Von Neumann algebras and factors

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Bounded operators

Theorem. Given a Hilbert space H, the linear operators $T : H \to H$ which are bounded, in the sense that

$$||T|| = \sup_{||x|| \leq 1} ||Tx||$$

is finite, form a complex algebra $B(H)$, which:

(1) Is complete with respect to $||.||$ (Banach algebra).

(2) Has an involution $T \to T^*$, $< Tx, y >=< x, T^*y >$.

The norm and involution are related by $||TT^*|| = ||T||^2$.

Remark. Assuming $H = l^2(I)$, we have $B(H) \subset M_I(\mathbb{C})$, via $T \to (< Te_j, e_i >)_{ij}$. The adjoint is then $(M^*)_{ij} = \bar{M}_{ji}$.

Operator algebras

Definition. A von Neumann algebra is an algebra of bounded operators $A \subset B(H)$ which is:

(1) Stable under the involution: $T \in A \implies T^* \in A$.

(2) Weakly closed: $T_n \in A, T_n x \to Tx \implies T \in A$.

Examples. The simplest examples are the matrix algebras $M_N(\mathbb{C})$, and their $*$-subalgebras $A \subset M_N(\mathbb{C})$. In particular, we have

$$A =< M >$$

the $*$-algebra generated by a matrix $M \in M_N(\mathbb{C})$. In the normal case, $MM^* = M^*M$, we have $< M >= \{P(M) | P \in \mathbb{C}[X]\}$.

Commutative algebras

Theorem. Given a measured space X, we have an algebra

$$L^\infty(X) \subset B(L^2(X))$$

with the functions $f \in L^\infty(X)$ acting via $T_f : g \to fg$.

Theorem. The commutative von Neumann algebras are those of the form $L^\infty(X)$, with X being a measured space.

Proof. Basic functional analysis and operator theory. The full statement involves as well a multiplicity, in regards with H.

Random matrices

Definition. A random matrix algebra is an algebra of type:

$$A = M_N(L^\infty(X))$$

The elements of A are called random matrices.

Theorem. The matrices $M \in A$ having i.i.d. normal entries, up to the constraint $M = M^*$, follow with $N \to \infty$ the semicircle law:

$$\gamma_t = \frac{1}{2\pi t}\sqrt{4t^2 - x^2}dx$$

Proof. Moment method. The Wick formula gives with $N \to \infty$ the Catalan numbers, which are the moments of γ_t.

Free probability

<u>Definition</u>. Two subalgebras $B, C \subset A$ are called:

(1) Independent, if $tr(b) = tr(c) = 0$ implies $tr(bc) = 0$.

(2) Free, if $tr(b_i) = tr(c_i) = 0$ implies $tr(b_1 c_1 b_2 c_2 \ldots) = 0$.

<u>Theorem</u>. If $x_1, x_2, x_3, \ldots \in A$ are independent/free, i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim \mathcal{N}(0,t)/\gamma_t$$

where $\mathcal{N}(0,t)/\gamma_t$ are the normal/Wigner semicircle laws.

<u>Proof</u>. Linearization of the convolution, $\log F/R$ transforms.

General theory

Theorem. For a $*$-algebra of operators $A \subset B(H)$, the following conditions are equivalent:

(1) A is weakly closed, i.e. is a von Neumann algebra.

(2) A is equal to its algebraic bicommutant, $A = A$''.

This is von Neumann's "bicommutant theorem". As a consequence, the von Neumann algebras appear as commutants, $A = P'$.

Comment. Von Neumann $\implies$ norm closed. The von Neumann algebras are the norm closed $*$-algebras with separable predual.

Finite dimensions

Theorem. Let $A \subset M_N(\mathbb{C})$ be a $*$-algebra.

(1) We have $1 = p_1 + \ldots + p_k$, with $p_i \in A$ minimal projections.

(2) The spaces $A_i = p_i A p_i$ are non-unital $*$-subalgebras of A.

(3) We have a non-unital $*$-algebra sum $A = A_1 \oplus \ldots \oplus A_k$.

(4) Unital $*$-algebra isomorphisms $A_i \simeq M_{N_i}(\mathbb{C})$, $N_i = rank(p_i)$.

(5) Thus, we can decompose $A \simeq M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$.

Proof. (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5).

Reduction theory

Theorem. When writing the center of the algebra as

$$Z(A) = L^\infty(X)$$

wihh X measured space, the algebra decomposes as

$$A = \int_X A_x \, dx$$

with the summands being "factors", $Z(A_x) = \mathbb{C}$.

Example. In finite dimensions the algebra must be

$$A = M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$$

and this is its decomposition as a sum of factors.

