Commuting squares, vertex and spin models

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Orthogonal MASA

Definition. A pair of orthogonal MASA is a pair of MASA

$$B, C \subset A$$

which are orthogonal: $tr(bc) = tr(b)tr(c)$, for any $b \in B, c \in C$.

Theorem. Up to a unitary, the pairs of orthogonal MASA in the simplest von Neumann factor, namely $M_N(\mathbb{C})$, are

$$A = \Delta \quad , \quad B = H\Delta H^*$$

with $\Delta =$ diagonal matrices, and $H \in M_N(\mathbb{C})$ Hadamard.

Hadamard matrices

The Fourier matrix, $F_N = (w^{ij})$ with $w = e^{2\pi i/N}$,

$$F_N = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & w & w^2 & \ldots & w^{N-1} \\ 1 & w^2 & w^4 & \ldots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{N-1} & w^{(2N-1)} & \ldots & w^{(N-1)^2} \end{pmatrix}$$

is Hadamard. More generally, given a finite abelian group G,

$$(F_G)_{i\chi} = \chi(i)$$

regarded via $G \simeq \widehat{G}$ as a matrix $F_G \in M_G(\mathbb{C})$, is Hadamard.

Expectations

<u>Theorem</u>. Given $H \in M_N(\mathbb{C})$ Hadamard, the associated pair of MASA form a subfactor-theoretic "commuting square"

$$\begin{array}{ccc} \Delta & \longrightarrow & M_N(\mathbb{C}) \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & H\Delta H^* \end{array}$$

in the sense that the expectations to the middle algebras commute, and their product is the expectation to the small algebra.

$\implies$ What is the subfactor associated to H?

Commuting squares

Definition. A commuting square in the sense of subfactor theory is a commuting diagram of FD algebras with traces,

$$\begin{array}{ccc} A & \longrightarrow & B \\ \uparrow & & \uparrow \\ C & \longrightarrow & D \end{array}$$

such that the expectations to the middle algebras commute, and their product is the expectation to the small algebra.

Basic construction

Theorem. Given a commuting square, by basic construction we obtain a Jones tower of commuting squares,

$$\begin{array}{ccccccc}
A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \cdots\cdots\to & A_\infty \\
\uparrow & & \uparrow & & \uparrow & & \\
C_0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \cdots\cdots\to & C_\infty
\end{array}$$

with the limiting algebras being hyperfinite II_1 factors,

$$A_\infty \simeq C_\infty \simeq R$$

and so with $C_\infty \subset A_\infty$ being a subfactor of R.

Ocneanu compactness

Theorem. In the context of a subfactor commuting square,

$$
\begin{array}{ccccccc}
A_{\infty 0} & & A_{\infty 1} & & A_{\infty 2} & & \\
\uparrow & & \uparrow & & \uparrow & & \\
A_{20} & \longrightarrow & A_{21} & \longrightarrow & A_{22} & \dashrightarrow & A_{2\infty} \\
\uparrow & & \uparrow & & \uparrow & & \\
A_{10} & \longrightarrow & A_{11} & \longrightarrow & A_{12} & \dashrightarrow & A_{1\infty} \\
\uparrow & & \uparrow & & \uparrow & & \\
A_{00} & \longrightarrow & A_{01} & \longrightarrow & A_{02} & \dashrightarrow & A_{0\infty}
\end{array}
$$

the horizontal planar algebra is $A'_{\infty 0} \cap A_{\infty k} = A'_{10} \cap A_{0k}$.

Further results

There are many examples, including those coming from Hadamard matrices, or from more general "vertex and spin models", or even more generally, from squares having $\mathbb{C}$ in the lower left corner:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & D \end{array}$$

All these subfactors have integer index. In non-integer index there are many interesting examples as well. In fact, any finite depth subfactor can be shown to appear from a commuting square.

Basic examples

Consider a vertex model or spin model commuting square, or more generally a commuting square having $\mathbb{C}$ in the lower left corner:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & D \end{array}$$

Our claim is that this produces a fixed point subfactor, coming from an action of a certain compact quantum group G.

