The story of particle physics

Teo Banica

Department of Mathematics, University of Cergy-Pontoise, F-95000 Cergy-Pontoise, France. teo.banica@gmail.com

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Abstract. This is an introduction to the elementary particles of theoretical physics, kept as down-to-earth as possible. We first discuss the electrons and photons, based on our daily knowledge of electricity, light and heat, and with some mathematics and physics explained too. Then we get into protons, neutrons and more complicated particles, appearing from radioactivity, experiments, stars and atomic bombs. Finally, we discuss quarks and the Standard Model, and also questions going beyond this, coming on one hand from the Big Bang, and on the other hand, from dark energy and matter.

Preface

What is an elementary particle? Good question, and your physics graduate friend would probably say no worries, come doing some studies in physics with us, first undergraduate in order to learn the basics, and get started with some quantum mechanics, then graduate in order to struggle a bit with quantum field theory, and in the end, when eventually graduating, you will surely know what an elementary particle is.

This sounds like a good plan, and although there is a lot of truth in it, there are actually some slight omissions too. The problem indeed is that Einstein himself didn't quite understand what an elementary particle is, and nor did Feynman and many others coming after, and in fact no one really knows what an elementary particle is, and countless scientists are actively working on this, now as I type, and as you read.

Nevermind. What does exist, beyond any doubt, and can certainly be told without many troubles, is the story of the elementary particles. That is, the story of elementary particles from the perspective of mankind, as opposed to the same story from the perspective of the particles themselves, which remains something elusive.

The present book will be here for that, telling this story. We will go through all sorts of discoveries of mankind, which happened via theory and experiments, and with all this accompanied by a minimal amount of mathematics and physics explanations.

More in detail now, the book is organized in 4 parts, the plan being as follows:

I. We will first discuss the electrons and photons, based on our daily knowledge of electricity, light and heat, and with the needed math and physics explained too.

II. Then we will get into protons, neutrons and more complicated particles, appearing from radioactivity, various experiments, stars and atomic bombs.

III. Then we will get deeper into the particle jungle, via advanced theory and experiments, featuring antiparticles, mesons, neutrinos, and many other beasts.

IV. Finally, we will discuss quarks and the Standard Model, and also questions going beyond this, coming from the Big Bang, and from dark energy and matter.

In the hope that you will like this text. There are of course many other books with the same purpose, explaining particle physics and the Standard Model to undergraduates, and I will leave it to you to decide if you find this too simple, and need a more advanced book to start with, or find this too complicated, and need a more elementary book to start with, or, who knows, maybe just agree with me. In any case, glad to see you here, welcome to the subject, and one way or another you will get to know more about it. And don't forget, later in a few years, to let us know what an elementary particle truly is.

Many thanks go to my mathematics and physics colleagues, and to my math students too, usually I teach them calculus at the university, and whenever things get a bit too abstract, or boring or complicated, I slightly deviate towards quantum physics and particles, and everyone suddenly gets very interested. Thanks as well to my cats, watching them daily escaping from our quantum physics theories has always been inspirational.

Contents

Part I

Electrons, photons

At first I was afraid
I was petrified
Kept thinking I could never live
Without you by my side

CHAPTER 1

Electricity

1a. Electrons, charges

Welcome to the elementary particles. As the name indicates, these are little particles, invisible to the naked eye, that our world is made of. We will attempt in this book to understand what these elementary particles are, what are their main properties, and also, with a bit of luck, perhaps fully understand too how usual matter, in all its wild diversity, solids, liquids, gases, not to talk about life, cells and animals, is made of them.

You surely know a bit about all this, but let's go slowly, from the beginnings. At the beginning was the Stone Age man, who after a 16 hour day of healthy work was asking himself, before going to sleep, the same question as we do, what is this world made of.

In answer, sand is made of grains of sand, and if you further grind these grains of sand, which is something our Stone Age man was certainly expert at, up to the point of having some very fine dust, whose particles are invisible to the naked eye, here are your elementary particles. Moreover, experience with grinding and remixing various substances suggets that there might be actually very few types of elementary particles, perhaps no more than 3-4 of them, and with everything lying in the precise formula of the mix. Add to this, as principle, that certain types of particles tend naturally to stick to other types of particles, so the final nature of the substance, solid, liquid or gas, lies again in the precise formula of the mix, and we have here our first theory of elementary particles.

This sounds quite good, and it is tempting to further explore this theory, allowing us the usage of some modern math tools, that Stone Age man was probably not aware of, as to get something fully modern and scientific out of it. Before that, however, it is perhaps wise to ask the cat, whose knows physics much better than me. And cat answers:

Cat 1.1. *What you say sounds way too peaceful, more like fluid mechanics. Matter is force, look for particles making things happen.*

Interesting advice, and I think I will just follow it. After all, Stone Age man was far more peaceful than we think, remember for instance that it goes back to that times that our human species was domesticated by felines. So, thanks cat, may the force be with us, and let us try this radically new approach to our elementary particle questions.

But, what to start with? I am afraid that we will have to leave behind our beloved Stone Age, and skip the Middle Ages too, and get directly to our modern world, with all its complications. And here, good news, we have right away two interesting phenomena, that we are all aware of, and that can serve as starting point, for our investigations:

(1) You enter a room, and turn on the light. Well, there should be some little beasts, travelling from the light bulb to your eyes, telling you that the bulb is there.

(2) You plug in some electric device, say an angle grinder. Well, here there should be some little beasts too, escaping from the power outlet, and making the disc turn.

Which sounds very good, obviously what we have here are two types of particles, making things happen, as in Cat 1.1, let's call them photons for (1) and electrons for (2), that we can try to understand first. And for matter, we can leave that for later.

Shall we start with photons? Unfortunately, these were quite common in the Middle Ages, and in the Stone Age too, as light coming from the Sun, or from fires, so we will be facing here the challenge of doing better than our ancestors, who were more healthy, hard-working and intelligent than us, and perhaps better not count on that.

So, getting started for good now, electrons. And here, you don't necessarily need a power outlet for having them, a basic Van de Graaff machine, or just rubbing some suitable materials together, will do. Let us record this finding as follows:

FACT 1.2. *Each piece of matter has a charge $q\in\mathbb R$, which is normally neutral, $q=0$, but that we can make positive or negative, by using various methods. We say that responsible for the charge is the amount of electrons present, as follows:*

1. *When the matter lacks electrons, the charge is positive, $q>0$.*
2. *When there are more electrons than needed, the charge is negative, $q<0$.*

And, good news, this will be the starting point for the considerations in this book, the electrons, as defined above. Of course you might say, for instance if you are a math student used to a fair amount of exactness, in your learning, that what we say in Fact 1.2 is a bit borderline, for something to be labeled as axiomatic. But well, physics is not mathematics, it's sort of harder, when it comes to having things started, and that's what we have. Of course we will be back to it, with axioms, later, that is promised.

Moving ahead now, as our first result, due to Coulomb, and that will come as a physics fact instead of a mathematics theorem, because, well, I must admit that what we have in Fact 1.2 is indeed more than borderline, as axiomatics for a theory, we have:

FACT 1.3 (Coulomb law). *Any pair of charges $q_1, q_2 \in \mathbb{R}$ is subject to a force as follows, which is attractive if $q_1q_2 < 0$ and repulsive if $q_1q_2 > 0$,*

$$||F|| = K \cdot \frac{|q_1q_2|}{d^2}$$

where $d > 0$ is the distance between the charges, and $K > 0$ is a certain constant.

Observe the amazing similarity with the Newton law for gravity. However, as we will discover soon, passed a few simple facts, things will be far more complicated here.

As in the gravity case, the force F appearing above is understood to be parallel to the vector $x_2 - x_1 \in \mathbb{R}^3$ joining as $x_1 \to x_2$ the locations $x_1, x_2 \in \mathbb{R}^3$ of our charges, and by taking into account the attraction/repulsion rules above, we have:

PROPOSITION 1.4. *The Coulomb force of q_1 at x_1 acting on q_2 at x_2 is*

$$F = K \cdot \frac{q_1q_2(x_2 - x_1)}{||x_2 - x_1||^3}$$

with $K > 0$ being the Coulomb constant, as above.

PROOF. We have indeed the following computation:

$$\begin{aligned} F &= sgn(q_1q_2) \cdot ||F|| \cdot \frac{x_2 - x_1}{||x_2 - x_1||} \\ &= sgn(q_1q_2) \cdot K \cdot \frac{|q_1q_2|}{||x_2 - x_1||^2} \cdot \frac{x_2 - x_1}{||x_2 - x_1||} \\ &= K \cdot \frac{q_1q_2(x_2 - x_1)}{||x_2 - x_1||^3} \end{aligned}$$

Thus, we are led to the formula in the statement. □

All the above looks quite encouraging, and it is tempting to try now to develop some Newton type theory for electricity, inspired from gravity, with some mathematics, ellipses and everything. And, why not with charges replaced by single electrons $-$, and their antiparticles protons $+$, as a matter of further clarifying our axiomatics and formalism.

