1}{2\pi}\sqrt{4-x^2}dx$$

called Wigner semicircle law. As for the odd moments of γ_1, these all vanish.

PROOF. The even moments can be computed with $x = 2\cos t$, and we get:

$$\begin{aligned} M_{2k} &= \frac{1}{\pi}\int_0^2 \sqrt{4-x^2}x^{2k}\,dx \\ &= \frac{1}{\pi}\int_0^{\pi/2}\sqrt{4-4\cos^2 t}\,(2\cos t)^{2k} 2\sin t\,dt \\ &= \frac{4^{k+1}}{\pi}\int_0^{\pi/2}\cos^{2k} t\sin^2 t\,dt \\ &= \frac{4^{k+1}}{\pi}\cdot\frac{\pi}{2}\cdot\frac{(2k)!!2!!}{(2k+3)!!} \\ &= 2\cdot 4^k\cdot\frac{(2k)!/2^k k!}{2^{k+1}(k+1)!} \\ &= C_k \end{aligned}$$

Here we have used the following well-known formula, due to Wallis, with the exponents being given by $\varepsilon(p) = 1$ if p is even, and $\varepsilon(p) = 0$ if p is odd:

$$\int_0^{\pi/2}\cos^p t\sin^q t\,dt = \left(\frac{\pi}{2}\right)^{\varepsilon(p)\varepsilon(q)}\frac{p!!q!!}{(p+q+1)!!}$$

As for the odd moments, these all vanish, because the density of γ_1 is even. □

More generally, we have the following result, involving a parameter $t > 0$:

PROPOSITION 3.11. *The numbers $t^k C_k$ are the even moments of*

$$\gamma_t = \frac{1}{2\pi t}\sqrt{4t-x^2}dx$$

called semicircle law on $[-2\sqrt{t}, 2\sqrt{t}]$. As for the odd moments of γ_t, these all vanish.

PROOF. This follows indeed from Proposition 3.10, with $x = \sqrt{t}y$. To be more precise, the even moments of the measure in the statement are given by:

$$\begin{aligned} M_{2k} &= \frac{1}{2\pi t}\int_{-2\sqrt{t}}^{2\sqrt{t}}\sqrt{4t-x^2}\,x^{2k}\,dx \\ &= \frac{1}{2\pi t}\int_{-1}^{1}\sqrt{4t-ty^2}\,(\sqrt{t}y)^{2k}\,\sqrt{t}\,dy \\ &= \frac{t^k}{2\pi}\int_{-1}^{1}\sqrt{4-y^2}\,y^{2k}\,dy \\ &= t^k C_k \end{aligned}$$

As for the odd moments, these all vanish, because the density of γ_t is an even function. Thus, we are led to the conclusion in the statement. □

Finally, as previously announced, such results can be recovered as well from some knowledge of the physics of our space-time, the result here being as follows:

THEOREM 3.12. *We have the following formula, which makes SU_2 isomorphic to the unit real sphere $S^3_{\mathbb{R}} \subset \mathbb{R}^3$, or sphere of the Euclidean space-time:*

$$SU_2 = \left\{ \begin{pmatrix} x+iy & z+it \\ -z+it & x-iy \end{pmatrix} \Big| \, x^2+y^2+z^2+t^2=1 \right\}$$

Also, the odd moments of the main character $\chi = 2x$ vanish, the even moments are the Catalan numbers, and the law of χ is the semicircle law γ_1.

PROOF. Obviously, many things going on here, the idea being as follows:

(1) For a 2×2 matrix of determinant 1 the unitarity condition $U^* = U^{-1}$ reads:

$$\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Thus $d = \bar{a}, c = -\bar{b}$, and we conclude that we have the following formula:

$$SU_2 = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \Big| \, |a|^2+|b|^2=1 \right\}$$

But with $a = x+iy$, $b = z+it$ this gives the formula of SU_2 in the statement.

(2) Regarding the rest, this depends a bit on your knowledge of physics. At the advanced level, the matrix coefficients of x,y,z,t in the formula of SU_2 are the Pauli spin matrices, and if you heard about these, you certainly heard about the Clebsch-Gordan rules too, which give via representation theory the Catalan number assertion.

(3) Alternatively, we can recover the Catalan numbers with some spherical integration know-how, as follows, but with this looking again as some physics witchcraft:

$$\begin{aligned}
\int_{S^3_{\mathbb{R}}} x^{2k} &= \frac{3!!(2k)!!}{(2k+3)!!} \\
&= 2\cdot\frac{3\cdot 5\cdot 7\ldots(2k-1)}{2\cdot 4\cdot 6\ldots(2k+2)} \\
&= 2\cdot\frac{(2k)!}{2^k k! 2^{k+1}(k+1)!} \\
&= \frac{1}{4^k}\cdot\frac{1}{k+1}\binom{2k}{k} \\
&= \frac{C_k}{4^k}
\end{aligned}$$

(4) To be more precise, we have used here, with $N=4$, the following generalization of the Wallis formula for trigonometric integrals from the proof of Proposition 3.10, which can be deduced from it, via spherical coordinates and Fubini:

$$\int_{S^{N-1}_\mathbb{R}} x_1^{k_1}\ldots x_N^{k_N}\,dx=\frac{(N-1)!!k_1!!\ldots k_N!!}{(N+\Sigma k_i-1)!!}$$

(5) Finally, the assertion regarding the semicircle law γ_1 can be recovered as well via some geometry and physics, or via a computation, as in Proposition 3.10. □

Summarizing, we have solved the moment problem for the Catalan numbers, with all sorts of solutions being available, depending on knowledge and taste. Now by putting everything together, we obtain the Wigner theorem, as follows:

THEOREM 3.13. *Given a sequence of Wigner random matrices*

$$Z_N\in M_N(L^\infty(X))$$

having independent G_t variables as entries, up to $Z_N=Z_N^*$, we have, with $N\to\infty$,

$$\frac{Z_N}{\sqrt{N}}\sim\gamma_t$$

the limiting measure being the Wigner semicircle law $\gamma_t=\frac{1}{2\pi t}\sqrt{4t-x^2}dx$.

PROOF. This follows indeed by combining our previous result, Theorem 3.5, with either of the above results, regarding the Wigner semicircle law. □

3c. Free probability

All that we have so far about random matrices, and more precisely our asymptotic results regarding the complex Gaussian and the Wigner matrices, remains quite mysterious. Following Voiculescu [87], [88], [89], we will present now a conceptual approach to all this, notably solving an open question that we were having, regarding the precise nature of the circular law. As a bonus, this will lead to some totally new results as well, and to be more precise, to "asymptotic freeness" results for our random matrices.

As a starting point, we have the following update, in the general setting of the C^*-algebras endowed with traces, of the classical notion of independence:

DEFINITION 3.14. *We call two subalgebras $B,C\subset A$ independent when the following condition is satisfied, for any $x\in B$ and $y\in C$:*

$$tr(xy)=tr(x)tr(y)$$

Equivalently, the following condition must be satisfied, for any $x\in B$ and $y\in C$:

$$tr(x)=tr(y)=0\implies tr(xy)=0$$

Also, $b,c\in A$ are called independent when $B=<b>$ and $C=<c>$ are independent.

It is possible to develop some theory here, but this leads of course to the usual CLT. As a much more interesting notion now, we have Voiculescu's freeness [87]:

Definition 3.15. *Given a pair (A,tr), we call two subalgebras $B,C\subset A$ free when the following condition is satisfied, for any $x_i\in B$ and $y_i\in C$:*

$$tr(x_i)=tr(y_i)=0\implies tr(x_1y_1x_2y_2\ldots)=0$$

Also, $b,c\in A$ are called free when $B=<b>$ and $C=<c>$ are free.

As a first observation, there is a certain lack of symmetry between Definition 3.14 and Definition 3.15, because the latter does not include an explicit formula for quantities of type $tr(x_1y_1x_2y_2\ldots)$. But this can be done, the precise result being as follows:

Proposition 3.16. *If $B,C\subset A$ are free, the restriction of tr to $<B,C>$ can be computed in terms of the restrictions of tr to B,C. To be more precise, we have*

$$tr(x_1y_1x_2y_2\ldots)=P\Big(\{tr(x_{i_1}x_{i_2}\ldots)\}_i,\{tr(y_{j_1}y_{j_2}\ldots)\}_j\Big)$$

where P is certain polynomial, depending on the length of $x_1y_1x_2y_2\ldots$, having as variables the traces of products $x_{i_1}x_{i_2}\ldots$ and $y_{j_1}y_{j_2}\ldots$, with $i_1<i_2<\ldots$ and $j_1<j_2<\ldots$

Proof. Our first claim is that we have $tr(xy)=tr(x)tr(y)$. Indeed, this follows from the following computation, with the convention $x'=x-tr(x)$:

$$\begin{aligned}tr(xy)&=tr[(x'+tr(x))(y'+tr(y))]\\&=tr(x'y')+t(x')tr(y)+tr(x)tr(y')+tr(x)tr(y)\\&=tr(x'y')+tr(x)tr(y)\\&=tr(x)tr(y)\end{aligned}$$

In general, the situation is of course more complicated than this, but the same trick applies. To be more precise, we can start our computation as follows:

$$\begin{aligned}tr(x_1y_1x_2y_2\ldots)&=tr\Big[(x_1'+tr(x_1))(y_1'+tr(y_1))(x_2'+tr(x_2))\ldots\Big]\\&=tr(x_1'y_1'x_2'y_2'\ldots)+\text{other terms}\\&=\text{other terms}\end{aligned}$$

Thus, we are led to a kind of recurrence, and this gives the result. □

Let us discuss now some examples of independence and freeness. We first have the following result, from [87], which is something elementary:

Proposition 3.17. *Given two algebras (A,tr) and (B,tr), the following hold:*

1. *A,B are independent inside their tensor product $A\otimes B$, endowed with its canonical tensor product trace, given on basic tensors by $tr(a\otimes b)=tr(a)tr(b)$.*
2. *A,B are free inside their free product $A*B$, endowed with its canonical free product trace, given by the formulae in Proposition 3.16.*

PROOF. Both the assertions are indeed clear from definitions, with just some standard discussion needed for (2), in connection with the free product trace. See [87]. ☐

More concretely now, we have the following result, also from Voiculescu [87]:

THEOREM 3.18. *We have the following results, valid for group algebras:*

(1) $C^*(\Gamma), C^*(\Lambda)$ *are independent inside* $C^*(\Gamma \times \Lambda)$.
(2) $C^*(\Gamma), C^*(\Lambda)$ *are free inside* $C^*(\Gamma * \Lambda)$.