Factors

Theorem. The factors, $Z(A) = \mathbb{C}$, fall into 3 classes:

(1) Type I. These are the usual matrix algebras $M_N(\mathbb{C})$ (type I_N), and the algebra $B(H)$, with H separable (type I_∞).

(2) Type II. These are the ∞D factors having a trace $tr : A \to \mathbb{C}$ (type II_1) and their tensor products with $B(H)$ (type II_∞).

(3) Type III. These fall into several classes, III_λ with $\lambda \in [0, 1]$, and appear from II_1 factors, via crossed product type constructions.

Proof. This is heavy, due to Murray and von Neumann, and then Connes, based on ideas of Tomita, Takesaki and others.

$\implies$ The II_1 factors are the "building blocks" of the theory.

Group duals

Definition. Consider a discrete group Γ.

(1) We endow $\mathbb{C}[\Gamma]$ with the involution $g^* = g^{-1}$.

(2) We embed $\mathbb{C}[\Gamma] \subset B(l^2(\Gamma))$ via $T_g : h \to gh$.

(3) We let $L(\Gamma)$ be the closure of $\mathbb{C}[\Gamma]$ inside $B(l^2(\Gamma))$.

Theorem. When Γ is abelian, we have an identification

$$L(\Gamma) = L^\infty(G)$$

where $G = \{\chi : \Gamma \to \mathbb{T}\}$ is the Pontrjagin dual of Γ.

$\implies$ In general, we can define $G = \widehat{\Gamma}$, abstract "noncommutative measured space", by the formula $L^\infty(G) = L(\Gamma)$.

Noncommutative geometry

<u>Theorem</u>. The group algebras $L(\Gamma)$ have traces, given by

$$tr(g) = \delta_{g1}$$

and so decompose into integrals of type I_N and II_1 factors.

<u>Theorem</u>. A group algebra $L(\Gamma)$ is a II_1 factor precisely when Γ has infinite conjugacy classes (ICC property).

<u>More</u>. The group duals $G = \widehat{\Gamma}$ are the "NC tori". One can talk as well about NC spheres, quantum groups, and so on, e.g. about

$$A = L^\infty(G/H)$$

with $H \subset G$ quantum groups, with $tr =$ uniform integration.

II_1 factors

Definition. A II_1 factor is a von Neumann algebra $A \subset B(H)$:

(1) Which is infinite dimensional, $\dim(A) = \infty$.

(2) Has trivial center, $Z(A) = \mathbb{C}$.

(3) And has a faithful positive unital trace, $tr : A \to \mathbb{C}$.

Theorem. The trace is unique.

Theorem. The trace of projections can take any value in $[0, 1]$.

$\Longrightarrow$ This is very interesting, "continuous dimension".

The factor R

Theorem. The following limiting von Neumann algebra,

$$R = \lim_{k\to\infty} M_{N_k}(\mathbb{C})$$

is a II_1 factor, independent of the limiting procedure.

Theorem. R is the unique "hyperfinite" II_1 factor. In fact, R is the "building block" for the hyperfinite von Neumann algebras.

Theorem. A group algebra $L(\Gamma)$ is hyperfinite precisely when Γ is amenable. If in addition we have ICC, then $L(\Gamma) \simeq R$.

Subfactors 1/2

Definition. Consider an inclusion of II_1 factors $A \subset B$.

(1) Its index is the number $[B : A] = \dim_A B \in [1, \infty]$, defined as a Murray-von Neumann "continuous dimension" quantity.

(2) The "basic construction" is $A \subset B \subset C$, by "reflection", with $C =< B, e >$, where $e : B \to A$ is the orthogonal projection.

Theorem. Let $A_0 \subset A_1$ be a subfactor of finite index $N \in [1, \infty)$, and consider its Jones tower, obtained by basic construction:

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \ldots$$

The Jones projections $e_1, e_2, e_3, \ldots$ generate then a copy of the Temperley-Lieb algebra TL_N, inside the ambient algebra $B(H)$.

Subfactors 2/2

Theorem. The index of subfactors is "quantized",

$$N \in \left\{ 4\cos^2\left(\frac{\pi}{n}\right) \Big| n \in \mathbb{N} \right\} \cup [4,\infty]$$

and all the admissible index values are attainable.

Theorem. Consider the commutants $P_k = A_0' \cap A_k$.

(1) The graded union $P = \cup_k P_k$ contains TL_N.

(2) P is a planar algebra, "its elements behave like diagrams".

(3) In the "amenable" case, P classifies the subfactor.

$\implies$ The philosophy is that $A \subset B$ appears via an action of an underlying "quantum group", of the most general type. Of particular interest is the hyperfinite case, $A \simeq B \simeq R$.