Fortunately cat, who is still present, quickly intervenes, and says:

CAT 1.5. *Don't even think about Newton type stuff, when charges move they produce magnetism, what you have in Fact 1.3 is just part of the story. And don't think either about talking about single electrons, Fact 1.3 is something of statistical nature.*

Which sounds quite frightening, if I understand well what we have to do is to build a theory of electrostatics, based on Fact 1.3, then further upgrade that into a theory of electromagnetism, and then, well, get beyond statistics, with some precise laws for the movements of the electrons, position, speed and everything, and why not about their shape too, are these round, or perhaps looking like strings, or other weird shapes.

Nevermind. We won't be scared by this, and slowly develop all the needed theory, but as a remark here for the math reader, coming as a continuation of the discussion started after Fact 1.2, sure I will keep my promise to come back with axioms, in due time, but I'm afraid that this will be probably towards the end of the book. Deal, I hope.

So, forgetting now about high hopes and abstractions, and getting back, with due modesty, to what we have in Fact 1.3, let us further explore the physics there, matter of having it perfectly understood. In relation with the value of the constant K appearing there, called Coulomb constant, things are a bit tricky, as follows:

FACT 1.6. *The Coulomb constant K is given by the formula*

$$K = 8.987\ 551\ 7923(14) \times 10^9$$

in standard units, with the charges being measured in coulombs C, given by

$$1C \simeq 6.241\ 509 \times 10^{18}\, e$$

where e is the elementary charge, namely minus that of an electron.

There are in fact several interesting things going on here. First, at the end you would say why not simply saying that e is the charge of the proton $+$, but the thing is that the proton $+$ and the electron $-$ do not have in fact the same exact charge, with sign switched, and the electron was preferred, as always, over the proton for formulating things.

Which takes us into the question of why the charge of the electron is $-$, instead of $+$. And there is a long story here, involving debates among the 18th century greats, and with a little bit of confusion being involved too, because the electrons $-$ are attracted by positive charges $q > 0$, and so observed around these positive charges $q > 0$, which might lead to the idea that they might have themselves a positive charge $+$, contributing to $q > 0$. Benjamin Franklin is generally credited for the $-$ convention.

Things were later restored in the early 20th century, with the atomic theory of Bohr and others, where electrons $-$ spin around a proton and neutron core $q > 0$, and with this picture, including the signs, looking like something very reasonable.

Passed all this, another peculiarity of Fact 1.6 comes in relation with the definition of the coulomb, which is in fact given by definition by an exact formula, namely:

$$1C = \frac{5 \times 10^{18}}{0.801\ 088\ 317}\, e$$

This in practice gives the following more precise formula for the coulomb, which shows that a charge of $1C$ is something fractionary, that cannot be realized in real life:

$$1C = 6241\ 509\ 074\ 460\ 762\ 607.776\, e$$

The problem comes from the following alternative definition of the coulomb, in terms of the ampere, which is something more complicated, that we will talk about later:

$$1C = 1A \cdot 1s$$

Hang on, we are not done yet. Adding to the confusion, the Coulomb constant is usually denoted K, but also k, or most often k_e, but in fact the most often is written in the following form, with ε_0 being the so-called permittivity of free space:

$$K = \frac{1}{4\pi\varepsilon_0}$$

And the story is not over here, because ε_0 itself is given by the following formula, with μ_0 being the magnetic permeability of free space, and c being the speed of light:

$$\varepsilon_0 = \frac{1}{\mu_0 c^2}$$

And we are surely still not done, because all the above discussion assumes that the other units that are used are standard, namely meter and second, and this is not always standard, due to the about 50 orders of magnitude physics has to deal with.

In any case, let us end this interesting discussion about units with something concrete, useful, and very illustrating, in relation with gravity, as follows:

Theorem 1.7. *The electrical repulsion between two electrons is about*

$$R = 10^{42}$$

times bigger than their gravitational attraction.

Proof. Consider indeed two electrons, having masses m, m and charges $-e, -e$. The magnitudes of the electric repulsion F_e and gravity attraction F_g are given by:

$$||F_e|| = \frac{Ke^2}{d^2} \quad , \quad ||F_g|| = \frac{Gm^2}{d^2}$$

Thus the ratio of forces R that we want to measure is given by:

$$R = \frac{||F_e||}{||F_g||} = \frac{Ke^2}{Gm^2}$$

Regarding now the data, this is as follows, with m at rest, and in standard units, namely meters and seconds, also kilograms, and including now coulombs too:

$$K = 8.897 \times 10^9 \quad , \quad G = 6.674 \times 10^{-11}$$

$$e = 1.602 \times 10^{-19} \quad , \quad m = 9.109 \times 10^{-31}$$

We obtain the following approximation for the ratio R considered above:

$$\begin{aligned} R &= \frac{8.897 \times 1.602^2}{6.674 \times 9.109^2} \times \frac{10^9 \times 10^{-38}}{10^{-11} \times 10^{-62}} \\ &= (4.123 \times 10^{-2}) \times 10^{44} \\ &\simeq 10^{42} \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

For adding to the picture, and in order to fully understand what that $R = 10^{42}$ number that we found truly means, let us complement the above result with:

PROPOSITION 1.8. *The universe, or at least the known universe, is about*

$$r = 10^{37}$$

bigger than a hydrogen atom, with this ratio being $10,000$ *smaller than R.*

PROOF. The radius of the hydrogen atom can be anywhere between $25 - 120$ pm, with 1 pm $= 10^{-12}$ m, depending on the convention used, with a commonly accepted figure being 53 pm, representing the mean distance between the proton and the electron. As for the radius of the known universe, again there is a story here, with a commonly accepted figure being 4.4×10^{26} m. Thus the ratio that we are interested in is:

$$r = \frac{4.4 \times 10^{26}}{53 \times 10^{-12}} \simeq 10^{37}$$

And this is 10,000 smaller than 10^{42}, as claimed. □

As a side comment, however, when speaking masses instead of sizes, the number $R = 10^{42}$ pales when compared to the mass of the known universe, counting ordinary mass only, accounting for 4.9%, divided by the mass of a hydrogen atom, which is:

$$\mathfrak{R} = \frac{1.5 \times 10^{53}}{1.8 \times 10^{-30}} \simeq 10^{83}$$

Getting back now to Theorem 1.7 as it is, let us point out that this is something not at all anecdotical, even in the context of the most abstract theoretical physics that you can ever imagine, not to say pure mathematics, because of the following rule of thumb, which is something widely agreed upon, by most of the scientists:

RULE 1.9. *Don't ever expect the mathematics and physics to be the same, over* 10 *orders of magnitude or so.*

In other words, with this in hand, Theorem 1.7 tells us a very interesting thing, namely that the mathematics and physics of the Coulomb force $F_e \sim 1/d^2$ will be in fact very different from the mathematics and physics of the Newton force $F_g \sim 1/d^2$. We will see in what follows that indeed it is so, but it is of course far better to be warned in advance of the potential difficulties on the way. So, Theorem 1.7 is something very smart.

As a further comment about this, as already mentioned, Rule 1.9 is something widely agreed upon, by applied mathematicians and physicists, chemists and engineers, and with that 10 orders of magnitude being usually replaced by something far sharper, of type 2-3. In theoretical physics and pure mathematics however, things can be quite wild, with quantum gravity research for instance trying to unify things which are about 50 orders of magnitude apart, no less than that. We will talk a bit about this later in this book.

Finally, you might perhaps still wonder if, even with Rule 1.9 and its potential consequences taken into account, all this detailed discussion about numbers and constants was really worth it. And here, I don't know what to say, personally I just love numbers and fractions, and back in the days, when I did my studies, the general rule was that who loves fractions goes to science, and who doesn't love them, goes to humanities.

But wait, cat just came back from his hunt, and when asked about this, says:

CAT 1.10. *Glad to hear that you like constants and fractions, and I'll have later a question for you, regarding the exact computation of a certain $\alpha \simeq 1/137$ constant.*

Interesting, he's probably talking here about the fine structure constant, whose exact computation is the main problem in mathematics and physics combined, and in science in general. At least according to what my physics colleagues discuss, in the coffee room at the university. More on this later in this book, when we will cross that $\alpha \simeq 1/137$ constant on our way, and let's hope of course that we will cross it sometimes soon.

1b. The Gauss law

Let us develop now the basic mathematics for electrostatics. We first have:

DEFINITION 1.11. *Given charges $q_1,\ldots,q_k \in \mathbb{R}$ located at positions $x_1,\ldots,x_k \in \mathbb{R}^3$, we define their electric field to be the vector function*

$$E(x) = K \sum_i \frac{q_i(x - x_i)}{||x - x_i||^3}$$

so that their force applied to a charge $Q \in \mathbb{R}$ positioned at $x \in \mathbb{R}^3$ is given by $F = QE$.

More generally, we will be interested in electric fields of various non-discrete configurations of charges, such as curves, surfaces and solid bodies. So, let us go ahead with:

DEFINITION 1.12. *The electric field of a charge configuration $L \subset \mathbb{R}^3$, with charge density function $\rho : L \to \mathbb{R}$, is the vector function*

$$E(x) = K \int_L \frac{\rho(z)(x - z)}{||x - z||^3}\, dz$$

so that the force of L applied to a charge Q positioned at x is given by $F = QE$.