PROOF. In order to prove these results, we can use the general results in Proposition 3.17, along with the following two isomorphisms, which are both standard:

$$C^*(\Gamma \times \Lambda) = C^*(\Lambda) \otimes C^*(\Gamma) \quad , \quad C^*(\Gamma * \Lambda) = C^*(\Lambda) * C^*(\Gamma)$$

Alternatively, we can check the independence and freeness formulae on group elements, which is something trivial, and then conclude by linearity. See [87]. ☐

We have already seen limiting theorems in classical probability, in chapter 2. In order to deal now with freeness, let us develop some tools. First, we have:

PROPOSITION 3.19. *We have a well-defined operation* $\boxplus$, *given by*

$$\mu_a \boxplus \mu_b = \mu_{a+b}$$

with a, b *being free, called free convolution.*

PROOF. We need to check here that if a, b are free, then the distribution μ_{a+b} depends only on the distributions μ_a, μ_b. But for this purpose, we can use the formula in Proposition 3.16. Indeed, by plugging in arbitrary powers of a, b as variables x_i, y_j, we obtain a family of formulae of the following type, with Q being certain polyomials:

$$tr(a^{k_1}b^{l_1}a^{k_2}b^{l_2}\ldots) = P\Big(\{tr(a^k)\}_k, \{tr(b^l)\}_l\Big)$$

Thus the moments of $a+b$ depend only on the moments of a, b, and the same argument shows that the same holds for $*$-moments, and this gives the result. ☐

In order to advance now, we would need an analogue of the Fourier transform, or rather of the log of the Fourier transform. Quite remarkably, such a transform exists indeed, the precise result here, due to Voiculescu [87], being as follows:

THEOREM 3.20. *Given a probability measure* μ, *define its* R*-transform as follows:*

$$G_\mu(\xi) = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t} \implies G_\mu\left(R_\mu(\xi) + \frac{1}{\xi}\right) = \xi$$

The free convolution operation is then linearized by the R*-transform.*

PROOF. This is something quite tricky, the idea being as follows:

(1) In order to model the free convolution, the best is to use creation operators on free Fock spaces, corresponding to the semigroup von Neumann algebras $L(\mathbb{N}^{*k})$. Indeed, we have some freeness here, a bit in the same way as in the free group algebras $C^*(F_k)$.

(2) The point now, motivating this choice, is that the variables of type $S^*+f(S)$, with $S\in L(\mathbb{N})$ being the shift, and with $f\in\mathbb{C}[X]$ being an arbitrary polynomial, are easily seen to model in moments all the possible distributions $\mu:\mathbb{C}[X]\to\mathbb{C}$.

(3) Now let $f,g\in\mathbb{C}[X]$ and consider the variables $S^*+f(S)$ and $T^*+g(T)$, where $S,T\in L(\mathbb{N}*\mathbb{N})$ are the shifts corresponding to the generators of $\mathbb{N}*\mathbb{N}$. These variables are free, and by using a 45° argument, their sum has the same law as $S^*+(f+g)(S)$.

(4) Thus the operation $\mu\to f$ linearizes the free convolution. We are therefore left with a computation inside $L(\mathbb{N})$, which is elementary, and whose conclusion is that $R_\mu=f$ can be recaptured from μ via the Cauchy transform G_μ, as in the statement. □

With the above linearization technology in hand, we can now establish the following remarkable free analogue of the CLT, also due to Voiculescu [87]:

THEOREM 3.21 (Free CLT). *Given self-adjoint variables $x_1,x_2,x_3,\ldots$, which are f.i.d., centered, with variance $t>0$, we have, with $n\to\infty$, in moments,*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n x_i\sim\gamma_t$$

where $\gamma_t=\frac{1}{2\pi t}\sqrt{4t-x^2}dx$ is the Wigner semicircle law of parameter t.

PROOF. We follow the same idea as in the proof of the CLT:

(1) At $t=1$, the R-transform of the variable in the statement can be computed by using the linearization property from Theorem 3.20, and is given by:

$$R(\xi)=nR_x\left(\frac{\xi}{\sqrt{n}}\right)\simeq\xi$$

(2) On the other hand, some standard computations show that the Cauchy transform of the Wigner law γ_1 satisfies the following equation:

$$G_{\gamma_1}\left(\xi+\frac{1}{\xi}\right)=\xi$$

Thus, by using Theorem 3.20, we have the following formula:

$$R_{\gamma_1}(\xi)=\xi$$

(3) We conclude that the laws in the statement have the same R-transforms, and so they are equal. The passage to the general case, $t>0$, is routine, by dilation. □

In the complex case now, we have a similar result, also from [87], as follows:

THEOREM 3.22 (Free CCLT). *Given random variables $x_1, x_2, x_3, \ldots$ which are f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$, in moments,*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim \Gamma_t$$

where $\Gamma_t = law\big((a+ib)/\sqrt{2}\big)$, with a, b being free, each following the Wigner semicircle law γ_t, is the Voiculescu circular law of parameter t.

PROOF. This follows indeed from the free CLT, established before, simply by taking real and imaginary parts of all the variables involved. □

Quite remarkably, the law Γ_t found in Theorem 3.22 coincides with the previous "circular law" found in chapter 2. In order to explain this, let us start with:

PROPOSITION 3.23. *Let H be the complex Hilbert space having as basis the colored integers $k = \circ \bullet \bullet \circ \ldots$, and consider the shift operators on this space:*

$$S : k \to \circ k \quad , \quad T : k \to \bullet k$$

We have then the following equalities of distributions,

$$S + S^* \sim \gamma_1 \quad , \quad S + T^* \sim \Gamma_1$$

with respect to the state $\varphi(T) =< Te, e >$, where e is the empty word.

PROOF. This is standard free probability, the idea being as follows:

(1) The first formula, namely $S + S^* \sim \gamma_1$, is something that we already met, in a slightly different formulation, when proving Theorem 3.21.

(2) As for the second formula, $S + T^* \sim \Gamma_1$, this follows from the first formula, by using the freeness results and the rotation tricks from the proof of Theorem 3.20. □

At the combinatorial level now, we have the following result:

PROPOSITION 3.24. *A variable $a \in A$ follows the law Γ_1 precisely when*

$$tr(a^k) = |\mathcal{NC}_2(k)|$$

for any colored integer $k = \circ \bullet \bullet \circ \ldots$

PROOF. By using Proposition 3.23, it is enough to do the computation in the model there. To be more precise, we can use the following explicit formulae for S, T:

$$S : k \to \circ k \quad , \quad T : k \to \bullet k$$

With these formulae in hand, our claim is that we have the following formula:

$$< (S + T^*)^k e, e >= |\mathcal{NC}_2(k)|$$

In order to prove this formula, we can proceed as for the semicircle laws. Indeed, let us expand the quantity $(S + T^*)^k$, and then apply the state φ. With respect to the previous computation, for the semicircle laws, what happens is that the contributions will come this time via the following formulae, which must succesively apply, as to collapse the whole product of S, S^*, T, T^* variables into a 1 quantity:

$$S^*S = 1 \quad , \quad T^*T = 1$$

As before, these applications of the rules $S^*S = 1$, $T^*T = 1$ must appear in a non-crossing manner, but what happens now, in contrast with the previous computation, where $S + S^*$ was self-adjoint, is that at each point where the exponent k has a $\circ$ entry we must use $T^*T = 1$, and at each point where the exponent k has a $\bullet$ entry we must use $S^*S = 1$. Thus the contributions, which are each worth 1, are parametrized by the partitions $\pi \in \mathcal{NC}_2(k)$. Thus, we obtain the above moment formula, as desired. □

More generally now, by rescaling, we have the following result:

Theorem 3.25. *A variable $a \in A$ is circular, $a \sim \Gamma_t$, precisely when its moments are given by the formula*

$$tr(a^k) = t^{|k|/2} |\mathcal{NC}_2(k)|$$

for any colored integer $k = \circ \bullet \bullet \circ \ldots$

Proof. This follows indeed from Proposition 3.24, by rescaling. Alternatively, we can get this as well directly, by suitably modifying Proposition 3.23 first. □

The above result is very interesting for us, because we can now formulate:

Theorem 3.26. *Given a sequence of complex Gaussian matrices $Z_N \in M_N(L^\infty(X))$, having independent G_t variables as entries, with $t > 0$, we have*

$$\frac{Z_N}{\sqrt{N}} \sim \Gamma_t$$

in the $N \to \infty$ limit, with the limiting measure being Voiculescu's circular law.

Proof. We know from chapter 2 that the asymptotic moments are:

$$M_k\left(\frac{Z_N}{\sqrt{N}}\right) \simeq t^{|k|/2} |\mathcal{NC}_2(k)|$$

Now by comparing with Theorem 3.25, this gives the result. □

There are many other things that can be said about free probability, in the spirit of the free CLT and CCLT, and we will get back to this regularly, in what follows.

3d. Asymptotic freeness

We have already seen some applications of free probability to random matrices. As a first true application now of all this, still following Voiculescu [89], we have:

THEOREM 3.27. *Given a family of sequences of Wigner matrices,*

$$Z_N^i \in M_N(L^\infty(X)) \quad , \quad i \in I$$

with pairwise independent entries, each following the complex normal law G_t, with $t > 0$, up to the constraint $Z_N^i = (Z_N^i)^*$, the rescaled sequences of matrices

$$\frac{Z_N^i}{\sqrt{N}} \in M_N(L^\infty(X)) \quad , \quad i \in I$$

become with $N \to \infty$ semicircular, each following the Wigner law γ_t, and free.

PROOF. We can assume that we are dealing with 2 sequences of matrices, Z_N, Z_N'. In order to prove the asymptotic freeness, consider the following matrix:

$$Y_N = \frac{1}{\sqrt{2}}(Z_N + iZ_N')$$

This is a complex Gaussian matrix, so by using Theorem 3.26, we have:

$$\frac{Y_N}{\sqrt{N}} \sim \Gamma_t$$

We are therefore in the situation where $(Z_N + iZ_N')/\sqrt{N}$, which has asymptotically semicircular real and imaginary parts, converges to the distribution of a free combination of such variables. Thus Z_N, Z_N' become asymptotically free, as desired. □

Getting now to the complex case, we have a similar result here, as follows:

THEOREM 3.28. *Given a family of sequences of complex Gaussian matrices,*

$$Z_N^i \in M_N(L^\infty(X)) \quad , \quad i \in I$$

with pairwise independent entries, each following the law G_t, with $t > 0$, the matrices

$$\frac{Z_N^i}{\sqrt{N}} \in M_N(L^\infty(X)) \quad , \quad i \in I$$

become with $N \to \infty$ circular, each following the Voiculescu law Γ_t, and free.

PROOF. This follows indeed from Theorem 3.27, which applies to the real and imaginary parts of our complex Gaussian matrices, and gives the result. □

As before with theoretical free probability, there are many other things that can be said, as a continuation of the above results. We will back to this, later on.

3e. Exercises

Exercises.

CHAPTER 4

Wishart matrices

4a. Positive matrices

We discuss in this chapter the complex Wishart matrices, which are the positive analogues of the Gaussian and Wigner matrices. These matrices were introduced and studied by Marchenko-Pastur in [**65**], not long after Wigner's paper [**97**], and are of interest in connection with many questions. They are constructed as follows:

Definition 4.1. *A complex Wishart matrix is a random matrix of type*

$$W = YY^* \in M_N(L^\infty(X))$$

with Y being a complex Gaussian matrix, with entries following the law G_t.

