Planar algebras and spectral measures

Teo Banica

"Introduction to subfactor theory", 4/6

07/20

Temperley-Lieb

Definition. $TL_N(k)$ is the formal span of the noncrossing pairings between k upper points and k lower points,

$$TL_N(k) = span(NC_2(k,k))$$

with product given on generators by vertical concatenation, with the convention that "things go downwards",

$$\pi\sigma = \begin{bmatrix} \sigma \\ \pi \end{bmatrix}$$

and with the rule for the circles that might appear in the middle:

$$\bigcirc = N$$

That is, each such circle counts for a multiplicative N factor.

Properties

The algebra $TL_N(k)$ has an involution, given by:

$$A^* = \forall$$

We have embeddings of unital $*$-algebras, as follows,

$$TL_N(k) \subset TL_N(k+1)$$

obtained by adding a string at right, and the union

$$TL_N = \bigcup_{k \in \mathbb{N}} TL_N(k)$$

is a graded $*$-algebra. There is a $\otimes$ operation as well.

Subfactors

Theorem. Consider a subfactor $A_0 \subset A_1$, of index $N \in [1, \infty)$, and build by "basic construction" the associated Jones tower:

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \ldots$$

The Jones projections $e_1, e_2, e_3, \ldots$ generate then a copy of TL_N.

Proof. The TL_N relations follow from a careful study of the basic construction, by translation. Since we have

$$tr(\pi) = N^{loops<\pi>}$$

which is faithful on TL_N, this representation is faithful.

Planar algebras

Theorem. The planar algebra structure of the algebra

$$TL_N =< e_1, e_2, e_3, \ldots >$$

extends into a planar algebra structure of $P = (P_k)$, where

$$P_k = A_0' \cap A_k$$

are the higher relative commutants (FD $*$-algebras).

Definition

A planar algebra is a collection of FD complex vector spaces

$$P = (P_k)_{k\in\mathbb{N}}$$

with an action on it of the diagrams consisting of:

(1) a big box, containing s small boxes,

(2) with $2k + \sum_{i=1}^{s} 2k_i$ points on these boxes,

(3) and with NC strings connecting these points.

That is, associated to any such diagram is a linear map

$$P_{k_1} \otimes \ldots \otimes P_{k_s} \to P$$

and the gluing of diagrams corresponds to the composition of maps.

Examples

(1) The Temperley-Lieb algebra TL_N. Here the linear generators $\pi \in NC_2(k,k)$ are put into boxes in the obvious way.

(2) The Fuss-Catalan algebra $FC_{N,M}$. Same technology as for TL_N, but this time the strings are colored, with two colors.

(3) The tensor planar algebra T_N. Here $T_N(k) = M_N(\mathbb{C})^{\otimes k}$, and the operations correspond to the usual tensor calculus.

(4) The spin planar algebra S_N. Here $S_N(k) = (\mathbb{C}^N)^{\otimes k}$, and the indices are doubled, before being put into boxes.

Theory

Theorem 1. The subfactors $A_0 \subset A_1$ having "finite depth" are classified by their planar algebras $P = (P_k)$.

Theorem 2. More generally, the "amenable" subfactors $A_0 \subset A_1$ are classified by their planar algebras $P = (P_k)$.

Theorem 3. In general, any planar algebra produces a subfactor (complementing "any subfactor produces a planar algebra").

TL subfactors

Theorem. The Temperley-Lieb subfactors exist for any admissible value of the index, namely

$$N \in \left\{ 4\cos^2\left(\frac{\pi}{n}\right) \middle| n \in \mathbb{N} \right\} \cup [4, \infty]$$

and can be explicitly constructed as subfactors of $L(F_\infty)$.

Question. What about subfactors of R?

FC subfactors

Theorem. In the presence of an intermediate subfactor,

$$A_0 \subset B \subset A_1$$

the corresponding planar algebra contains the FC one:

$$FC \subset P$$

FC subfactors can be obtained by composing TL subfactors.

Question. Same as before, what about R?

Tensor subfactors

Theorem. The planar algebra of a Wassermann type subfactor

$$A^G \subset (M_N(\mathbb{C}) \otimes A)^G$$

is a subalgebra of the corresponding tensor planar algebra

$$\left(End(u^{\otimes k})\right)_{k\in\mathbb{N}} \subset T_N$$

and any subalgebra of T_N appears in this way.

Comment. This follows from Tannaka, and the correspondence is not bijective, because we have to lift the projective version.

Spin subfactors

Theorem. The planar algebra of subfactor of type

$$A^G \subset (\mathbb{C}^N \otimes A)^G$$

is a subalgebra of the corresponding spin planar algebra

$$\left(Fix(u^{\otimes k})\right)_{k\in\mathbb{N}} \subset S_N$$

and any subalgebra of S_N appears in this way.

Comment. Once again follows from Tannaka. The correspondence is now bijective, because $G \subset S_N^+$ implies $1 \in u$.

Invariants

The good. The spectral measure of a planar algebra $P = (P_k)$ is the real probability measure μ having as moments:

$$M_k = \dim(P_k)$$

The bad. The Poincaré series of P is the following series, with $z \in \mathbb{C}$, which is the Stieltjes transform of μ:

$$f(z) = \sum_{k=0}^{\infty} \dim(P_k) z^k$$

The ugly. The principal graph of P is the Bratelli diagram of

$$P_0 \subset P_1 \subset P_2 \subset \ldots$$

with the reflections coming from basic constructions removed.

Examples 1/2

(1) TL. Here we obtain the Marchenko-Pastur law

$$\pi = \frac{1}{2\pi}\sqrt{4x^{-1}-1}dx$$

also known as free Poisson law, or squared semicircle law.

(2) FC. Here we obtain the real free Bessel law

$$\beta = \pi_{\varepsilon_2}$$

which appears as a compound free Poisson measure.

Examples 2/2

(3) Tensor subfactors. Here we obtain the character law

$$\mu = law(\chi\chi^*)$$

with $\chi = Tr(u)$, assuming that $G \to PU_n$ comes via $ad(u)$.

(4) Spin subfactors. Here we obtain the character law

$$\mu = law(\chi)$$

with $\chi = Tr(u)$, where u corresponds to the action $G \curvearrowright \mathbb{C}^N$.

Questions

1. In the tensor and spin algebra context, we can truncate,

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

with respect to a parameter $t > 0$. What about in general?

2. In index 4, Jones' manipulation on the Poincaré series,

$$\Theta(q) = q + \frac{1-q}{1+q} f\left(\frac{q}{(1+q)^2}\right)$$

blows up the spectral measure on $\mathbb{T}$. What about in general?