It is most convenient now to forget about the charges, and focus on the corresponding electric fields E. These fields are by definition vector functions $E:\mathbb{R}^3\to\mathbb{R}^3$, with the convention that they take $\pm\infty$ values at the places where the charges are located, and intuitively, are best represented by their field lines, constructed as follows:

DEFINITION 1.13. *The field lines of $E:\mathbb{R}^3\to\mathbb{R}^3$ are the oriented curves*

$$\gamma\subset\mathbb{R}^3$$

pointing at every point $x\in\mathbb{R}^3$ at the direction of the field, $E(x)\in\mathbb{R}^3$.

As a basic example here, for one charge the field lines are the half-lines emanating from its position, oriented according to the sign of the charge:

$$\begin{matrix}\nwarrow&\uparrow&\nearrow\\ \leftarrow&\oplus&\rightarrow\\ \swarrow&\downarrow&\searrow\end{matrix}\qquad\qquad\begin{matrix}\searrow&\downarrow&\swarrow\\ \rightarrow&\ominus&\leftarrow\\ \nearrow&\uparrow&\nwarrow\end{matrix}$$

For two charges now, if these are of opposite signs, $+$ and $-$, you get a picture that you are very familiar with, namely that of the field lines of a bar magnet:

$$\begin{matrix}\nearrow&\nearrow&\rightarrow&\rightarrow&\rightarrow&\rightarrow&\searrow&\searrow\\ \nwarrow&\uparrow&\nearrow&\rightarrow&\rightarrow&\searrow&\downarrow&\swarrow\\ \leftarrow&\oplus&\rightarrow&\rightarrow&\rightarrow&\rightarrow&\ominus&\leftarrow\\ \swarrow&\downarrow&\searrow&\rightarrow&\rightarrow&\nearrow&\uparrow&\nwarrow\\ \searrow&\searrow&\rightarrow&\rightarrow&\rightarrow&\rightarrow&\nearrow&\nearrow\end{matrix}$$

If the charges are $+,+$ or $-,-$, you get something of similar type, but repulsive this time, with the field lines emanating from the charges being no longer shared:

$$\begin{matrix}\leftarrow&\nwarrow&\nwarrow\\ &\uparrow&\nearrow\\ \leftarrow&\oplus&\\ &\downarrow&\searrow\\ \leftarrow&\swarrow&\swarrow\end{matrix}\qquad\qquad\begin{matrix}\nearrow&\nearrow&\rightarrow\\ \nwarrow&\uparrow&\\ &\oplus&\rightarrow\\ \swarrow&\downarrow&\\ \searrow&\searrow&\rightarrow\end{matrix}$$

The field lines obviously do not encapsulate the whole information about the field, with the direction of each vector $E(x)\in\mathbb{R}^3$ being there, but with the magnitude $||E(x)||\geq 0$ of this vector missing. However, in practice, when drawing, when picking up uniformly radially spaced field lines around each charge, and with the number of these lines being proportional to the magnitude of the charge, and then completing the picture, the density of the field lines around each point $x\in\mathbb{R}$ will give you then the magnitude $||E(x)||\geq 0$ of the field there, up to a scalar. With this being, of course, very practical.

Let us summarize these observations as a mathematical result, follows:

PROPOSITION 1.14. *Given an electric field $E:\mathbb{R}^3\to\mathbb{R}^3$, the knowledge of its field lines is the same as the knowledge of the composition*

$$nE:\mathbb{R}^3\to\mathbb{R}^3\to S$$

where $S\subset\mathbb{R}^3$ is the unit sphere, and $n:\mathbb{R}^3\to S$ is the rescaling map, namely:

$$n(x)=\frac{x}{||x||}$$

However, in practice, when the field lines are accurately drawn, the density of the field lines gives you the magnitude of the field, up to a scalar.

PROOF. Thus follows indeed from the above discussion. It is possible to be a bit more mathematical here, but we will not really need this, in what follows. □

Let us introduce now a key definition, as follows:

DEFINITION 1.15. *The flux of an electric field $E:\mathbb{R}^3\to\mathbb{R}^3$ through a surface $S\subset\mathbb{R}^3$, assumed to be oriented, is the quantity*

$$\Phi_E(S)=\int_S < E(x),n(x) > dx$$

with $n(x)$ being unit vectors orthogonal to S, following the orientation of S. Intuitively, the flux measures the signed number of field lines crossing S.

Here by orientation of S we mean precisely the choice of unit vectors $n(x)$ as above, orthogonal to S, which must vary continuously with x. For instance a sphere has two possible orientations, one with all these vectors $n(x)$ pointing inside, and one with all these vectors $n(x)$ pointing outside. More generally, any surface has locally two possible orientations, so if it is connected, it has two possible orientations. In what follows the convention is that the closed surfaces are oriented with each $n(x)$ pointing outside.

As a first illustration, let us do a basic computation, as follows:

PROPOSITION 1.16. *For a point charge $q\in\mathbb{R}$ at the center of a sphere S,*

$$\Phi_E(S)=\frac{q}{\varepsilon_0}$$

where the constant is $\varepsilon_0=1/(4\pi K)$, independently of the radius of S.

PROOF. Assuming that S has radius r, we have the following computation:

$$\begin{aligned}
\Phi_E(S) &= \int_S < E(x), n(x) > dx \\
&= \int_S \left\langle \frac{Kqx}{r^3}, \frac{x}{r} \right\rangle dx \\
&= \int_S \frac{Kq}{r^2}\, dx \\
&= \frac{Kq}{r^2} \times 4\pi r^2 \\
&= 4\pi Kq
\end{aligned}$$

Thus with $\varepsilon_0 = 1/(4\pi K)$ as above, we obtain the result. □

More generally now, we have the following result:

THEOREM 1.17. *The flux of a field E through a sphere S is given by*

$$\Phi_E(S) = \frac{Q_{enc}}{\varepsilon_0}$$

where Q_{enc} is the total charge enclosed by S, and $\varepsilon_0 = 1/(4\pi K)$.

PROOF. This can be done in several steps, as follows:

(1) Before jumping into computations, let us do some manipulations. First, by discretizing the problem, we can assume that we are dealing with a system of point charges. Moreover, by additivity, we can assume that we are dealing with a single charge. And if we denote by $q \in \mathbb{R}$ this charge, located at $v \in \mathbb{R}^3$, we want to prove that we have the following formula, where $B \subset \mathbb{R}^3$ denotes the ball enclosed by S:

$$\Phi_E(S) = \frac{q}{\varepsilon_0}\, \delta_{v \in B}$$

(2) By linearity we can assume that we are dealing with the unit sphere S. Moreover, by rotating we can assume that our charge q lies on the Ox axis, that is, that we have $v = (r, 0, 0)$ with $r \geq 0$, $r \neq 1$. The formula that we want to prove becomes:

$$\Phi_E(S) = \frac{q}{\varepsilon_0}\, \delta_{r<1}$$

(3) Let us start now the computation. With $u=(x,y,z)$, we have:

$$\begin{aligned}
\Phi_E(S) &= \int_S < E(u),u > du\\
&= \int_S \left\langle \frac{Kq(u-v)}{||u-v||^3},u\right\rangle du\\
&= Kq\int_S \frac{< u-v,u >}{||u-v||^3}\,du\\
&= Kq\int_S \frac{1-< v,u >}{||u-v||^3}\,du\\
&= Kq\int_S \frac{1-rx}{(1-2xr+r^2)^{3/2}}\,du
\end{aligned}$$

(4) In order to compute the above integral, we can use spherical coordinates for the unit sphere S. Our integral from (3) becomes in this way:

$$\begin{aligned}
\Phi_E(S) &= Kq\int_S \frac{1-rx}{(1-2xr+r^2)^{3/2}}\,du\\
&= Kq\int_0^{2\pi}\int_0^{\pi} \frac{1-r\cos s}{(1-2r\cos s+r^2)^{3/2}}\cdot\sin s\,ds\,dt\\
&= 2\pi Kq\int_0^{\pi} \frac{(1-r\cos s)\sin s}{(1-2r\cos s+r^2)^{3/2}}\,ds\\
&= \frac{q}{2\varepsilon_0}\int_0^{\pi} \frac{(1-r\cos s)\sin s}{(1-2r\cos s+r^2)^{3/2}}\,ds
\end{aligned}$$

(5) The point now is that the integral on the right can be computed with the change of variables $x=\cos s$. Indeed, we have $dx=-\sin s\,ds$, and we obtain:

$$\begin{aligned}
\int_0^{\pi} \frac{(1-r\cos s)\sin s}{(1-2r\cos s+r^2)^{3/2}}\,ds &= \int_{-1}^{1} \frac{1-rx}{(1-2rx+r^2)^{3/2}}\,dx\\
&= \left[\frac{x-r}{\sqrt{1-2rx+r^2}}\right]_{-1}^{1}\\
&= \frac{1-r}{\sqrt{1-2r+r^2}}-\frac{-1-r}{\sqrt{1+2r+r^2}}\\
&= \frac{1-r}{|1-r|}+1\\
&= 2\delta_{r<1}
\end{aligned}$$

Thus, we are led to the formula in the statement. □

As a comment here, at $r=1$, which is normally avoided by our problematics, the integral I_r computed in (5) above converges too, and can be evaluated as follows:

$$I_1=\left[\frac{x-1}{\sqrt{2-2x}}\right]_{-1}^1=\left[-\sqrt{\frac{1-x}{2}}\,\right]_{-1}^1=1$$

Thus, we have the correct middle step between the $0,2$ values of the integral I_r, and getting back now to the flux, at $r=1$ we formally have $\Phi_E(S)=q/(2\varepsilon_0)$, which again is the correct middle step between the $0,q/\varepsilon_0$ values of the flux.