Due to the formula $W = YY^*$, the Wishart matrices are positive, in the abstract positivity sense of chapter 1. Before getting into their study, let us first develop some more theory for the positive matrices and operators. We first have the following result:

Theorem 4.2. *For an operator $T \in B(H)$, the following are equivalent:*

(1) $< Tx, x > \geq 0$, *for any $x \in H$.*
(2) T *is normal, and* $\sigma(T) \subset [0, \infty)$.
(3) $T = S^2$, *for some $S \in B(H)$ satisfying $S = S^*$.*
(4) $T = R^*R$, *for some $R \in B(H)$.*

If these conditions are satisfied, we call T positive, and write $T \geq 0$.

Proof. We have already seen some implications in chapter 1, but the best is to forget the few partial results that we know, and prove everything, as follows:

(1) $\implies$ (2) Assuming $< Tx, x > \geq 0$, with $S = T - T^*$ we have:

$$\begin{aligned} < Sx, x > &= < Tx, x > - < T^*x, x > \\ &= < Tx, x > - < x, Tx > \\ &= < Tx, x > - \overline{< Tx, x >} \\ &= 0 \end{aligned}$$

The next step is to use a polarization trick, as follows:

$$\begin{aligned} < Sx, y > &= < S(x+y), x+y > - < Sx, x > - < Sy, y > - < Sy, x > \\ &= - < Sy, x > \\ &= < y, Sx > \\ &= \overline{< Sx, y >} \end{aligned}$$

Thus we must have $< Sx, y > \in \mathbb{R}$, and with $y \to iy$ we obtain $< Sx, y > \in i\mathbb{R}$ too, and so $< Sx, y >= 0$. Thus $S = 0$, which gives $T = T^*$. Now since T is self-adjoint, it is normal as claimed. Moreover, by self-adjointness, we have:

$$\sigma(T) \subset \mathbb{R}$$

In order to prove now that we have indeed $\sigma(T) \subset [0, \infty)$, as claimed, we must invert $T + \lambda$, for any $\lambda > 0$. For this purpose, observe that we have:

$$\begin{aligned} < (T+\lambda)x, x > &= < Tx, x > + < \lambda x, x > \\ &\geq < \lambda x, x > \\ &= \lambda ||x||^2 \end{aligned}$$

But this shows that $T + \lambda$ is injective. In order to prove now the surjectivity, and the boundedness of the inverse, observe first that we have:

$$\begin{aligned} Im(T+\lambda)^\perp &= \ker(T+\lambda)^* \\ &= \ker(T+\lambda) \\ &= \{0\} \end{aligned}$$

Thus $Im(T + \lambda)$ is dense. On the other hand, observe that we have:

$$\begin{aligned} ||(T+\lambda)x||^2 &= < Tx + \lambda x, Tx + \lambda x > \\ &= ||Tx||^2 + 2\lambda < Tx, x > + \lambda^2 ||x||^2 \\ &\geq \lambda^2 ||x||^2 \end{aligned}$$

Thus for any vector in the image $y \in Im(T + \lambda)$ we have:

$$||y|| \geq \lambda \Big|\Big|(T+\lambda)^{-1} y\Big|\Big|$$

As a conclusion to what we have so far, $T + \lambda$ is bijective and invertible as a bounded operator from H onto its image, with the following norm bound:

$$||(T+\lambda)^{-1}|| \leq \lambda^{-1}$$

But this shows that $Im(T + \lambda)$ is complete, hence closed, and since we already knew that $Im(T + \lambda)$ is dense, our operator $T + \lambda$ is surjective, and we are done.

(2) $\implies$ (3) Since T is normal, and with spectrum contained in $[0, \infty)$, we can use the continuous functional calculus formula for the normal operators from chapter 5, with the function $f(x) = \sqrt{x}$, as to construct a square root $S = \sqrt{T}$.

(3) $\implies$ (4) This is trivial, because we can set $R = S$.

(4) $\implies$ (1) This is clear, because we have the following computation:

$$< R^*Rx, x >=< Rx, Rx >= ||Rx||^2$$

Thus, we have the equivalences in the statement. □

In analogy with what happens in finite dimensions, where among the positive matrices $A \geq 0$ we have the strictly positive ones, $A > 0$, given by the fact that the eigenvalues are strictly positive, we have as well a "strict" version of the above result, as follows:

THEOREM 4.3. *For an operator $T \in B(H)$, the following are equivalent:*

1. *T is positive and invertible.*
2. *T is normal, and $\sigma(T) \subset (0, \infty)$.*
3. *$T = S^2$, for some $S \in B(H)$ invertible, satisfying $S = S^*$.*
4. *$T = R^*R$, for some $R \in B(H)$ invertible.*

If these conditions are satisfied, we call T strictly positive, and write $T > 0$.

PROOF. Our claim is that the above conditions (1-4) are precisely the conditions (1-4) in Theorem 4.2, with the assumption "T is invertible" added. Indeed:

(1) This is clear by definition.

(2) In the context of Theorem 4.2 (2), namely when T is normal, and $\sigma(T) \subset [0, \infty)$, the invertibility of T, which means $0 \notin \sigma(T)$, gives $\sigma(T) \subset (0, \infty)$, as desired.

(3) In the context of Theorem 4.2 (3), namely when $T = S^2$, with $S = S^*$, by using the basic properties of the functional calculus for normal operators, the invertibility of T is equivalent to the invertibility of its square root $S = \sqrt{T}$, as desired.

(4) In the context of Theorem 4.2 (4), namely when $T = RR^*$, the invertibility of T is equivalent to the invertibility of R. This can be either checked directly, or deduced via the equivalence (3) $\iff$ (4) from Theorem 4.2, by using the above argument (3). □

As a subtlety now, we have the following complement to the above result:

PROPOSITION 4.4. *For a strictly positive operator, $T > 0$, we have*

$$< Tx, x >> 0 \quad , \quad \forall x \neq 0$$

but the converse of this fact is not true, unless we are in finite dimensions.

PROOF. We have several things to be proved, the idea being as follows:

(1) Regarding the main assertion, the inequality can be deduced as follows, by using the fact that the operator $S=\sqrt{T}$ is invertible, and in particular injective:

$$\begin{aligned} <Tx,x> &= <S^2x,x> \\ &= <Sx,S^*x> \\ &= <Sx,Sx> \\ &= ||Sx||^2 \\ &> 0 \end{aligned}$$

(2) In finite dimensions, assuming $<Tx,x>>0$ for any $x\neq 0$, we know from Theorem 4.2 that we have $T\geq 0$. Thus we have $\sigma(T)\subset[0,\infty)$, and assuming by contradiction $0\in\sigma(T)$, we obtain that T has $\lambda=0$ as eigenvalue, and the corresponding eigenvector $x\neq 0$ has the property $<Tx,x>=0$, contradiction. Thus $T>0$, as claimed.

(3) Finally, regarding the counterexample for the converse, in infinite dimensions, consider the following operator on $l^2(\mathbb{N})$:

$$T=\begin{pmatrix}1\\&\frac{1}{2}\\&&\frac{1}{3}\\&&&\ddots\end{pmatrix}$$

Then T is well-defined and bounded, and we have $<Tx,x>>0$, for any vector $x\neq 0$. However, T is not invertible, and so the converse does not hold, as stated. □

As a last topic regarding positivity, let us discuss polar decomposition. We first have:

PROPOSITION 4.5. *Given an operator $T\in B(H)$, we can construct a positive operator $|T|\in B(H)$ as follows, by using the fact that T^*T is positive:*

$$|T|=\sqrt{T^*T}$$

The square of this operator is then $|T|^2=T^*T$. In the case $H=\mathbb{C}$, we obtain in this way the usual absolute value of the complex numbers:

$$|z|=\sqrt{z\bar{z}}$$

More generally, in the case where $H=\mathbb{C}^N$ is finite dimensional, we obtain in this way the usual moduli of the complex matrices $A\in M_N(\mathbb{C})$.

PROOF. We have several things to be proved, the idea being as follows:

(1) The first assertion follows from Theorem 4.2. Indeed, according to (4) there the operator T^*T is indeed positive, and then according to (2) there we can extract the square root of this latter positive operator, by applying to it the function $\sqrt{z}$.

(2) By functional calculus we have then $|T|^2=T^*T$, as desired.

(3) In the case $H=\mathbb{C}$, we obtain indeed the absolute value of complex numbers.

(4) In the case where the space H is finite dimensional, $H=\mathbb{C}^N$, we obtain indeed the usual moduli of the complex matrices $A\in M_N(\mathbb{C})$. □

Regarding now the polar decomposition formula, let us start with a weak version of this statement, regarding the invertible operators, as follows:

Theorem 4.6. *We have the polar decomposition formula*

$$T=U\sqrt{T^*T}$$

with U being a unitary, for any $T\in B(H)$ invertible.

Proof. According to our definition of $|T|=\sqrt{T^*T}$, we have:

$$\begin{aligned} < |T|x,|T|y > &= < x,|T|^2y > \\ &= < x,T^*Ty > \\ &= < Tx,Ty > \end{aligned}$$

Thus we can define a unitary operator $U\in B(H)$ as follows:

$$U(|T|x)=Tx$$

But this formula shows that we have $T=U|T|$, as desired. □

Observe that we have uniqueness in the above result, in what regards the choice of the unitary $U\in B(H)$, due to the fact that we can write this unitary as follows:

$$U=T(\sqrt{T^*T})^{-1}$$

More generally now, we have the following result:

Theorem 4.7. *We have the polar decomposition formula*

$$T=U\sqrt{T^*T}$$

with U being a partial isometry, for any $T\in B(H)$.

Proof. As before, in the proof of Theorem 4.6, we have the following equality, valid for any two vectors $x,y\in H$:

$$< |T|x,|T|y >=< Tx,Ty >$$

We conclude that the following linear application is well-defined, and isometric:

$$U:Im|T|\to Im(T)\quad,\quad |T|x\to Tx$$

By continuity we can extend this map U into an isometry, as follows:

$$U:\overline{Im|T|}\to\overline{Im(T)}\quad,\quad |T|x\to Tx$$

Moreover, we can further extend U into a partial isometry $U:H\to H$, by setting $Ux=0$, for any $x\in\overline{Im|T|}^\perp$, and with this convention, the result follows. □

4b. Marchenko-Pastur

We discuss now the complex Wishart matrices, which are the positive analogues of the Gaussian and Wigner matrices. These matrices were introduced and studied by Marchenko-Pastur in [**65**], not long after Wigner's paper [**97**], and are of interest in connection with many questions. They are constructed as follows:

Definition 4.8. *A complex Wishart matrix is a random matrix of type*

$$W = YY^* \in M_N(L^\infty(X))$$

with Y being a complex Gaussian matrix, with entries following the law G_t.