Hopf images

<u>Theorem</u>. Any commuting square having $\mathbb{C}$ in the lower left corner, as above, can be written in the following way,

$$\begin{array}{ccc} A \otimes_G \mathbb{C} & \longrightarrow & A \otimes_G D \\ \uparrow & & \uparrow \\ \mathbb{C} \otimes_G \mathbb{C} & \longrightarrow & \mathbb{C} \otimes_G D \end{array}$$

with G being a compact quantum group, obtained from

$$B = A \otimes_G D$$

by performing a certain Hopf image factorization procedure.

Jones tower

Theorem. The horizontal Jones tower is given by

$$\begin{array}{ccccccc}
A\otimes_G\mathbb{C} & \longrightarrow & A\otimes_G D & \longrightarrow & A\otimes_G E & \cdots\cdots\to & A\otimes_G R\\
\uparrow & & \uparrow & & \uparrow & & \\
\mathbb{C}\otimes_G\mathbb{C} & \longrightarrow & \mathbb{C}\otimes_G D & \longrightarrow & \mathbb{C}\otimes_G E & \cdots\cdots\to & \mathbb{C}\otimes_G R
\end{array}$$

and a similar result holds for the vertical Jones tower.

Proof. The computations here are very similar to those needed when computing the Jones tower of fixed point subfactors.

Conclusion

Theorem. The subfactor associated to a commuting square of the following form, having $\mathbb{C}$ in the lower left corner,

$$\begin{array}{ccc} A & \longrightarrow & B \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & D \end{array}$$

is a fixed point subfactor, coming from an action of a certain compact quantum group G, obtained by solving

$$B = A \otimes_G D$$

via a Hopf image factorization construction. The corresponding planar algebra can be computed as well in terms of G.

Hadamard models

<u>Definition</u>. Given an Hadamard matrix $H \in M_N(\mathbb{C})$, we associate to it the smallest subgroup $G \subset S_N^+$ producing a factorization

$$\begin{array}{ccc} C(S_N^+) & \xrightarrow{\quad\quad\pi\quad\quad} & M_N(\mathbb{C}) \\ & \searrow \quad\quad \nearrow & \\ & C(G) & \end{array}$$

of the representation mapping $u_{ij} \in C(S_N^+)$ to the projections

$$P_{ij} = Proj\left(\frac{H_i}{H_j}\right)$$

where $H_1, \ldots, H_N \in \mathbb{C}^N$ are the rows of H, regarded inside $\mathbb{T}^N$.

General theory

Theorem. The above quantum group $G \subset S_N^+$ is the one wich produces the subfactor, and computes the planar algebra.

Theorem. For a Fourier matrix F_G of a finite abelian group G, the above construction produces G itself, acting on itself.

Theorem. For a product of Hadamard matrices, $H = H' \otimes H''$, we obtain a product of quantum groups, $G = G' \times G$".

Diță deformations

Theorem. Given two finite abelian groups G, H, with $|G| = M$, $|H| = N$, consider the main character χ of the quantum group associated to the Diță deformation $\mathcal{F}_{G \times H}$. We have then

$$law\left(\frac{\chi}{N}\right) = \left(1 - \frac{1}{M}\right)\delta_0 + \frac{1}{M}\,\pi_t$$

in moments, with $M = tN \to \infty$, where π_t is the free Poisson law of parameter $t > 0$. In addition, this holds for any generic fiber.

Proof. Long story here (B, BB, B, B).

Further results

All this fits into the matrix modelling theory for quantum groups. The central object here is the stationary matrix model

$$\pi : C(S_4^+) \to M_4(C(SU_2))$$

given on the standard coordinates by the formula

$$\pi(u_{ij}) = [x \to Proj(c_i x c_j)]$$

where $x \in SU_2$, and c_1, c_2, c_3, c_4 are the Pauli matrices. There is a natural generalization of this, involving the Weyl matrices.