Even more generally now, we have the following result, due to Gauss:

THEOREM 1.18 (Gauss law). *The flux of a field E through a surface S is given by*

$$\Phi_E(S)=\frac{Q_{enc}}{\varepsilon_0}$$

where Q_{enc} is the total charge enclosed by S, and $\varepsilon_0=1/(4\pi K)$.

PROOF. This basically follows from Theorem 1.17, or even from Proposition 1.16, by adding to the results there a number of new ingredients, as follows:

(1) Our first claim is that given a closed surface S, with no charges inside, the flux through it of any choice of external charges vanishes:

$$\Phi_E(S)=0$$

This claim is indeed supported by the intuitive interpretation of the flux, as corresponding to the signed number of field lines crossing S. Indeed, any field line entering as $+$ must exit somewhere as $-$, and vice versa, so when summing we get 0.

(2) In practice now, in order to prove this rigorously, there are several ways. A first argument, which is quite elementary, is the one used by Feynman in [**34**], based on the fact that, due to $F\sim 1/d^2$, local deformations of S will leave invariant the flux, and so in the end we are left with a rotationally invariant surface, where the result is clear.

(3) A second argument, which basically uses the same idea, but is perhaps a bit more robust, is by redoing the computations in the proof of Theorem 1.17, by assuming this time that the integration takes place on an arbitrary surface as follows:

$$S_\lambda=\left\{\lambda(u)u\Big|u\in S\right\}$$

To be more precise, here $\lambda:S\to(0,\infty)$ is a certain function, defining the surface, whose derivatives will appear both in the construction of the normal vectors $n(x)$ with $x=\lambda(u)u$, and in the Jacobian of the change of variables $x\to u$, and in the end, when integrating over S as in the proof of Theorem 1.17, this function λ dissapears.

(4) A third argument, used by basically all electrodynamics books at the graduate level, and by some undergraduate books too, is by using heavy calculus, namely partial integration in 3D, and we will discuss this later, more in detail, a bit later.

(5) A fourth argument is by following the idea in (1), namely carefully axiomatizing the field lines, and their relation with the field, and then obtaining $\Phi_E(S) = 0$ by using the in-and-out trick in (1), as explained for instance by Griffiths in [42].

(6) To summmarize, we are led to the conclusion that given a closed surface S, with no charges inside, the flux through it of any choice of external charges vanishes:

$$\Phi_E(S) = 0$$

(7) The point now is that, with this and Proposition 1.16 in hand, we can finish by using a standard math trick. Let us assume indeed, by discretizing, that our system of charges is discrete, consisting of enclosed charges $q_1, \ldots, q_k \in \mathbb{R}$, and an exterior total charge Q_{ext}. We can surround each of $q_1, \ldots, q_k$ by small disjoint spheres $U_1, \ldots, U_k$, chosen such that their interiors do not touch S, and we have:

$$\begin{aligned}
\Phi_E(S) &= \Phi_E(S - \cup U_i) + \Phi_E(\cup U_i) \\
&= 0 + \Phi_E(\cup U_i) \\
&= \sum_i \Phi_E(U_i) \\
&= \sum_i \frac{q_i}{\varepsilon_0} \\
&= \frac{Q_{enc}}{\varepsilon_0}
\end{aligned}$$

(8) To be more precise, in the above the union $\cup U_i$ is a usual disjoint union, and the flux is of course additive over components. As for the difference $S - \cup U_i$, this is by definition the disjoint union of S with the disjoint union $\cup(-U_i)$, with each $-U_i$ standing for U_i with orientation reversed, and since this difference has no enclosed charges, the flux through it vanishes by (6). Finally, the end makes use of Proposition 1.16. □

We have the following point of view on the Gauss formula, more conceptual:

THEOREM 1.19 (Gauss). *Given an electric potential E, its divergence is given by*

$$< \nabla, E >= \frac{\rho}{\varepsilon_0}$$

where ρ denotes as usual the charge distribution. Also, we have

$$\nabla \times E = 0$$

meaning that the curl of E vanishes.

PROOF. The first formula, called Gauss law in differential form, follows from:

$$\begin{aligned}\int_B <\nabla, E> &= \int_S <E(x), n(x)> dx\\ &= \Phi_E(S)\\ &= \frac{Q_{enc}}{\varepsilon_0}\\ &= \int_B \frac{\rho}{\varepsilon_0}\end{aligned}$$

As a side remark, the Gauss law in differential form can be established as well directly, with the computation, involving a Dirac mass, being as follows:

$$\begin{aligned}<\nabla, E>(x) &= \left\langle \nabla, K\int_{\mathbb{R}^3} \frac{\rho(z)(x-z)}{||x-z||^3}\,dz\right\rangle\\ &= K\int_{\mathbb{R}^3}\left\langle \nabla, \frac{x-z}{||x-z||^3}\right\rangle \rho(z)\,dz\\ &= K\int_{\mathbb{R}^3} 4\pi\delta_x\cdot\rho(z)dz\\ &= 4\pi K\int_{\mathbb{R}^3}\delta_x\,\rho(z)dz\\ &= \frac{\rho(x)}{\varepsilon_0}\end{aligned}$$

Regarding the curl, by discretizing and linearity we can assume that we are dealing with a single charge q, positioned at 0. We have, by using spherical coordinates r, s, t:

$$\begin{aligned}\int_a^b <E(x), dx> &= \int_a^b \left\langle \frac{Kqx}{||x||^3}, dx\right\rangle\\ &= \int_a^b \left\langle \frac{Kq}{r^2}\cdot\frac{x}{||x||}, dx\right\rangle\\ &= \int_a^b \frac{Kq}{r^2}\,dr\\ &= \left[-\frac{Kq}{r}\right]_a^b\\ &= Kq\left(\frac{1}{r_a}-\frac{1}{r_b}\right)\end{aligned}$$

In particular the integral of E over any closed loop vanishes, and by using now the Stokes theorem, we conclude that the curl of E vanishes, as stated. $\square$

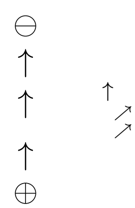
1c. Magnetic fields

Just by feeding a light bulb with a battery, and looking at the cables, and playing a bit with them, we are led to the following interesting conclusion:

FACT 1.20. *Parallel electric currents in opposite directions repel, and parallel electric currents in the same direction attract.*

We can in fact say even more, by further playing with the cables, armed this time with a compass. The conclusion is that each cable produces some kind of "magnetic field" around it, which interestingly, is not oriented in the direction of the current, but is rather orthogonal to it, given by the right-hand rule, as follows:

FACT 1.21 (Right-hand rule). *An electric current produces a magnetic field B which is orthogonal to it, whose direction is given by the right-hand rule,*



namely wrap your right hand around the cable, with the thumb pointing towards the direction of the current, and the movement of your wrist will give you the direction of B.

This is something even more interesting than Fact 1.20. Indeed, not only moving charges produce something new, that we'll have to investigate, but they know well about 3D, and more specifically about orientation there, left and right, even if living in 1D.

And isn't this amazing. Let us summarize this discussion with:

FACT 1.22. *Charges are smart, they know about $3D$, and about left and right.*

With this discussed, let us go ahead and investigate the charge smartness, and more specifically the magnetic fields discovered above. In order to evaluate the properties of the magnetic fields B coming from electric currents, the simplest way is that of making them act on exterior charges Q. And we have here the following formula:

FACT 1.23 (Lorentz force law). *The magnetic force on a charge Q, moving with velocity v in a magnetic field B, is as follows, with $\times$ being a vector product:*

$$F_m = (v \times B)Q$$

In the presence of both electric and magnetic fields, the total force on Q is

$$F = (E + v \times B)Q$$

where E is the electric field.

Here the occurrence of the vector product $\times$ is not surprising, due to the fact that the right-hand rule appears both in Fact 1.21, and in the definition of $\times$. In fact, the Lorentz force law is just a fancy mathematical reformulation of Fact 1.21, telling us that, once the magnetic fields B duly axiomatized, and with this being a remaining big problem, their action on exterior charges Q will be proportional to the charge, $F_m \sim Q$, and with the orientation and magnitude coming from the 3D of the right-hand rule in Fact 1.21.