There are in fact several possible definitions for the complex Wishart matrices, with some being more clever and useful that some other. To start with, we will use the above definition, which comes naturally out of what we know about the Gaussian and Wigner matrices. Once such matrices studied, we will talk about their versions, too.

Observe that, due to the formula $W = YY^*$, the complex Wishart matrices are obviously positive, in the sense of the positivity notion discussed above:

$$W \geq 0$$

Due to this key positivity property, and to the otherwise "randomness" of W, such matrices are useful in many down-to-earth contexts. More on this later.

As usual with the random matrices, we will be interested in computing the asymptotic laws of our Wishart matrices W, suitably rescaled, in the $N \to \infty$ limit. Quite surprisingly, the computation here leads to the Catalan numbers, but not exactly in the same way as for the Wigner matrices, the precise result being as follows:

Theorem 4.9. *Given a sequence of complex Wishart matrices*

$$W_N = Y_N Y_N^* \in M_N(L^\infty(X))$$

with Y_N being $N \times N$ complex Gaussian of parameter $t > 0$, we have

$$M_k\left(\frac{W_N}{N}\right) \simeq t^k C_k$$

for any exponent $k \in \mathbb{N}$, in the $N \to \infty$ limit.

Proof. There are several possible proofs for this result, as follows:

(1) A first method is by using the result that we have from chapter 2, for the Gaussian matrices Y_N. Indeed, we know from there that we have the following formula, valid for any colored integer $K = \circ\bullet\bullet\circ\ldots$, in the $N \to \infty$ limit:

$$M_K\left(\frac{Y_N}{\sqrt{N}}\right) \simeq t^{|K|/2} |\mathcal{NC}_2(K)|$$

With $K=\circ\bullet\circ\bullet\ldots$, alternating word of length $2k$, with $k\in\mathbb N$, this gives:

$$M_k\left(\frac{Y_NY_N^*}{N}\right)\simeq t^k|\mathcal{NC}_2(K)|$$

Thus, in terms of the Wishart matrix $W_N=Y_NY_N^*$ we have, for any $k\in\mathbb N$:

$$M_k\left(\frac{W_N}{N}\right)\simeq t^k|\mathcal{NC}_2(K)|$$

The point now is that, by doing some combinatorics, we have:

$$|\mathcal{NC}_2(K)|=|NC_2(2k)|=C_k$$

Thus, we are led to the formula in the statement.

(2) A second method, that we will explain now as well, is by proving the result directly, starting from definitions. The matrix entries of our matrix $W=W_N$ are given by:

$$W_{ij}=\sum_{r=1}^NY_{ir}\bar{Y}_{jr}$$

Thus, the normalized traces of powers of W are given by the following formula:

$$\begin{aligned}
tr(W^k)&=\frac{1}{N}\sum_{i_1=1}^N\ldots\sum_{i_k=1}^NW_{i_1i_2}W_{i_2i_3}\ldots W_{i_ki_1}\\
&=\frac{1}{N}\sum_{i_1=1}^N\ldots\sum_{i_k=1}^N\sum_{r_1=1}^N\ldots\sum_{r_k=1}^NY_{i_1r_1}\bar{Y}_{i_2r_1}Y_{i_2r_2}\bar{Y}_{i_3r_2}\ldots Y_{i_kr_k}\bar{Y}_{i_1r_k}
\end{aligned}$$

By rescaling now W by a $1/N$ factor, as in the statement, we obtain:

$$tr\left(\left(\frac{W}{N}\right)^k\right)=\frac{1}{N^{k+1}}\sum_{i_1=1}^N\ldots\sum_{i_k=1}^N\sum_{r_1=1}^N\ldots\sum_{r_k=1}^NY_{i_1r_1}\bar{Y}_{i_2r_1}Y_{i_2r_2}\bar{Y}_{i_3r_2}\ldots Y_{i_kr_k}\bar{Y}_{i_1r_k}$$

By using now the Wick rule, we obtain the following formula for the moments, with $K=\circ\bullet\circ\bullet\ldots$, alternating word of length $2k$, and with $I=(i_1r_1,i_2r_1,\ldots,i_kr_k,i_1r_k)$:

$$\begin{aligned}
M_k\left(\frac{W}{N}\right)&=\frac{t^k}{N^{k+1}}\sum_{i_1=1}^N\ldots\sum_{i_k=1}^N\sum_{r_1=1}^N\ldots\sum_{r_k=1}^N\#\left\{\pi\in\mathcal P_2(K)\Big|\pi\leq\ker I\right\}\\
&=\frac{t^k}{N^{k+1}}\sum_{\pi\in\mathcal P_2(K)}\#\left\{i,r\in\{1,\ldots,N\}^k\Big|\pi\leq\ker I\right\}
\end{aligned}$$

In order to compute this quantity, we use the standard bijection $\mathcal{P}_2(K) \simeq S_k$. By identifying the pairings $\pi \in \mathcal{P}_2(K)$ with their counterparts $\pi \in S_k$, we obtain:

$$M_k\left(\frac{W}{N}\right) = \frac{t^k}{N^{k+1}}\sum_{\pi\in S_k}\#\left\{i,r\in\{1,\ldots,N\}^k\Big|i_s=i_{\pi(s)+1},r_s=r_{\pi(s)},\forall s\right\}$$

Now let $\gamma \in S_k$ be the full cycle, $\gamma = (1\,2\,\ldots\,k)$. The general factor in the product computed above is then 1 precisely when following two conditions are satisfied:

$$\gamma\pi \leq \ker i \quad , \quad \pi \leq \ker r$$

Counting the number of free parameters in our moment formula, we obtain:

$$M_k\left(\frac{W}{N}\right) = t^k\sum_{\pi\in S_k}N^{|\pi|+|\gamma\pi|-k-1}$$

The point now is that the last exponent is well-known to be ≤ 0, with equality precisely when the permutation $\pi \in S_k$ is geodesic, which in practice means that π must come from a noncrossing partition. Thus we obtain, in the $N \to \infty$ limit:

$$M_k\left(\frac{W}{N}\right) \simeq t^k C_k$$

Thus, we are led to the conclusion in the statement. □

As a consequence of the above result, we have a new look on the Catalan numbers, which is more adapted to our present Wishart matrix considerations, as follows:

Proposition 4.10. *The Catalan numbers $C_k = |NC_2(2k)|$ appear as well as*

$$C_k = |NC(k)|$$

where $NC(k)$ is the set of all noncrossing partitions of $\{1,\ldots,k\}$.

Proof. This follows indeed from the proof of Theorem 4.9. □

The direct explanation for the above formula, relating noncrossing partitions and pairings, comes from the following result, which is very useful, and good to know:

Proposition 4.11. *We have a bijection between noncrossing partitions and pairings*

$$NC(k) \simeq NC_2(2k)$$

which is constructed as follows:

1. *The application $NC(k) \to NC_2(2k)$ is the "fattening" one, obtained by doubling all the legs, and doubling all the strings as well.*
2. *Its inverse $NC_2(2k) \to NC(k)$ is the "shrinking" application, obtained by collapsing pairs of consecutive neighbors.*

Proof. The fact that the two operations in the statement are indeed inverse to each other is clear, by computing the corresponding two compositions, with the remark that the construction of the fattening operation requires the partitions to be noncrossing. □

Getting back now to Wishart matrices, at $t = 1$ we are led to the question of finding the law having the Catalan numbers as moments. We have here:

Proposition 4.12. *The real measure having the Catalan numbers as moments is*

$$\pi_1 = \frac{1}{2\pi}\sqrt{4x^{-1} - 1}\,dx$$

called Marchenko-Pastur law of parameter 1*.*

Proof. We can prove this in two different ways, as follows:

(1) By using the Stieltjes inversion formula. In order to apply this formula, we need a simple formula for the Cauchy transform. For this purpose, our starting point will be the formula from chapter 3 for the generating series of the Catalan numbers, namely:

$$\sum_{k=0}^\infty C_k z^k = \frac{1 - \sqrt{1 - 4z}}{2z}$$

By using this formula with $z = \xi^{-1}$, we obtain the following formula:

$$\begin{aligned} G(\xi) &= \xi^{-1}\sum_{k=0}^\infty C_k \xi^{-k} \\ &= \xi^{-1}\cdot\frac{1 - \sqrt{1 - 4\xi^{-1}}}{2\xi^{-1}} \\ &= \frac{1}{2}\left(1 - \sqrt{1 - 4\xi^{-1}}\right) \\ &= \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\xi^{-1}} \end{aligned}$$

With this formula in hand, let us apply now the Stieltjes inversion formula, from chapter 3. The first term, namely 1/2, which is trivial, will not contribute to the density. As for the second term, which is something non-trivial, this will contribute to the density, the rule here being that the square root $\sqrt{1 - 4\xi^{-1}}$ will be replaced by the "dual" square root $\sqrt{4x^{-1} - 1}\,dx$, and that we have to multiply everything by $-1/\pi$. Thus, by Stieltjes inversion we obtain the density in the statement, namely:

$$\begin{aligned} d\mu(x) &= -\frac{1}{\pi}\cdot -\frac{1}{2}\sqrt{4x^{-1} - 1}\,dx \\ &= \frac{1}{2\pi}\sqrt{4x^{-1} - 1}\,dx \end{aligned}$$

(2) Alternatively, if the above was too complicated, we can simply cheat. Indeed, the moments of the law π_1 in the statement can be computed with $x = 4\cos^2 t$, as follows:

$$\begin{aligned} M_k &= \frac{1}{2\pi}\int_0^4 \sqrt{4x^{-1}-1}\,x^k dx \\ &= \frac{1}{2\pi}\int_0^{\pi/2} \frac{\sin t}{\cos t}\cdot(4\cos^2 t)^k \cdot 2\cos t\sin t\,dt \\ &= \frac{4^{k+1}}{\pi}\int_0^{\pi/2}\cos^{2k} t\sin^2 t\,dt \\ &= \frac{4^{k+1}}{\pi}\cdot\frac{\pi}{2}\cdot\frac{(2k)!!2!!}{(2k+3)!!} \\ &= 2\cdot 4^k\cdot\frac{(2k)!/2^k k!}{2^{k+1}(k+1)!} \\ &= C_k \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Now back to the Wishart matrices, we are led to the following result:

THEOREM 4.13. *Given a sequence of complex Wishart matrices*

$$W_N = Y_N Y_N^* \in M_N(L^\infty(X))$$

with Y_N being $N\times N$ complex Gaussian of parameter 1, we have

$$\frac{W_N}{N} \sim \frac{1}{2\pi}\sqrt{4x^{-1}-1}\,dx$$

with $N\to\infty$, with the limiting measure being the Marchenko-Pastur law π_1.

