Subfactors and the Temperley-Lieb algebra

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II_1 factors

Definition. A II_1 factor is a von Neumann algebra, i.e. a weakly closed $*$-algebra of operators $A \subset B(H)$, which:

(1) Is infinite dimensional, $\dim(A) = \infty$.

(2) Has trivial center, $Z(A) = \mathbb{C}$.

(3) And has a trace, $tr : A \to \mathbb{C}$.

– As basic example, we have the Murray-von Neumann hyperfinite factor R, obtained as inductive limit of matrix algebras.

– We have as well the group von Neumann algebras $L(\Gamma)$, with Γ being a discrete group having infinite conjugacy classes (ICC).

Comment. When Γ has the ICC property and is amenable we have $L(\Gamma) \simeq R$. In general, it is not known when $L(\Gamma) \simeq L(\Lambda)$.

Theory

Theorem 1. The trace of the II_1 factors

$$tr : A \to \mathbb{C}$$

which is by definition faithful, positive and unital, is unique.

Theorem 2. The traces of the projections $p \in A$,

$$p^2 = p^* = p$$

take as values all the numbers in $[0, 1]$.

Subfactors

Definition. A subfactor is an inclusion of II_1 factors

$$A \subset B$$

inside $B(H)$. The index of such an inclusion is the number

$$N = \dim_A B \in [1, \infty]$$

which is a Murray-von Neumann "continuous dimension" quantity.

Theorem. When the index is an integer, $N \in \mathbb{N}$, we have a basis

$$\{e_1, \ldots, e_N\} \subset B$$

of B over A, which is orthogonal with respect to tr.

Examples 1/4

Given a finite group G, acting on a von Neumann algebra P, we have the following inclusion:

$$P \subset P \rtimes G$$

When P is a II_1 factor and $G \to Aut(P)$ is minimal, this is a subfactor of index $N = |G|$, called depth 2 subfactor.

Examples 2/4

Given a compact group G, acting on a von Neumann algebra P, and given with a finite index subgroup $H \subset G$, we have:

$$P^G \subset P^H$$

When P is a II_1 factor and $G \to Aut(P)$ is minimal, this is a subfactor of index $N = [G : H]$, called subgroup subfactor.

Examples 3/4

Given a compact group G, acting on a von Neumann algebra P, and given with a unitary representation $G \to PU_n$, we have:

$$P^G \subset (M_n(\mathbb{C}) \otimes P)^G$$

When P is a II_1 factor and $G \to Aut(P)$ is minimal, this is a subfactor of index $N = n^2$, called Wassermann subfactor.

Examples 4/4

Given a discrete group $\Gamma = < g_1, \ldots, g_n >$, acting on a von Neumann algebra Q, we have the following inclusion:

$$\left\{ diag(g_1(q), \ldots, g_n(q)) \middle| q \in Q \right\} \subset M_n(Q)$$

When Q is a II_1 factor and $\Gamma \subset Aut(Q)$ is outer, this is a subfactor of index $N = n^2$, called diagonal subfactor.

Basic construction

<u>Theorem</u>. Given a subfactor $A \subset B$, we consider the projection

$$e : B \to A \quad , \quad e \in B(H)$$

with respect to the trace, we construct the algebra

$$C = < B, e > \subset B(H)$$

which is a factor, and we obtain in this way a new subfactor,

$$A \subset \underline{B \subset C}$$

having the same index. This is the Jones "basic construction".

Examples

(1) Depth two, $P \subset P \rtimes G$. We obtain a crossed product by $\widehat{G}$, the dual group when G is abelian, and a quantum group in general.

(2) Subgroup, $P^G \subset P^H$. In the simplest case, where $H = \{1\}$, and the subfactor is $P^G \subset P$, we obtain $P \subset P \rtimes G$.

(3) Wassermann, $P^G \subset M_n(P)^G$. Here we obtain $M_{n^2}(P)^G$, with $G \to PU_{n^2}$ being the tensor square of $G \to PU_n$.

(4) Diagonal, $Q^{\curvearrowright\Gamma} \subset M_n(Q)$. Here we obtain $M_{n^2}(Q)$, with the embedding $M_n(Q) \subset M_{n^2}(Q)$ being twisted by Γ.

$\implies$ The basic construction $A \subset B \subset C$ is a "reflection".

The tower

Definition. Starting from the initial subfactor, relabeled

$$A_0 \subset A_1$$

we obtain by basic construction a whole tower of factors

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \ldots$$

with $e_1, e_2, e_3, \ldots$ being the Jones projections, at each step.

Temperley-Lieb

Theorem. In the context of a Jones tower as above,

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \ldots$$

the Jones projections $e_1, e_2, e_3, \ldots$ generate a copy of TL_N.

Proof. The TL_N relations follow from a careful study of

$$A \subset B \subset C$$

and also of $A \subset B \subset C \subset D$, by translation. Since we have

$$tr(\pi) = N^{loops<\pi>}$$

which is faithful on TL_N, this representation is faithful.

Index theorem

Theorem. The index of subfactors is "quantized", as follows:

$$N \in \{4\cos^2(\pi/n) | n \in \mathbb{N}\} \cup [4, \infty]$$

Moreover, all values are attained, and we have ADE at $N \leq 4$.

Proof. According to the above, we have a representation:

$$TL_N \subset B(H)$$

At $N < 4$ this is not always possible, due to positivity reasons.

Planar algebras

Recall that TL_N is a "planar" algebra, appearing as the span

$$TL_N = span(NC_2)$$

the product being vertical concatenation ($\downarrow$), with $\bigcirc = N$.

<u>Theorem</u>. Given $A_0 \subset A_1$, the planar algebra structure of

$$< e_1, e_2, e_3, \ldots >= TL_N$$

extends into a planar algebra structure of $P = (P_k)$, where

$$P_k = A_0' \cap A_k$$

are the higher relative commutants (FD complex vector spaces).

Example

Consider a Wassermann subfactor, coming from $\pi : G \to PU_n$:

$$P^G \subset M_n(P)^G$$

The Jones tower for this subfactor comes then from $\{\pi^{\otimes k}\}$:

$$P^G \subset M_n(P)^G \subset M_{n^2}(P)^G \subset M_{n^3}(P)^G \subset \ldots$$

As for the higher relative commutants, these are as follows:

$$\mathbb{C} \subset End(\pi) \subset End(\pi^{\otimes 2}) \subset End(\pi^{\otimes 3}) \subset \ldots$$

Thus, we obtain the usual planar operations on $End(\pi^{\otimes k})$.

Classification

Theorem 1. The subfactors $A \subset B$ having "finite depth" are classified by their planar algebras $P = (P_k)$.

Theorem 2. More generally, the "amenable" subfactors $A \subset B$ are classified by their planar algebras $P = (P_k)$.

Theorem 3. In general, any planar algebra produces a subfactor (complementing "any subfactor produces a planar algebra").