As an interesting application of the Lorentz force law, we have:

THEOREM 1.24. *Magnetic forces do not work.*

PROOF. This might seem quite surprising, but the math is there, as follows:

$$\begin{aligned} dW_m &= < F_m, dx > \\ &= < (v \times B)Q, v\,dt > \\ &= Q < v \times B, v > dt \\ &= 0 \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Moving ahead now, let us talk axiomatization of electric currents, including units. We have here the following definition, clarifying our previous discussion about coulombs:

DEFINITION 1.25. *The electric currents I are measured in amperes, given by:*

$$1A = 1C/s$$

As a consequence, the coulomb is given by $1C = 1A \times 1s$.

With this notion in hand, let us keep building the math and physics of magnetism. So, assume that we are dealing with an electric current I, producing a magnetic field B. In this context, the Lorentz force law from Fact 1.23 takes the following form:

$$F_m = \int (dx \times B)I$$

The current being typically constant along the wire, this reads:

$$F_m = I \int dx \times B$$

We can deduce from this the following result:

THEOREM 1.26. *The volume current density J satisfies*

$$< \nabla, J >= -\dot{\rho}$$

called continuity equation.

PROOF. We have indeed the following computation, for any surface S enclosing a volume V, based on the Lorentz force law, and on the overall chage conservation:

$$\begin{aligned}\int_V <\nabla, J> &= \int_S <J, n(x)> dx \\ &= -\frac{d}{dt}\int_V \rho \\ &= -\int_V \dot{\rho}\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Moving ahead now, let us formulate the following definition:

DEFINITION 1.27. *The realm of magnetostatics is that of the steady currents,*

$$\dot{\rho} = 0 \quad , \quad \dot{J} = 0$$

in analogy with electrostatics, dealing with fixed charges.

As a first observation, for steady currents the continuity equation reads:

$$<\nabla, J> = 0$$

We have here a bit of analogy between electrostatics and magnetostatics, and with this in mind, let us look for equations for the magnetic field B. We have:

FACT 1.28 (Biot-Savart law). *The magnetic field of a steady line current is given by*

$$B = \frac{\mu_0}{4\pi}\int \frac{I \times x}{||x||^3}$$

where μ_0 is a certain constant, called the magnetic permeability of free space.

This law not only gives us all we need, for studying steady currents, and we will talk about this in a moment, with math and everything, but also makes an amazing link with the Coulomb force law, due to the following fact, which is also part of it:

FACT 1.29 (Biot-Savart, continued). *The electric permittivity of free space ε_0 and the magnetic permeability of free space μ_0 are related by the formula*

$$\varepsilon_0 \mu_0 = \frac{1}{c^2}$$

where c is as usual the speed of light.

This is something truly remarkable, and very deep, that will have numerous consequences, in what follows, be that for investigating phenomena like radiation, or for making the link with Einstein's relativity theory, both crucially involving c.

But, first of all, this is certainly an invitation to rediscuss units and constants, as a continuation of our previous discussion on this topic. In what regards the units, we won't be impressed by the ampere, and keep using the coulomb, as a main unit:

CONVENTIONS 1.30. *We keep using standard units, namely meters, kilograms, seconds, along with the coulomb, defined by the following exact formula*

$$1C = \frac{5 \times 10^{18}}{0.801\ 088\ 317}\ e$$

with e being minus the charge of the electron, which in practice means:

$$1C \simeq 6.241 \times 10^{18}\, e$$

We will also use the ampere, defined as $1A = 1C/s$, for measuring currents.

In what regards constants, however, time to do some cleanup. We have been boycotting for some time already the Coulomb constant K, and using instead $\varepsilon_0 = 1/(4\pi K)$, due to the ubiquitous 4π factor, first appearing as the area of the unit sphere, $A = 4\pi$, in the computation for the Gauss law for the unit sphere. Together with Fact 1.29, this suggests using the numbers ε_0, μ_0 as our new constants, by always keeping in mind $\varepsilon_0\mu_0 = 1/c^2$, and by having of course c as constant too, and we are led in this way into:

CONVENTIONS 1.31. *We use from now on as constants the electric permittivity of free space ε_0 and the magnetic permeability of free space μ_0, given by*

$$\varepsilon_0 = 8.854\ 187\ 8128(13) \times 10^{-12}$$

$$\mu_0 = 1.256\ 637\ 062\ 12(19) \times 10^{-6}$$

as well as the speed of light, given by the following exact formula,

$$c = 299\,792\,458$$

which are related by $\varepsilon_0\mu_0 = 1/c^2$, and with the Coulomb constant being $K = 1/(4\pi\varepsilon_0)$.

Observe in passing that we are not messing up our figures, which can be quite often the case in this type of situation, because according to our data, and by truncating instead of rounding, as busy theoretical physicists usually do, we have:

$$\varepsilon_0\mu_0 c^2 = 8.854 \times 1.256 \times 2.997^2 \times 10^{16-12-6} = 0.998$$

Getting back now to theory and math, the Biot-Savart law has as consequence:

THEOREM 1.32. *We have the following formula:*

$$< \nabla, B >= 0$$

That is, the divergence of the magnetic field vanishes.

PROOF. We recall that the Biot-Savart law tells us that the magnetic field B of a steady line current I is given by the following formula:

$$B = \frac{\mu_0}{4\pi} \int \frac{I \times x}{||x||^3}$$

By applying the divergence operator to this formula, we obtain:

$$\begin{aligned} < \nabla, B > &= \frac{\mu_0}{4\pi} \int \left\langle \nabla, \frac{I \times x}{||x||^3} \right\rangle \\ &= \frac{\mu_0}{4\pi} \int \left\langle \nabla \times J, \frac{x}{||x||^3} \right\rangle - \left\langle \nabla \times \frac{x}{||x||^3}, J \right\rangle \\ &= \frac{\mu_0}{4\pi} \int \left\langle 0, \frac{x}{||x||^3} \right\rangle - \langle 0, J \rangle \\ &= 0 \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

Regarding now the curl, we have here a similar result, as follows:

THEOREM 1.33 (Ampère law). *We have the following formula,*

$$\nabla \times B = \mu_0 J$$

computing the curl of the magnetic field.

PROOF. Again, we use the Biot-Savart law, telling us that the magnetic field B of a steady line current I is given by the following formula:

$$B = \frac{\mu_0}{4\pi} \int \frac{I \times x}{||x||^3}$$

By applying the curl operator to this formula, we obtain:

$$\begin{aligned} \nabla \times B &= \frac{\mu_0}{4\pi} \int \nabla \times \frac{I \times x}{||x||^3} \\ &= \frac{\mu_0}{4\pi} \int \left\langle \nabla, \frac{x}{||x||^3} \right\rangle J - < \nabla, J > \frac{x}{||x||^3} \\ &= \frac{\mu_0}{4\pi} \int 4\pi \delta_x \cdot J - \frac{\mu_0}{4\pi} \cdot 0 \\ &= \mu_0 \int \delta_x \cdot J \\ &= \mu_0 J \end{aligned}$$

Thus, we are led to the conclusion in the statement. □

As a conclusion to all this, the equations of magnetostatics are as follows:

THEOREM 1.34. *The equations of magnetostatics are*

$$< \nabla, B >= 0 \quad , \quad \nabla \times B = \mu_0 J$$

with the second equation being the Ampère law.

PROOF. This follows indeed from the above discussion, and more specifically from Theorem 1.32 and Theorem 1.33, which both follow from the Biot-Savart law. □

Observe the obvious analogy with the Gauss equations of electrostatics, namely:

$$< \nabla, E >= \frac{\rho}{\varepsilon_0} \quad , \quad \nabla \times E = 0$$

As a conclusion to all this, looks like someone has played here with basic 3D math, vectors, products and so on, and messed them up, as for electrostatics to become magnetostatics, and vice versa. More on this later, when talking about unification.

1d. Maxwell equations

Quite remarkably, and at the origin of all modern theory of electromagnetism, and of any type of modern electrical engineering too, we have:

FACT 1.35 (Faraday laws). *The following happen:*

(1) *Moving a wire loop γ through a magnetic field B produces a current through γ.*
(2) *Keeping γ fixed, but changing the strength of B, produces too current through γ.*

In order to understand what is going on here, let us start with the simplest electric loop that we know, namely a battery feeding a light bulb:

$$\begin{matrix}
\oplus & \to & \to & \to & \to & \to & \to & \to & \searrow \\
\star & & & & & & & & \odot \\
\ominus & \leftarrow & \leftarrow & \leftarrow & \leftarrow & \leftarrow & \leftarrow & \leftarrow & \swarrow
\end{matrix}$$

Here the star stands for the fact that we don't really know what happens inside the battery, typically a complicated chemical process. Nor we will actually worry about the bulb, let us simply assume that this bulb does not exist at all. We will be interested in the force driving the current around the loop, and we have here:

PROPOSITION 1.36. *When writing the force driving the current through a loop γ as*

$$F = F_\star + F_e$$

with $F_\star$ coming from the source, and F_e coming from the loop, the quantity

$$\mathcal{E} = \int_\gamma < F(x), dx >$$

called electromotive force, or emf of the loop, is simply obtained by integrating $F_\star$.