PROOF. This follows indeed from Theorem 4.9 and Proposition 4.12. □

We have as well a parametric version of the above result, as follows:

THEOREM 4.14. *Given a sequence of complex Wishart matrices*

$$W_N = Y_N Y_N^* \in M_N(L^\infty(X))$$

with Y_N being $N\times N$ complex Gaussian of parameter $t>0$, we have

$$\frac{W_N}{tN} \sim \frac{1}{2\pi}\sqrt{4x^{-1}-1}\,dx$$

with $N\to\infty$, with the limiting measure being the Marchenko-Pastur law π_1.

PROOF. This follows again from Theorem 4.9 and Proposition 4.12. To be more precise, recall the main formula from Theorem 4.9, for the matrices as above, namely:

$$M_k\left(\frac{W_N}{N}\right) \simeq t^k C_k$$

By dividing by t^k, this formula can be written as follows:

$$M_k\left(\frac{W_N}{tN}\right)\simeq C_k$$

Now by using Proposition 4.12, we are led to the conclusion in the statement. □

Summarizing, we have deduced the Marchenko-Pastur theorem from the result for Gaussian matrices, via some moment combinatorics.

It is possible as well to be a bit more direct here, by passing through the Wigner theorem from chapter 3, and then recovering the Marchenko-Pastur law directly from the Wigner semicircle law, by performing a kind of square operation. But this is more or less the same thing as we did above. We will be back to this, in a more general setting.

4c. Parametric version

We discuss now a generalization of the above results, motivated by a whole array of concrete questions, and bringing into the picture a "true" parameter $t>0$, which is different from the parameter $t>0$ used above, which is something quite trivial.

For this purpose, let us go back to the definition of the Wishart matrices. There were as follows, with Y being a $N\times N$ matrix with i.i.d. entries, each following the law G_t:

$$W=YY^*$$

The point now is that, more generally, we can use in this $W=YY^*$ construction a $N\times M$ matrix Y with i.i.d. entries, each following the law G_t, with $M\in\mathbb N$ being arbitrary. Thus, we have a new parameter, and by ditching the old parameter $t>0$, which was something not very interesting, we are led to the following definition, which is the "true" definition of the Wishart matrices, from [65] and the subsequent literature:

Definition 4.15. *A complex Wishart matrix is a $N\times N$ matrix of the form*

$$W=YY^*$$

where Y is a $N\times M$ matrix with i.i.d. entries, each following the law G_1.

As before with our previous Wishart matrices, that the new ones generalize, up to setting $t=1$, we have $W\geq 0$, by definition. Due to this property, and to the otherwise "randomness" of W, these matrices are useful in many contexts. More on this later.

In order to see what is going on, combinatorially, let us compute moments. The result here is substantially more interesting than that for the previous Wishart matrices, with the new revelant numeric parameter being now the number $t=M/N$, as follows:

THEOREM 4.16. *Given a sequence of complex Wishart matrices*

$$W_N = Y_N Y_N^* \in M_N(L^\infty(X))$$

with Y_N being $N \times M$ complex Gaussian of parameter 1, we have

$$M_k\left(\frac{W_N}{N}\right) \simeq \sum_{\pi \in NC(k)} t^{|\pi|}$$

for any exponent $k \in \mathbb{N}$, in the $M = tN \to \infty$ limit.

PROOF. This is something which is very standard, as follows:

(1) Before starting, let us clarify the relation with our previous Wishart matrix results. In the case $M = N$ we have $t = 1$, and the formula in the statement reads:

$$M_k\left(\frac{W_N}{N}\right) \simeq |NC(k)|$$

Thus, what we have here is the previous Wishart matrix formula, in full generality, at the value $t = 1$ of our old parameter $t > 0$.

(2) Observe also that by rescaling, we can obtain if we want from this the previous Wishart matrix formula, in full generality, at any value $t > 0$ of our old parameter. Thus, things fine, we are indeed generalizing what we did before.

(3) In order to prove now the formula in the statement, we proceed as usual, by using the Wick formula. The matrix entries of our Wishart matrix $W = W_N$ are given by:

$$W_{ij} = \sum_{r=1}^{M} Y_{ir} \bar{Y}_{jr}$$

Thus, the normalized traces of powers of W are given by the following formula:

$$\begin{aligned} tr(W^k) &= \frac{1}{N} \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} W_{i_1 i_2} W_{i_2 i_3} \ldots W_{i_k i_1} \\ &= \frac{1}{N} \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} \sum_{r_1=1}^{M} \cdots \sum_{r_k=1}^{M} Y_{i_1 r_1} \bar{Y}_{i_2 r_1} Y_{i_2 r_2} \bar{Y}_{i_3 r_2} \ldots Y_{i_k r_k} \bar{Y}_{i_1 r_k} \end{aligned}$$

By rescaling now W by a $1/N$ factor, as in the statement, we obtain:

$$tr\left(\left(\frac{W}{N}\right)^k\right) = \frac{1}{N^{k+1}} \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} \sum_{r_1=1}^{M} \cdots \sum_{r_k=1}^{M} Y_{i_1 r_1} \bar{Y}_{i_2 r_1} Y_{i_2 r_2} \bar{Y}_{i_3 r_2} \ldots Y_{i_k r_k} \bar{Y}_{i_1 r_k}$$

(4) By using now the Wick rule, we obtain the following formula for the moments, with $K = \circ \bullet \circ \bullet \ldots$, alternating word of lenght $2k$, and $I = (i_1r_1, i_2r_1, \ldots, i_kr_k, i_1r_k)$:

$$\begin{aligned} M_k\left(\frac{W}{N}\right) &= \frac{1}{N^{k+1}}\sum_{i_1=1}^N\ldots\sum_{i_k=1}^N\sum_{r_1=1}^M\ldots\sum_{r_k=1}^M \#\left\{\pi \in \mathcal{P}_2(K)\Big|\pi \le \ker I\right\} \\ &= \frac{1}{N^{k+1}}\sum_{\pi\in\mathcal{P}_2(K)} \#\left\{i \in \{1,\ldots,N\}^k, r \in \{1,\ldots,M\}^k\Big|\pi \le \ker I\right\} \end{aligned}$$

(5) In order to compute this quantity, we use the standard bijection $\mathcal{P}_2(K) \simeq S_k$. By identifying the pairings $\pi \in \mathcal{P}_2(K)$ with their counterparts $\pi \in S_k$, we obtain:

$$M_k\left(\frac{W}{N}\right) = \frac{1}{N^{k+1}}\sum_{\pi\in S_k} \#\left\{i \in \{1,\ldots,N\}^k, r \in \{1,\ldots,M\}^k\Big|i_s = i_{\pi(s)+1}, r_s = r_{\pi(s)}\right\}$$

Now let $\gamma \in S_k$ be the full cycle, which is by definition the following permutation:

$$\gamma = (1\,2\,\ldots\,k)$$

The general factor in the product computed above is then 1 precisely when following two conditions are simultaneously satisfied:

$$\gamma\pi \le \ker i \quad , \quad \pi \le \ker r$$

Counting the number of free parameters in our expectation formula, we obtain:

$$M_k\left(\frac{W}{N}\right) = \frac{1}{N^{k+1}}\sum_{\pi\in S_k} N^{|\gamma\pi|}M^{|\pi|} = \sum_{\pi\in S_k} N^{|\gamma\pi|-k-1}M^{|\pi|}$$

(6) Now by using the same arguments as in the case $M = N$, from the proof of Theorem 4.9, we conclude that in the $M = tN \to \infty$ limit the permutations $\pi \in S_k$ which matter are those coming from noncrossing partitions, and so that we have:

$$M_k\left(\frac{W}{N}\right) \simeq \sum_{\pi\in NC(k)} N^{-|\pi|}M^{|\pi|} = \sum_{\pi\in NC(k)} t^{|\pi|}$$

We are therefore led to the conclusion in the statement. □

In order to recapture now the density out of the moments, we can of course use the Stieltjes inversion formula, but the computations here are a bit opaque. So, inspired from what happens at $t = 1$, let us cheat a bit, and formulate a nice definition, as follows:

Definition 4.17. *The Marchenko-Pastur law π_t of parameter $t > 0$ is given by:*

$$a \sim \gamma_t \implies a^2 \sim \pi_t$$

That is, π_t the law of the square of a variable following the law γ_t.

This is certainly very nice, and we know that at $t = 1$ we obtain indeed the Marchenko-Pastur law π_1, as constructed above. In general, we have the following result:

PROPOSITION 4.18. *The Marchenko-Pastur law of parameter $t>0$ is*

$$\pi_t=\max(1-t,0)\delta_0+\frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx$$

the support being $[0,4t^2]$, and the moments of this measure are

$$M_k=\sum_{\pi\in NC(k)}t^{|\pi|}$$

exactly as for the asymptotic moments of the complex Wishart matrices.

PROOF. This follows as usual, by doing some computations, either combinatorics, or calculus. To be more precise, we have three formulae for π_t to be connected, namely the one in Definition 4.17, and the two ones from the present statement, and the connections between them can be established exactly as we did before, at $t=1$. □

Summarizing, we have now a definition for the Marchenko-Pastur law π_t, which is quite elegant, via Definition 4.17, but which still requires some computations, performed in the proof of Proposition 4.18. We will see in a moment that an even more elegant definition for π_t, out of its particular case π_1 which was well understood, simply obtained by considering the corresponding 1-parameter free convolution semigroup. We will also see that π_t appears as the "free version" of the Poisson law p_t, and that this can be even taken as a definition for π_t, if we really want to. More on this later.

Now back to the complex Wishart matrices that we are interested in, in this chapter, we can now formulate a final result regarding them, as follows:

THEOREM 4.19. *Given a sequence of complex Wishart matrices*

$$W_N=Y_NY_N^*\in M_N(L^\infty(X))$$

with Y_N being $N\times M$ complex Gaussian of parameter 1, we have

$$\frac{W_N}{N}\sim\max(1-t,0)\delta_0+\frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx$$

with $M=tN\to\infty$, with the limiting measure being the Marchenko-Pastur law π_t.

PROOF. This follows indeed from Theorem 4.16 and Proposition 4.18. □

As it was the case with the Gaussian and Wigner matrices, there are many other things that can be said about the complex Wishart matrices, as variations of the above. We refer here to the standard random matrix literature [**2**], [**66**], [**69**], [**91**]. We will be back to this right below, in the remainder of this chapter, with some wizarding computations from [**5**], and then more systematically later, when doing more specialized free probability.

4d. Shifted semicircles

Our goal now, in the remainder of this chapter, will be that of explaining a surprising result, due to Aubrun [5], stating that when suitably block-transposing the entries of a complex Wishart matrix, we obtain as asymptotic distribution a shifted version of Wigner's semicircle law. Following [5], [12], let us start with the following definition:

Definition 4.20. *The partial transpose of a complex Wishart matrix W of parameters (dn, dm) is the matrix*

$$\tilde{W} = (id \otimes t)W$$

where id is the identity of $M_d(\mathbb{C})$, and t is the transposition of $M_n(\mathbb{C})$.

In more familiar terms of bases and indices, the standard decomposition $\mathbb{C}^{dn} = \mathbb{C}^d \otimes \mathbb{C}^n$ induces an algebra decomposition $M_{dn}(\mathbb{C}) = M_d(\mathbb{C}) \otimes M_n(\mathbb{C})$, and with this convention made, the partial transpose matrix $\tilde{W}$ constructed above has entries as follows:

$$\tilde{W}_{ia,jb} = W_{ib,ja}$$

Our goal in what follows will be that of computing the law of $\tilde{W}$, first when d, n, m are fixed, and then in the $d \to \infty$ regime. For this purpose, we will need a number of standard facts regarding the noncrossing partitions. Let us start with:

Proposition 4.21. *For a permutation $\sigma \in S_p$, we have the formula*

$$|\sigma| + \#\sigma = p$$

where $|\sigma|$ is the number of cycles of σ, and $\#\sigma$ is the minimal $k \in \mathbb{N}$ such that σ is a product of k transpositions. Also, the following formula defines a distance on S_p,

$$(\sigma, \pi) \to \#(\sigma^{-1}\pi)$$

and the set of permutations $\sigma \in S_p$ which saturate the triangular inequality

$$\#\sigma + \#(\sigma^{-1}\gamma) = \#\gamma = p - 1$$

where $\gamma \in S_p$ is a full cycle, is in bijection with the set $NC(p)$.

Proof. All this is standard, as explained before. □

We also recall a well-known bijection between $NC(p)$ and the set $NC_2(2p)$ of noncrossing pairings of $2p$ elements. To a noncrossing partition $\pi \in NC(p)$ we associate an element $\widetilde{\pi} \in NC_2(2p)$ as follows. For each block $\{i_1, i_2, \ldots, i_k\}$ of π, we add the pairings $\{2i_1 - 1, 2i_k\}$, $\{2i_1, 2i_2 - 1\}$, $\{2i_2, 2i_3 - 1\}$, $\ldots$, $\{2i_{k-1}, 2i_k - 1\}$ to $\widetilde{\pi}$. The inverse operation is given by collapsing the elements $2i - 1, 2i \in \{1, \ldots, 2p\}$ to a single element $i \in \{1, \ldots, p\}$. We have the following formula, where $\vee$ is the join operation on $NC_2(2p)$, and $\rho_{12} = (12)(34)\ldots(2p-1, 2p)$ is the fattened identity permutation:

$$|\pi| = |\widetilde{\pi} \vee \rho_{12}|$$

Similarly, we have the formula $|\pi\gamma| = |\widetilde{\pi} \vee \rho_{14}|$, where ρ_{14} is the pairing corresponding to the fattening of the inverse full cycle $\gamma^{-1}(i) = i - 1$, which pairs an element $2i$ with $2(i-1) - 1 = 2i - 3$, or, equivalently, an element $i \in \{1, \ldots, 2p\}$ with $i + (-1)^{i+1}3$.

We will need the following well-known result:

PROPOSITION 4.22. *The number $||\pi||$ of blocks having even size is given by*

$$1 + ||\pi|| = |\pi\gamma|$$

for every noncrossing partition $\pi \in NC(p)$.

PROOF. We use a recurrence over the number of blocks of π. If π has just one block, its associated geodesic permutation is γ and we have:

$$|\gamma^2| = \begin{cases} 1 & (p \text{ odd}) \\ 2 & (p \text{ even}) \end{cases}$$

For partitions π with more than one block, we can assume without loss of generality that $\pi = \hat{1}_k \sqcup \pi'$, where $\hat{1}_k$ is a contiguous block of size k. Recall that the number of blocks of the permutation $\pi\gamma$ is given by the following formula, where $\rho_{14} \in P_2(2p)$ is the pair partition which pairs an element i with $i + (-1)^{i+1}3$:

$$|\pi\gamma| = |\widetilde{\pi} \vee \rho_{14}|$$

If k is an even number, $k = 2r$, consider the following partition, which contains the block $(1\,4\,5\,8\,\ldots 4r-3\,4r)$, along with the blocks coming from elements of the form $4i+2, 4i+3$ from $\{1, \ldots, 4r\}$ and from π':

$$\sigma = \widetilde{\hat{1}_{2r} \sqcup \pi'} \vee \rho_{14}$$

We can count the blocks of the join of two partitions by drawing them one beneath the other and counting the number of connected components of the curve, without taking into account the possible crossings. We conclude that we have the following formula, where ρ_{14}' is ρ_{14} restricted to the set $\{2k+1, 2k+2 \ldots, 2p\}$:

$$|\widetilde{\pi} \vee \rho_{14}| = 1 + |\widetilde{\pi'} \vee \rho_{14}'|$$

If k is odd, $k = 2r + 1$, there is no extra block appearing, so we have:

$$|\widetilde{\pi} \vee \rho_{14}| = |\widetilde{\pi'} \vee \rho_{14}'|$$

Thus, we are led to the conclusion in the statement. □

We can now investigate the block-transposed Wishart matrices, and we have:

THEOREM 4.23. *For any $p \geq 1$ we have the formula*

$$\lim_{d\to\infty}(\mathbb{E}\circ tr)\big(m\tilde{W}\big)^p = \sum_{\pi\in NC(p)} m^{|\pi|}n^{||\pi||}$$

where $|.|$ and $||.||$ are the number of blocks, and the number of blocks of even size.

PROOF. The matrix elements of the partial transpose matrix are given by:

$$\tilde{W}_{ia,jb} = W_{ib,ja} = (dm)^{-1}\sum_{k=1}^{d}\sum_{c=1}^{m} G_{ib,kc}\bar{G}_{ja,kc}$$

This gives the following formula:

$$\begin{aligned} tr(\tilde{W}^p) &= (dn)^{-1}(dm)^{-p}\sum_{i_1,\ldots,i_p=1}^{d}\sum_{a_1,\ldots,a_p=1}^{n}\prod_{s=1}^{p} W^{\Gamma}_{i_sa_s,i_{s+1}a_{s+1}} \\ &= (dn)^{-1}(dm)^{-p}\sum_{i_1,\ldots,i_p=1}^{d}\sum_{a_1,\ldots,a_p=1}^{n}\prod_{s=1}^{p} W_{i_sa_{s+1},i_{s+1}a_s} \\ &= (dn)^{-1}(dm)^{-p}\sum_{i_1,\ldots,i_p=1}^{d}\sum_{a_1,\ldots,a_p=1}^{n}\prod_{s=1}^{p}\sum_{j_1,\ldots,j_p=1}^{d}\sum_{b_1,\ldots,b_p=1}^{m} G_{i_sa_{s+1},j_sb_s}\bar{G}_{i_{s+1}a_s,j_sb_s} \end{aligned}$$

After interchanging the product with the last two sums, the average of the general term can be computed by the Wick rule, namely:

$$\mathbb{E}\left(\prod_{s=1}^{p} G_{i_sa_{s+1},j_sb_s}\bar{G}_{i_{s+1}a_s,j_sb_s}\right) = \sum_{\pi\in S_p}\prod_{s=1}^{p}\delta_{i_s,i_{\pi(s)+1}}\delta_{a_{s+1},a_{\pi(s)}}\delta_{j_s,j_{\pi(s)}}\delta_{b_s,b_{\pi(s)}}$$

Let $\gamma\in S_p$ be the full cycle $\gamma=(1\,2\,\ldots\,p)^{-1}$. The general factor in the above product is 1 if and only if the following four conditions are simultaneously satisfied:

$$\gamma^{-1}\pi \leq \ker i \quad , \quad \pi\gamma \leq \ker a \quad , \quad \pi \leq \ker j \quad , \quad \pi \leq \ker b$$

Counting the number of free parameters in the above equation, we obtain:

$$\begin{aligned} (\mathbb{E}\circ tr)(\tilde{W}^p) &= (dn)^{-1}(dm)^{-p}\sum_{\pi\in S_p} d^{|\pi|+|\gamma^{-1}\pi|}m^{|\pi|}n^{|\pi\gamma|} \\ &= \sum_{\pi\in S_p} d^{|\pi|+|\gamma^{-1}\pi|-p-1}m^{|\pi|-p}n^{|\pi\gamma|-1} \end{aligned}$$

The exponent of d in the last expression on the right is:

$$\begin{aligned} N(\pi) &= |\pi|+|\gamma^{-1}\pi|-p-1 \\ &= p-1-(\#\pi+\#(\gamma^{-1}\pi)) \\ &= p-1-(\#\pi+\#(\pi^{-1}\gamma)) \end{aligned}$$

As explained in the beginning of this section, this quantity is known to be ≤ 0, with equality iff π is geodesic, hence associated to a noncrossing partition. Thus:

$$(\mathbb{E}\circ tr)(\tilde{W}^p)=(1+O(d^{-1}))m^{-p}n^{-1}\sum_{\pi\in NC(p)}m^{|\pi|}n^{|\pi\gamma|}$$

Together with $|\pi\gamma|=||\pi||+1$, this gives the result. □

We would like now to find an equation for the moment generating function of the asymptotic law of $m\tilde{W}$. This moment generating function is defined by:

$$F(z)=\lim_{d\to\infty}(\mathbb{E}\circ tr)\left(\frac{1}{1-zm\tilde{W}}\right)$$

We have the following result, regarding this moment generating function:

THEOREM 4.24. *The moment generating function of $m\tilde{W}$ satisfies the equation*

$$(F-1)(1-z^2F^2)=mzF(1+nzF)$$

in the $d\to\infty$ limit.