PROOF. We have indeed the following computation, based on the fact that F_e being an electrostatic force, its integral over the loop vanishes:

$$\begin{aligned}\mathcal{E} &= \int_\gamma < F(x), dx > \\ &= \int_\gamma < F_\star(x), dx > + \int_\gamma < F_e(x), dx > \\ &= \int_\gamma < F_\star(x), dx > +0 \\ &= \int_\gamma < F_\star(x), dx >\end{aligned}$$

Thus, we have our result, and with the remark of course that the emf $\mathcal{E} \in \mathbb{R}$ is not really a force, but this is the standard terminology, and we will use it. ☐

In relation now with the Faraday principles from Fact 1.35, these can be fine-tuned, and reformulated in terms of the emf, in the following way:

FACT 1.37 (Faraday). *The emf of a loop γ moving through a magnetic field B is*

$$\mathcal{E} = -\dot{\Phi}$$

where Φ is the flux of the field B through the loop γ, given by:

$$\Phi = \int_\gamma < B(x), dx >$$

As for the emf of a fixed loop γ in a changing magnetic field B, this is

$$\mathcal{E} = -\int_\gamma < \dot{B}(x), dx >$$

which by Stokes is equivalent to the Faraday law $\Delta \times E = -\dot{B}$.

All the above is very useful in electromechanics, for construcing electric motors. Getting back now to theory, the above considerations lead to the following conclusion:

FACT 1.38 (Faraday). *In the context of moving chages, the electrostatics law*

$$\nabla \times E = 0$$

must be replaced by the following equation,

$$\nabla \times E = -\dot{B}$$

called Faraday law.

Along the same lines, and following now Maxwell, there is a correction as well to be made to the main law of magnetostatics, namely the Ampère law, as follows:

FACT 1.39 (Maxwell). *In the context of moving chages, the Ampère law*

$$\nabla \times B = \mu_0 J$$

must be replaced by the following equation,

$$\nabla \times B = \mu_0(J + \varepsilon_0 \dot{E})$$

called Ampère law with Maxwell correction term.

Now by putting everything together, and perhaps after doublecheking as well, with all sorts of experiments, that the remaining electrostatics and magnetostatics laws, that we have not modified, work indeed fine in the dynamic setting, we obtain:

THEOREM 1.40 (Maxwell). *Electrodynamics is governed by the formulae*

$$< \nabla, E >= \frac{\rho}{\varepsilon_0} \quad , \quad < \nabla, B >= 0$$

$$\nabla \times E = -\dot{B} \quad , \quad \nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \dot{E}$$

called Maxwell equations.

PROOF. This follows indeed from the above, the details being as follows:

(1) The first equation is the Gauss law, that we know well.

(2) The second equation is something anonymous, that we know well too.

(3) The third equation is a previously anonymous law, modified into Faraday's law.

(4) And the fourth equation is the Ampère law, as modified by Maxwell. □

The Maxwell equations are in fact not the end of everything, because in the context of the 2-body problem, they must be replaced by quantum mechanics. More later.

1e. Exercises

Exercises:

EXERCISE 1.41.

EXERCISE 1.42.

EXERCISE 1.43.

EXERCISE 1.44.

EXERCISE 1.45.

EXERCISE 1.46.

EXERCISE 1.47.

EXERCISE 1.48.

Bonus exercise.

CHAPTER 2

Light and heat

2a. Radiation

We have accumulated so far some good knowledge of the electron, which is the elementary particle responsible for electricity and magnetism. Or rather, to be fully correct, we know now the exact statistical properties of the big groups of electrons, probably millions or so, kept static or moving, with these being the Maxwell equations.

In this chapter we talk about the other elementary particle that we were having in mind, namely the photon, which is by definition responsible for light. But since light is obviously related to heat, with light heating the materials that it touches, and also with heating, or at least heating a lot, producing light, we are right away in front of a dillema: in order to get started, shall we talk about light, or about heat?

Not very clear all this, and as usual in such difficult situations, we will ask the cat. And cat, who is sunbathing outside, in the Summer afternoon, looks at me from behind his Ray Bans, sips from his cocktail, thinks for a while, and eventually declares:

CAT 2.1. *Start with what you know, the Maxwell equations, involving the speed of light* $c = 299,792,458$ *via Biot-Savart, that is pretty decent for a slow human.*

Thanks cat, enjoy the end of the afternoon, good piece of advice that seems to be, and let us get to work. So, forgetting now about what we wanted to do, photon either way, light or heat, let's leave that for later, let us focus on the Maxwell equations, and try to make appear c. And here, surprise, things are quite simple, as follows:

THEOREM 2.2. *In regions of space where there is no charge or current present the Maxwell equations for electrodynamics read*

$$< \nabla, E > = < \nabla, B > = 0$$

$$\nabla \times E = -\dot{B} \quad , \quad \nabla \times B = \dot{E}/c^2$$

and both the electric field E *and magnetic field* B *are subject to the wave equation*

$$\ddot{\varphi} = c^2 \Delta \varphi$$

with $\Delta = \sum_i d^2/dx_i^2$ *being the Laplace operator, and* c *the speed of light.*

PROOF. Under the circumstances in the statement, namely no charge or current present, the Maxwell equations simply read, taking into account $\mu_0\varepsilon_0 = 1/c^2$:

$$< \nabla, E >=< \nabla, B >= 0$$

$$\nabla \times E = -\dot{B} \quad , \quad \nabla \times B = \dot{E}/c^2$$

Now by applying the curl operator to the last two equations, we obtain:

$$\nabla \times (\nabla \times E) = -\nabla \times \dot{B} = -(\nabla \times B)' = -\ddot{E}/c^2$$

$$\nabla \times (\nabla \times B) = \nabla \times \dot{E}/c^2 = (\nabla \times E)'/c^2 = -\ddot{B}/c^2$$

But the double curl operator is subject to the following formula:

$$\nabla \times (\nabla \times \varphi) = \nabla < \nabla, \varphi > -\Delta\varphi$$

Now by using the first two equations, we are led to the conclusion in the statement. □

So, what is light? Light is the wave predicted by Theorem 2.2, travelling in vacuum at the maximum possible speed, c, and with an important extra property being that it depends on a real positive parameter, that can be called, upon taste, frequency, wavelength, or color. And in what regards the creation of light, the mechanism here is as follows:

FACT 2.3. *An accelerating or decelerating charge produces electromagnetic radiation, called light, whose frequency and wavelength can be explicitely computed.*

This phenomenon can be observed is a variety of situations, such as the usual light bulbs, where electrons get decelerated by the filament, acting as a resistor, or in usual fire, which is a chemical reaction, with the electrons moving around, as they do in any chemical reaction, or in more complicated machinery like nuclear plants, particle accelerators, and so on, leading there to all sorts of eerie glows, of various colors.

Getting back now to Fact 2.3, in its general form, as stated above, this is something which can be deduced via some math, based on the Maxwell equations.

Many other things involving c can be said, regarding the Maxwell equations, with for instance the invariance of these equations under the Lorentz transformations of space-time, and with these eventually coming, via Einstein's theory of relativity, from $v < c$. But all this is rather advanced material, for later. For now, the above will do.

2b. Color, polarization

Moving ahead, let us go back to the wave equation found in Theorem 2.2, and try to understand its simplest solutions. In 1D, the situation is as follows:

Theorem 2.4. *The 1D wave equation, with speed v, namely*

$$\ddot{\varphi} = v^2 \frac{d^2\varphi}{dx^2}$$

has as basic solutions the following functions,

$$\varphi(x) = A\cos(kx - wt + \delta)$$

with A being called amplitude, $kx-wt+\delta$ being called the phase, k being the wave number, w being the angular frequency, and δ being the phase constant. We have

$$\lambda = \frac{2\pi}{k} \quad , \quad T = \frac{2\pi}{kv} \quad , \quad \nu = \frac{1}{T} \quad , \quad w = 2\pi\nu$$

relating the wavelength λ, period T, frequency ν, and angular frequency w. Moreover, any solution of the wave equation appears as a linear combination of such basic solutions.

Proof. There are several things going on here, the idea being as follows:

(1) Our first claim is that the function φ in the statement satisfies indeed the wave equation, with speed $v = w/k$. For this purpose, observe that we have:

$$\ddot{\varphi} = -w^2\varphi \quad , \quad \frac{d^2\varphi}{dx^2} = -k^2\varphi$$

Thus, the wave equation is indeed satisfied, with speed $v = w/k$:

$$\ddot{\varphi} = \left(\frac{w}{k}\right)^2 \frac{d^2\varphi}{dx^2} = v^2 \frac{d^2\varphi}{dx^2}$$

(2) Regarding now the other things in the statement, all this is basically terminology, which is very natural, when thinking how $\varphi(x) = A\cos(kx - wt + \delta)$ propagates.