PROOF. We use the formula in Theorem 4.23. If we denote by $N(p,b,e)$ the number of partitions in $NC(p)$ having b blocks and e even blocks, we have:

$$\begin{aligned}F&=1+\sum_{p=1}^\infty\sum_{\pi\in NC(p)}z^pm^{|\pi|}n^{||\pi||}\\&=1+\sum_{p=1}^\infty\sum_{b=0}^\infty\sum_{e=0}^\infty z^pm^bn^eN(p,b,e)\end{aligned}$$

Let us try to find a recurrence formula for the numbers $N(p,b,e)$. If we look at the block containing 1, this block must have $r\geq 0$ other legs, and we get:

$$\begin{aligned}N(p,b,e)&=\sum_{r\in 2\mathbb{N}}\sum_{p=\Sigma p_i+r+1}\sum_{b=\Sigma b_i+1}\sum_{e=\Sigma e_i}N(p_1,b_1,e_1)\ldots N(p_{r+1},b_{r+1},e_{r+1})\\&+\sum_{r\in 2\mathbb{N}+1}\sum_{p=\Sigma p_i+r+1}\sum_{b=\Sigma b_i+1}\sum_{e=\Sigma e_i+1}N(p_1,b_1,e_1)\ldots N(p_{r+1},b_{r+1},e_{r+1})\end{aligned}$$

Here $p_1,\ldots,p_{r+1}$ are the number of points between the legs of the block containing 1, so that we have $p=(p_1+\ldots+p_{r+1})+r+1$, and the whole sum is split over two cases, r even or odd, because the parity of r affects the number of even blocks of our partition. Now by multiplying everything by a $z^pm^bn^e$ factor, and by carefully distributing the various powers of z,m,b on the right, we obtain the following formula:

$$\begin{aligned}z^pm^bn^eN(p,b,e) &= m\sum_{r\in 2\mathbb{N}}z^{r+1}\sum_{p=\Sigma p_i+r+1}\sum_{b=\Sigma b_i+1}\sum_{e=\Sigma e_i}\prod_{i=1}^{r+1}z^{p_i}m^{b_i}n^{e_i}N(p_i,b_i,e_i)\\ &+ mn\sum_{r\in 2\mathbb{N}+1}z^{r+1}\sum_{p=\Sigma p_i+r+1}\sum_{b=\Sigma b_i+1}\sum_{e=\Sigma e_i+1}\prod_{i=1}^{r+1}z^{p_i}m^{b_i}n^{e_i}N(p_i,b_i,e_i)\end{aligned}$$

Let us sum now all these equalities, over all $p\geq 1$ and over all $b,e\geq 0$. According to the definition of F, at left we obtain $F-1$. As for the two sums appearing on the right, that is, at right of the two z^{r+1} factors, when summing them over all $p\geq 1$ and over all $b,e\geq 0$, we obtain in both cases F^{r+1}. So, we have the following formula:

$$\begin{aligned}F-1 &= m\sum_{r\in 2\mathbb{N}}(zF)^{r+1}+mn\sum_{r\in 2\mathbb{N}+1}(zF)^{r+1}\\ &= m\,\frac{zF}{1-z^2F^2}+mn\,\frac{z^2F^2}{1-z^2F^2}\\ &= mzF\,\frac{1+nzF}{1-z^2F^2}\end{aligned}$$

But this gives the formula in the statement, and we are done. □

Our goal now will be that of further processing the formula in Theorem 4.24, as to reach to a formula for the density of the corresponding law. This is something quite tricky, and as a first result here, we can reformulate Theorem 4.24 as follows:

THEOREM 4.25. *The Cauchy transform of $m\tilde{W}$ satisfies the equation*

$$(\xi G-1)(1-G^2)=mG(1+nG)$$

in the $d\to\infty$ limit. Moreover, this equation simply reads

$$R=\frac{m}{2}\left(\frac{n+1}{1-z}-\frac{n-1}{1+z}\right)$$

with the substitutions $G\to z$ and $\xi\to R+z^{-1}$.

PROOF. We have two assertions to be proved, the first one being standard, and the second one being something quite magic, the idea being as follows:

(1) Consider the equation of F, found in Theorem 4.24, namely:

$$(F-1)(1-z^2F^2)=mzF(1+nzF)$$

With $z\to\xi^{-1}$ and $F\to\xi G$, so that $zF\to G$, we obtain, as desired:

$$(\xi G-1)(1-G^2)=mG(1+nG)$$

(2) Thus, we have our equation for the Cauchy transform, and with this in hand, we can try to go ahead, and use somehow the Stieltjes inversion formula, in order to reach

to a formula for the density. This is certainly possible, but our claim is that we can do better, by performing first some clever manipulations on the Cauchy transform.

(3) To be more precise, let us look at the equation of the Cauchy transform that we have. With the substitutions $\xi \to K$ and $G \to z$, this equation becomes:

$$(zK-1)(1-z^2) = mz(1+nz)$$

The point now is that with $K \to R + z^{-1}$ this latter equation becomes:

$$zR(1-z^2) = mz(1+nz)$$

But the solution of this latter equation is trivial to compute, given by:

$$R = m\,\frac{1+nz}{1-z^2} = \frac{m}{2}\left(\frac{n+1}{1-z} - \frac{n-1}{1+z}\right)$$

Thus, we are led to the conclusion in the statement. □

Now recall from chapter 3 that we have the following notion:

DEFINITION 4.26. *Given a real probability measure μ, define its R-transform by:*

$$G_\mu(\xi) = \int_{\mathbb{R}} \frac{d\mu(t)}{\xi - t} \implies G_\mu\left(R_\mu(\xi) + \frac{1}{\xi}\right) = \xi$$

That is, the R-transform is the inverse of the Cauchy transform, up to a ξ^{-1} factor.

Getting back now to our questions, we would like to find the probability measure having as R-transform the function in Theorem 4.25. But here, we can only expect to find some kind of modification of the Marchenko-Pastur law, so as a first piece of work, let us just compute the R-transform of the Marchenko-Pastur law. We have here:

PROPOSITION 4.27. *The R-transform of the Marchenko-Pastur law π_t is*

$$R_{\pi_t}(\xi) = \frac{t}{1-\xi}$$

for any $t > 0$.

PROOF. This can be done in two steps, as follows:

(1) At $t = 1$, we know that the moments of π_1 are the Catalan numbers, $M_k = C_k$, and we obtain that the Cauchy transform is given by the following formula:

$$G(\xi) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\xi^{-1}}$$

Now with $R(\xi)=\frac{1}{1-\xi}$ being the function in the statement, at $t=1$, we have:

$$\begin{aligned}G\left(R(\xi)+\frac{1}{\xi}\right) &= G\left(\frac{1}{1-\xi}+\frac{1}{\xi}\right)\\ &= G\left(\frac{1}{\xi-\xi^2}\right)\\ &= \frac{1}{2}-\frac{1}{2}\sqrt{1-4\xi+4\xi^2}\\ &= \frac{1}{2}-\frac{1}{2}(1-2\xi)\\ &= \xi\end{aligned}$$

Thus, the function $R(\xi)=\frac{1}{1-\xi}$ is indeed the R-transform of π_1, in the above sense.

(2) In the general case, $t>0$, the proof is similar, by using the moment formula for π_t, that we know from the above. We will be back to this with full details later, and with more about these laws too, when doing more in detail free probability. □

All this is very nice, and we can now further build on Theorem 4.25, as follows:

THEOREM 4.28. *The R-transform of $m\tilde{W}$ is given by*

$$R=R_{\pi_s}-R_{\pi_t}$$

in the $d\to\infty$ limit, where $s=m(n+1)/2$ and $t=m(n-1)/2$.

PROOF. We know from Theorem 4.25 that the R-transform of $m\tilde{W}$ is given by:

$$R=\frac{m}{2}\left(\frac{n+1}{1-z}-\frac{n-1}{1+z}\right)$$

By using now the formula in Proposition 4.27, this gives the result. □

We can now recover the original result of Aubrun [5], as follows:

THEOREM 4.29. *For a block-transposed Wishart matrix $\tilde{W}=(id\otimes t)W$ we have, in the $n=\beta m\to\infty$ limit, with $\beta>0$ fixed, the formula*

$$\frac{\tilde{W}}{d}\sim\gamma_\beta^1$$

with γ_β^1 being the shifted version of the semicircle law γ_β, with support centered at 1.

PROOF. This follows from Theorem 4.28. Indeed, in the $n=\beta m\to\infty$ limit, with $\beta>0$ fixed, we are led to the following formula for the Stieltjes transform:

$$f(x)=\frac{\sqrt{4\beta-(1-x)^2}}{2\beta\pi}$$

But this is the density of the shifted semicircle law having support as follows:

$$S = [1 - 2\sqrt{\beta}, 1 + 2\sqrt{\beta}]$$

Thus, we are led to the conclusion in the statement. See [5], [12]. □

Here we have used some standard free probability results at the end, which can be proved by direct computations, and we will be back to this later.

4e. Exercises

Exercises.

Part II

Heavy theorems

When love takes over
You know you can't deny
When love takes over
Cause something's here tonight

CHAPTER 5

Circular law

5a.

5b.

5c.

5d.

5e. Exercises

CHAPTER 6

Tracy-Widom law

6a.

6b.

6c.

6d.

6e. Exercises

CHAPTER 7

Fine fluctuations

7a.

7b.

7c.

7d.

7e. Exercises

CHAPTER 8

Asymptotic freeness

8a.

8b.

8c.

8d.

8e. Exercises

Part III

Wild combinatorics

What are we supposed to do
After all that we've been through
When everything that felt so right is wrong
Now that the love is gone

CHAPTER 9

Fast spinning

9a.

9b.

9c.

9d.

9e. Exercises

CHAPTER 10

Patterned matrices

10a.

10b.

10c.

10d.

10e. Exercises

CHAPTER 11

Block modifications

11a.

11b.

11c.

11d.

11e. Exercises

CHAPTER 12

Symplectic case

12a.

12b.

12c.

12d.

12e. Exercises

Part IV

The operator wars

You shoot me down, but I won't fall
I am titanium
You shoot me down, but I won't fall
I am titanium

CHAPTER 13

Factors, subfactors

13a.

13b.

13c.

13d.

13e. Exercises

CHAPTER 14

Microstates, entropy

14a.

14b.

14c.

14d.

14e. Exercises

CHAPTER 15

A trip to wonderland

15a.

15b.

15c.

15d.

15e. Exercises

CHAPTER 16

Matrix models

16a.

16b.

16c.

16d.

16e. Exercises

Bibliography

[1] G.W. Anderson and B. Farrell, Asymptotically liberating sequences of random unitary matrices, *Adv. Math.* **255** (2014), 381–413.

[2] G.W. Anderson, A. Guionnet and O. Zeitouni, An introduction to random matrices, Cambridge Univ. Press (2010).

[3] M. Anshelevich, Free Meixner states, *Comm. Math. Phys.* **276** (2007), 863–899.

[4] O. Arizmendi, I. Nechita and C. Vargas, On the asymptotic distribution of block-modified random matrices, *J. Math. Phys.* **57** (2016), 1–27.

[5] G. Aubrun, Partial transposition of random states and non-centered semicircular distributions, *Random Matrices Theory Appl.* **1** (2012), 125–145.

[6] J. Baik, P. Deift and T. Suidan, Combinatorics and random matrix theory, AMS (2016).

[7] T. Banica, Linear algebra and group theory (2023).

[8] T. Banica, Principles of operator algebras (2023).

[9] T. Banica, Methods of free probability (2023).

[10] T. Banica, S.T. Belinschi, M. Capitaine and B. Collins, Free Bessel laws, *Canad. J. Math.* **63** (2011), 3–37.

[11] T. Banica and S. Curran, Decomposition results for Gram matrix determinants, *J. Math. Phys.* **51** (2010), 1–14.

[12] T. Banica and I. Nechita, Asymptotic eigenvalue distributions of block-transposed Wishart matrices, *J. Theoret. Probab.* **26** (2013), 855–869.

[13] T. Banica and I. Nechita, Block-modified Wishart matrices and free Poisson laws, *Houston J. Math.* **41** (2015), 113–134.

[14] R.J. Baxter, Exactly solved models in statistical mechanics, Academic Press (1982).

[15] S.T. Belinschi and H. Bercovici, Partially defined semigroups relative to multiplicative free convolution, *Int. Math. Res. Not.* **2** (2005), 65–101.

[16] S.T. Belinschi, M. Bożejko, F. Lehner and R. Speicher, The normal distribution is $\boxplus$-infinitely divisible, *Adv. Math.* **226** (2011), 3677–3698.

[17] H. Bercovici and V. Pata, Stable laws and domains of attraction in free probability theory, *Ann. of Math.* **149** (1999), 1023–1060.

[18] P. Biane, M. Capitaine and A. Guionnet, Large deviation bounds for matrix Brownian motion, *Invent. Math.* **152** (2003), 433–459.

[19] D. Bisch and V.F.R. Jones, Algebras associated to intermediate subfactors, *Invent. Math.* **128** (1997), 89–157.

[20] G. Blower, Random matrices: high dimensional phenomena, Cambridge Univ. Press (2009).

[21] A. Borodin, I. Corwin and A. Guionnet, eds., Random matrices, AMS (2019).

[22] A. Bose and M. Bhattacharjee, Large covariance and autocovariance matrices, CRC Press (2018).

[23] A. Bose, Patterned random matrices, CRC Press (2018).

[24] A. Bose and K. Saha, Random circulant matrices, CRC Press (2018).

[25] A. Bose, Random matrices and non-commutative probability, CRC Press (2021).

[26] B. Collins, Moments and cumulants of polynomial random variables on unitary groups, the Itzykson-Zuber integral, and free probability, *Int. Math. Res. Not.* **17** (2003), 953–982.

[27] B. Collins and S. Matsumoto, On some properties of orthogonal Weingarten functions, *J. Math. Phys.* **50** (2009), 1–18.

[28] B. Collins and I. Nechita, Random quantum channels I: graphical calculus and the Bell state phenomenon, *Comm. Math. Phys.* **297** (2010), 345–370.

[29] B. Collins and I. Nechita, Random quantum channels II: entanglement of random subspaces, Rényi entropy estimates and additivity problems, *Adv. Math.* **226** (2011), 1181–1201.

[30] B. Collins and I. Nechita, Gaussianization and eigenvalue statistics for random quantum channels (III), *Ann. Appl. Probab.* **21** (2011), 1136–1179.

[31] B. Collins and P. Śniady, Integration with respect to the Haar measure on unitary, orthogonal and symplectic groups, *Comm. Math. Phys.* **264** (2006), 773–795.

[32] A. Connes, Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$, *Ann. of Math.* **104** (1976), 73–115.

[33] A. Connes, Noncommutative geometry, Academic Press (1994).

[34] S. Curran, Quantum rotatability, *Trans. Amer. Math. Soc.* **362** (2010), 4831–4851.

[35] S. Curran and R. Speicher, Quantum invariant families of matrices in free probability, *J. Funct. Anal.* **261** (2011), 897–933.

[36] P. Deift, Orthogonal polynomials and random matrices: a Riemann-Hilbert approach, AMS (1999).

[37] P. Di Francesco, Meander determinants, *Comm. Math. Phys.* **191** (1998), 543–583.

[38] P. Di Francesco, P. Mathieu and D. Sénéchal, Conformal field theory, Springer (1996).

[39] I. Dumitriu and A. Edelman, Matrix models for beta ensembles, *J. Math. Phys.* **43** (2002), 5830–5847.

[40] R. Durrett, Probability: theory and examples, Cambridge Univ. Press (1990).

[41] K. Dykema, Free products of hyperfinite von Neumann algebras and free dimension, *Duke Math. J.* **69** (1993), 97–119.

[42] B. Eynard, Counting surfaces, Birkhäuser (2016).

[43] W. Feller, An introduction to probability theory and its applications, Wiley (1950).

[44] M. Février and A. Nica, Infinitesimal non-crossing cumulants and free probability of type B, *J. Funct. Anal.* **258** (2010), 2983–3023.

[45] P. Flajolet and R. Sedgewick, Analytic combinatorics, Cambridge Univ. Press (2009).

[46] M. Fukuda and P. Śniady, Partial transpose of random quantum states: exact formulas and meanders, *J. Math. Phys.* **54** (2013), 1–31.

[47] P. Graczyk, G. Letac and H. Massam, The complex Wishart distribution and the symmetric group, *Ann. Statist.* **31** (2003), 287–309.

[48] A. Guionnet, V.F.R. Jones and D. Shlyakhtenko, Random matrices, free probability, planar algebras and subfactors, *Quanta of maths* **11** (2010), 201–239.

[49] A. Guionnet, M. Krishnapur and O. Zeitouni, The single ring theorem, *Ann. of Math.* **174** (2011), 1189–1217.

[50] R. Gurau, Random tensors, Oxford Univ. Press (2016).
[51] U. Haagerup, On Voiculescu's R and S transforms for free non-commuting random variables, *Fields Inst. Comm.* **12** (1997), 127–148.
[52] U. Haagerup and S. Thorbjørnsen, Random matrices with complex Gaussian entries, *Exposition. Math* **21** (2003), 293–337.
[53] F. Hiai and D. Petz, The semicircle law, free random variables and entropy, AMS (2000).
[54] R.A. Horn and C.R. Johnson, Matrix analysis, Cambridge Univ. Press (1985).
[55] K. Johansson, Shape fluctuations and random matrices, *Comm. Math. Phys.* **209** (2000), 437–476.
[56] V.F.R. Jones, Index for subfactors, *Invent. Math.* **72** (1983), 1–25.
[57] V.F.R. Jones, Planar algebras I (1999).
[58] V.F.R. Jones, The annular structure of subfactors, *Monogr. Enseign. Math.* **38** (2001), 401–463.
[59] K. Jung, Amenability, tubularity, and embeddings into R^ω, *Math. Ann.* **338** (2007), 241–248.
[60] C. Köstler, R. Speicher, A noncommutative de Finetti theorem: invariance under quantum permutations is equivalent to freeness with amalgamation, *Comm. Math. Phys.* **291** (2009), 473–490.
[61] T.M. Liggett, Interacting particle systems, Springer (1985).
[62] W. Liu, General de Finetti type theorems in noncommutative probability, *Comm. Math. Phys.* **369** (2019), 837–866.
[63] G. Livan, M. Novaes and P. Vivo, Introduction to random matrices: theory and practice, Springer (2018).
[64] C. Male, Traffic distributions and independence: permutation invariant random matrices and the three notions of independence, AMS (2020).
[65] V.A. Marchenko and L.A. Pastur, Distribution of eigenvalues in certain sets of random matrices, *Mat. Sb.* **72** (1967), 507–536.
[66] M.L. Mehta, Random matrices, Elsevier (1967).
[67] J.A. Mingo and A. Nica, Annular noncrossing permutations and partitions, and second-order asymptotics for random matrices, *Int. Math. Res. Not.* **28** (2004), 1413–1460.
[68] J.A. Mingo and M. Popa, Freeness and the transposes of unitarily invariant random matrices, *J. Funct. Anal.* **271** (2016), 883–921.
[69] J.A. Mingo and R. Speicher, Free probability and random matrices, Springer (2017).
[70] F.J. Murray and J. von Neumann, On rings of operators. IV, *Ann. of Math.* **44** (1943), 716–808.
[71] A. Nica and R. Speicher, Lectures on the combinatorics of free probability, Cambridge Univ. Press (2006).
[72] A. Ocneanu, Quantized groups, string algebras and Galois theory for algebras, *London Math. Soc. Lect. Notes* **136** (1988), 119–172.
[73] S. Popa, Classification of amenable subfactors of type II, *Acta Math.* **172** (1994), 163–255.
[74] S. Popa and D. Shlyakhtenko, Universal properties of $L(F_\infty)$ in subfactor theory, *Acta Math.* **191** (2004), 225–257.
[75] M. Potters and J.P. Bouchaud, A first course in random matrix theory, Cambridge Univ. Press (2020).
[76] S. Raum and M. Weber, The full classification of orthogonal easy quantum groups, *Comm. Math. Phys.* **341** (2016), 751–779.
[77] K. Schmüdgen, The moment problem, Springer (2017).

[78] H. Schultz, Non-commutative polynomials of independent Gaussian random matrices. The real and symplectic cases, *Probab. Theory Related Fields* **131** (2005), 261–309.

[79] D. Shlyakhtenko, Some applications of freeness with amalgamation, *J. Reine Angew. Math.* **500** (1998), 191–212.

[80] R. Speicher, Multiplicative functions on the lattice of noncrossing partitions and free convolution, *Math. Ann.* **298** (1994), 611–628.

[81] T. Tao, Topics in random matrix theory, AMS (2012).

[82] T. Tao and V.H. Vu, Additive combinatorics, Cambridge Univ. Press (2016).

[83] P. Tarrago and J. Wahl, Free wreath product quantum groups and standard invariants of subfactors, *Adv. Math.* **331** (2018), 1–57.

[84] P. Tarrago and M. Weber, Unitary easy quantum groups: the free case and the group case, *Int. Math. Res. Not.* **18** (2017), 5710–5750.

[85] N.H. Temperley and E.H. Lieb, Relations between the "percolation" and "colouring" problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the "percolation" problem, *Proc. Roy. Soc. London* **322** (1971), 251–280.

[86] C.A. Tracy and H. Widom, Level-spacing distributions and the Airy kernel, *Comm. Math. Phys.* **159** (1994), 151–174.

[87] D.V. Voiculescu, Addition of certain noncommuting random variables, *J. Funct. Anal.* **66** (1986), 323–346.

[88] D.V. Voiculescu, Multiplication of certain noncommuting random variables, *J. Operator Theory* **18** (1987), 223–235.

[89] D.V. Voiculescu, Limit laws for random matrices and free products, *Invent. Math.* **104** (1991), 201–220.

[90] D.V. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory, *Comm. Math. Phys.* **155** (1993), 71–92.

[91] D.V. Voiculescu, K.J. Dykema and A. Nica, Free random variables, AMS (1992).

[92] J. von Neumann, Mathematical foundations of quantum mechanics, Princeton Univ. Press (1955).

[93] S. Wang, Quantum symmetry groups of finite spaces, *Comm. Math. Phys.* **195** (1998), 195–211.

[94] A. Wassermann, Coactions and Yang-Baxter equations for ergodic actions and subfactors, *London Math. Soc. Lect. Notes* **136** (1988), 203–236.

[95] D. Weingarten, Asymptotic behavior of group integrals in the limit of infinite rank, *J. Math. Phys.* **19** (1978), 999–1001.

[96] H. Weyl, The theory of groups and quantum mechanics, Princeton Univ. Press (1931).

[97] E. Wigner, Characteristic vectors of bordered matrices with infinite dimensions, *Ann. of Math.* **62** (1955), 548–564.

[98] S.L. Woronowicz, Compact matrix pseudogroups, *Comm. Math. Phys.* **111** (1987), 613–665.

[99] J. Zinn-Justin, From random walks to random matrices, Oxford Univ. Press (2019).

[100] P. Zinn-Justin, Jucys-Murphy elements and Weingarten matrices, *Lett. Math. Phys.* **91** (2010), 119–127.