(3) Finally, the last assertion is something standard, coming from Fourier analysis, that we will not really need, in what follows. □

As a first observation, the above result invites the use of complex numbers. Indeed, we can write the solutions that we found in a more convenient way, as follows:

$$\varphi(x) = Re\left[A\, e^{i(kx-wt+\delta)}\right]$$

And we can in fact do even better, by absorbing the quantity $e^{i\delta}$ into the amplitude A, which becomes now a complex number, and writing our formula as:

$$\varphi = Re(\widetilde{\varphi}) \quad , \quad \widetilde{\varphi} = \widetilde{A}e^{i(kx-wt)}$$

Moving ahead now towards electromagnetism and 3D, let us formulate:

Definition 2.5. *A monochromatic plane wave is a solution of the 3D wave equation which moves in only 1 direction, making it in practice a solution of the 1D wave equation, and which is of the special from found in Theorem 2.4, with no frequencies mixed.*

In other words, we are making here two assumptions on our wave. First is the 1-dimensionality assumption, which gets us into the framework of Theorem 2.4. And second is the assumption, in connection with the Fourier decomposition result from the end of Theorem 2.4, that our solution is of "pure" type, meaning a wave having a well-defined wavelenght and frequency, instead of being a "packet" of such pure waves.

All this is still mathematics, and making now the connection with physics and electromagnetism, and more specifically with Theorem 2.4 and Fact 2.3, we have:

FACT 2.6. *Physically speaking, a monochromatic plane wave is the electromagnetic radiation appearing as in Theorem 2.2 and Fact 2.3, via equations of type*

$$E = Re(\widetilde{E}) \quad : \quad \widetilde{E} = \widetilde{E}_0 \, e^{i(<k,x>-wt)}$$

$$B = Re(\widetilde{B}) \quad : \quad \widetilde{B} = \widetilde{B}_0 \, e^{i(<k,x>-wt)}$$

with the wave number being now a vector, $k \in \mathbb{R}^3$. Moreover, it is possible to add to this an extra parameter, accounting for the possible polarization of the wave.

To be more precise, what we are doing here is to import the conclusions of our mathematical discussion so far, from Theorem 2.4 and Definition 2.5, into the context of our original physics discussion, from Fact 2.3. And also to add an extra twist coming from physics, and more specifically from the notion of polarization.

In any case, we have now a decent intuition about what light is, and more on this later, and let us discuss now the examples. The idea is that we have various types of light, depending on frequency and wavelength. These are normally referred to as "electromagnetic waves", but for keeping things simple and luminous, we will keep using the familiar term "light". The classification, in a rough form, is as follows:

Frequency	Type	Wavelength
	$-$	
$10^{18} - 10^{20}$	γ rays	$10^{-12} - 10^{-10}$
$10^{16} - 10^{18}$	X − rays	$10^{-10} - 10^{-8}$
$10^{15} - 10^{16}$	UV	$10^{-8} - 10^{-7}$
	$-$	
$10^{14} - 10^{15}$	blue	$10^{-7} - 10^{-6}$
$10^{14} - 10^{15}$	yellow	$10^{-7} - 10^{-6}$
$10^{14} - 10^{15}$	red	$10^{-7} - 10^{-6}$
	$-$	
$10^{11} - 10^{14}$	IR	$10^{-6} - 10^{-3}$
$10^{9} - 10^{11}$	microwave	$10^{-3} - 10^{-1}$
$1 - 10^{9}$	radio	$10^{-1} - 10^{8}$

Observe the tiny space occupied by the visible light, all colors there, and the many more missing, being squeezed under the $10^{14} - 10^{15}$ frequency banner. Here is a zoom on that part, with of course the remark that all this, colors, is something subjective:

Frequency THz $= 10^{12}$ Hz	Color	Wavelength nm $= 10^{-9}$ m
	–	
$670 - 790$	violet	$380 - 450$
$620 - 670$	blue	$450 - 485$
$600 - 620$	cyan	$485 - 500$
$530 - 600$	green	$500 - 565$
$510 - 530$	yellow	$565 - 590$
$480 - 510$	orange	$590 - 625$
$400 - 480$	red	$625 - 750$

Outside visible light we have, as you probably know it, UV on higher frequencies, and IR on lower frequencies. At the high frequency end we have X-rays, that you surely know about too, and γ rays, which are usually associated with various bad things, such as thunderstorms, solar flares, and small bugs with our nuclear energy technology.

As for the lower frequency end of the scale, first we have microwaves, but if you love physics and chemistry you should learn some cooking, that's first-class chemistry, that you can practice every day. And then we have all sorts of radio wavelenghts, including FM, followed by AM, and then by several more obscure low-frequency waves.

2c. Basic optics

Back now to our business, with all the above in hand, we can do some optics. Light usually comes in "bundles", with waves of several wavelenghts coming at the same time, from the same source, and the first challenge is that of separating these wavelenghts. In order to discuss this, let us start with the following fact:

FACT 2.7. *Inside a linear, homogeneous medium, where there is no free charge or current present, the Maxwell equations for electrodynamics read*

$$< \nabla, E > = < \nabla, B > = 0$$

$$\nabla \times E = -\dot{B} \quad , \quad \nabla \times B = \varepsilon\mu\dot{E}$$

with E, B being as before the electric and the magnetic field, and with $\varepsilon > \varepsilon_0$ and $\mu > \mu_0$ being the electric permittivity and magnetic permeability of the medium.

Observe that this is precisely the first part of Theorem 2.2, with the vacuum constants ε_0, μ_0 being replaced by their versions ε, μ, concerning the medium in question. In what regards now the second part of Theorem 2.2, we have:

THEOREM 2.8. *Inside a linear, homogeneous medium, where there is no free charge or free current present, both E and B are subject to the wave equation*

$$\ddot{\varphi} = v^2 \Delta \varphi$$

with v being the speed of light inside the medium, given by

$$v = \frac{c}{n} \quad : \quad n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}$$

with the quantity on the right $n > 1$ being called refraction index of the medium.

PROOF. This is something that we know well in vacuum, from the above, and the proof in general is identical, with the resulting speed being:

$$v = \frac{1}{\sqrt{\varepsilon\mu}}$$

But this formula can be written is a more familiar from, as above. □

As a first observation here, while the above is something quite trivial, mathematically speaking, from the physical viewpoint we are here into complicated things. Materials can be transparent or opaque, with the distinction between them being something very subtle, and advanced, and Theorem 2.8 obviously deals with the transparent case.

In short, we are here inside advanced materials theory, that we cannot really understand, with our knowledge so far. In what follows we will be interested in transparent materials only, such as glass. Regarding the other materials, such as rock, let us just mention that light dissapears inside them, converted into heat. Of course glass heats too when light crosses it, with this being related to $v < c$ inside it. More on this later.

Next in line, and for interest for us, we have:

FACT 2.9. *When travelling through a material, and hitting a new material, some of the light gets reflected, at the same angle, and some of it gets refracted, at a different angle, depending both on the old and the new material, and on the wavelength.*

Again, this is something deep, and very old as well, and there are many things that can be said here, ranging from various computations based on the Maxwell equations, to all sorts of considerations belonging to advanced materials theory.

As a basic formula here, we have the famous Snell law, which relates the incidence angle θ_1 to the refraction angle θ_2, via the following simple formula:

$$\frac{\sin \theta_2}{\sin \theta_1} = \frac{n_1(\lambda)}{n_2(\lambda)}$$

Here $n_i(\lambda)$ are the refraction indices of the two materials, adjusted for the wavelength, and with this adjustment for wavelength being the whole point, which is something quite complicated. For an introduction to all this, we refer for instance to Griffiths [42].

As a simple consequence of the above, we have:

THEOREM 2.10. *Light can be decomposed, by using a prism.*

PROOF. This follows from Fact 2.9. Indeed, when hitting a piece of glass, provided that the hitting angle is not 90°, the light will decompose over the wavelenghts present, with the corresponding refraction angles depending on these wavelengths. And we can capture these split components at the exit from the piece of glass, again deviated a bit, provided that the exit surface is not parallel to the entry surface. And the simplest device doing the job, that is, having two non-parallel faces, is a prism. □

With this in hand, we can now talk about spectroscopy:

FACT 2.11. *We can study events via spectroscopy, by capturing the light the event has produced, decomposing it with a prism, carefully recording its "spectral signature", consisting of the wavelenghts present, and their density, and then doing some reverse engineering, consisting in reconstructing the event out of its spectral signature.*

This is the main principle of spectroscopy, and applications, of all kinds, abound. In practice, the mathematical tool needed for doing the "reverse engineering" mentioned above is the Fourier transform, which allows the decomposition of packets of waves, into monochromatic components. Finally, let us mention too that, needless to say, the event can be reconstructed only partially out of its spectral signature. More on this later.

2d. Max Planck

In relation now with heat, consider a black body, that is, a body at thermal equilibrium, at temperature T. This body radiates heat, and we are interested in computing the energy density of the radiation $\mathcal{E}(\nu,T)$, around a given frequency ν of this radiation.

Quite surprisingly, the intuitive and honest modelling of the problem, and the subsequent math, done honestly too, lead to a spectacularly wrong result, as follows:

THEOREM 2.12. *We have the Rayleigh-Jeans formula for the energy density*

$$\mathcal{E}(\nu,T)=\frac{8\pi bT}{c^3}\,\nu^2$$

where b is the Boltzmann constant, leading globally to the divergent integral

$$\mathcal{E}=\frac{8\pi bTV}{c^3}\int_0^\infty\nu^2\,d\nu$$

over a volume V, with this divergence phenomenon being called UV catastrophe.

PROOF. This is arguably the most famous wrong result in the history of physics, so we will spend some time in trying to understand its proof. And with the comment that this will be no waste of time, because the fix, found later by Max Planck, uses exactly the same ideas and computations, but with an unexpected twist at the end.

(1) Our starting point are the equations for the electromagnetic radiation, that we will now regard as heat, as formulated before, namely:

$$E = Re(\widetilde{E}) \quad : \quad \widetilde{E} = e_n\, e^{i(<k_n,x>-w_n t)}$$

$$B = Re(\widetilde{B}) \quad : \quad \widetilde{B} = b_n\, e^{i(<k_n,x>-w_n t)}$$

Here n is a certain parameter, that will appear later on, and that we can for the moment ignore. Now inserting this data into the Maxwell equations gives the following formulae, connecting the parameters, that we will use several times in what follows:

$$k_n \times b_n + \frac{w_n}{c}\, e_n = 0$$

$$k_n \times e_n - \frac{w_n}{c}\, b_n = 0$$

$$< k_n, e_n >=< k_n, b_n >= 0$$

(2) Let us compute the electromagnetic energy in a finite volume $V = L^3$. We will use here the well-known fact, coming from classical electrodynamics, that the energy density in radiation is $(||E||^2 + ||B||^2)/8\pi$. Thus, the energy we are looking for is given by:

$$\mathcal{E} = \frac{1}{8\pi}\int_V ||E||^2 + ||B||^2$$

(3) In order to compute this integral, let us better model our question. Due to obvious periodicity reasons, the wave number k and the angular frequency w must be of the following form, with $n \in \mathbb{Z}^3$ being a vector with integer components:

$$k_n = \frac{2\pi}{L}\cdot n \quad , \quad w_n = c||k_n||$$

Thus, the electric and magnetic fields in our enclosure $V = L^3$ appear as linear combinations as follows, for certain vectors $e_n, b_n \perp n$, related by the formulae in (1):

$$E = Re(\widetilde{E}) \quad : \quad \widetilde{E} = \sum_n e_n\, e^{i(<k_n,x>-w_n t)}$$

$$B = Re(\widetilde{B}) \quad : \quad \widetilde{B} = \sum_n b_n\, e^{i(<k_n,x>-w_n t)}$$

(4) According to the above formula of E, we have:

$$\begin{aligned}
||E||^2 &= ||Re(\widetilde{E})||^2 \\
&= \frac{1}{4}\left|\left|\sum_n e_n\, e^{i(<k_n,x>-w_nt)} + \bar{e}_n\, e^{-i(<k_n,x>-w_nt)}\right|\right|^2 \\
&= \frac{1}{4}\sum_{nm} < e_n, e_m > e^{i(<k_n-k_m,x>-(w_n-w_m)t)} \\
&+ \frac{1}{4}\sum_{nm} < e_n, \bar{e}_m > e^{i(<k_n+k_m,x>-(w_n+w_m)t)} \\
&+ \frac{1}{4}\sum_{nm} < \bar{e}_n, e_m > e^{i(-<k_n+k_m,x>+(w_n+w_m)t)} \\
&+ \frac{1}{4}\sum_{nm} < \bar{e}_n, \bar{e}_m > e^{i(-<k_n-k_m,x>+(w_n-w_m)t)}
\end{aligned}$$

(5) Now by integrating, we obtain the following formula:

$$\begin{aligned}
\frac{1}{V}\int_V ||E||^2 &= \frac{1}{4}\sum_n < e_n, e_n > +\frac{1}{4}\sum_n < e_n, \bar{e}_{-n} > e^{-2iw_nt} \\
&+ \frac{1}{4}\sum_n < \bar{e}_n, e_{-n} > e^{2iw_nt} + \frac{1}{4}\sum_n < \bar{e}_n, \bar{e}_n >
\end{aligned}$$

(6) Similarly, according to the above formula of B, we have:

$$\begin{aligned}
\frac{1}{V}\int_V ||B||^2 &= \frac{1}{4}\sum_n < b_n, b_n > +\frac{1}{4}\sum_n < b_n, \bar{b}_{-n} > e^{-2iw_nt} \\
&+ \frac{1}{4}\sum_n < \bar{b}_n, b_{-n} > e^{2iw_nt} + \frac{1}{4}\sum_n < \bar{b}_n, \bar{b}_n >
\end{aligned}$$

(7) Before summing the integrals that we found, let us use the formulae connecting the parameters k_n, e_n, b_n found in (1) above, namely:

$$k_n \times b_n + \frac{w_n}{c}\, e_n = 0$$

$$k_n \times e_n - \frac{w_n}{c}\, b_n = 0$$

$$< k_n, e_n >=< k_n, b_n >= 0$$

By using these formulae, we first obtain the following identity:

$$\begin{aligned} < b_n, b_n > &= \frac{c^2}{w_n^2} < k_n \times e_n, k_n \times e_n > \\ &= \frac{c^2||k_n||^2}{w_n^2} < e_n, e_n > \\ &= < e_n, e_n > \end{aligned}$$

Similarly, we have we well the following identity:

$$\begin{aligned} < b_n, \bar{b}_{-n} > &= \frac{c^2}{w_n^2} < k_n \times e_n, k_{-n} \times \bar{e}_n > \\ &= -\frac{c^2||k_n||^2}{w_n^2} < e_n, \bar{e}_{-n} > \\ &= - < e_n, \bar{e}_{-n} > \end{aligned}$$

Also similarly, we have as well the following identity:

$$\begin{aligned} < \bar{b}_n, b_{-n} > &= \frac{c^2}{w_n^2} < k_n \times \bar{e}_n, k_{-n} \times e_n > \\ &= -\frac{c^2||k_n||^2}{w_n^2} < \bar{e}_n, e_{-n} > \\ &= - < \bar{e}_n, e_{-n} > \end{aligned}$$

Finally, we have as well the following identity:

$$\begin{aligned} < \bar{b}_n, \bar{b}_n > &= \frac{c^2}{w_n^2} < k_n \times \bar{e}_n, k_n \times \bar{e}_n > \\ &= \frac{c^2||k_n||^2}{w_n^2} < \bar{e}_n, \bar{e}_n > \\ &= < \bar{e}_n, \bar{e}_n > \end{aligned}$$

(8) We conclude that when summing the integrals computed in (5) and (6), all the terms involving phases will cancel, and we obtain the following formula:

$$\frac{1}{V}\int_V ||E||^2 + ||B||^2 = \frac{1}{2}\sum_n < e_n, e_n > + \frac{1}{2}\sum_n < \bar{e}_n, \bar{e}_n >$$

Now by multiplying everything by $V/8\pi$, as explained in (2), we obtain:

$$\mathcal{E} = \frac{V}{16\pi}\sum_n (< e_n, e_n > + < \bar{e}_n, \bar{e}_n >)$$

(9) The point now is that, by computing this sum, we are led to the Rayleigh-Jeans formula in the statement for the corresponding radiation energy density, namely:

$$\mathcal{E}(\nu,T)=\frac{8\pi bT}{c^3}\,\nu^2$$

(10) And this is certainly wrong, because the total energy which is radiated by our black body, all over the frequency spectrum, follows to be:

$$\mathcal{E}=\frac{8\pi bTV}{c^3}\int_0^\infty\nu^2\,d\nu=\infty$$

More precisely, the Rayleigh-Jeans formula works quite well all across the frequency spectrum, in particular fitting well with the known data, expect for the UV range, where things diverge. And with this phenomenon being called "UV catastrophe". □

Fortunately, the solution to the UV catastrophe, and to the black body problem in general, was found a few years later by Max Planck, his result being as follows:

THEOREM 2.13. *The correct formula for the black body radiation, obtained by assuming that energy is quantized, is the Planck formula*

$$\mathcal{E}(\nu,T)\,d\nu=\frac{8\pi h}{c^3}\cdot\frac{\nu^3d\nu}{e^{h\nu/bt}-1}$$

with h being a new constant, called Planck constant, given by

$$h=6.626\ 070\ 15\times10^{-34}$$

as per the latest SI regulations. The Planck formula fits with all known data, fits as well with the Rayleigh-Jeans formula outside the UV range, and globally leads to

$$\mathcal{E}=\int_0^\infty\mathcal{E}(\nu,T)\,d\nu=aT^4$$

with the radiation energy constant on the right being given by:

$$a=\frac{16\pi^8b^4}{15h^3c^3}$$

PROOF. This is something quite technical, obtained along the lines of the proof of Theorem 2.12, by counting in a new way, by assuming that energy is quantized. □

Regarding applications, a very interesting continuation of Planck's work concerns the black body radiation of the early universe, with the microwave part of it, via a Doppler shift, still permeating the space that we live in. And with this phenomenon, called "cosmic microwave background", being at the origin of modern cosmology. More on this later.

2e. Exercises

Exercises:

Exercise 2.14.

Exercise 2.15.

Exercise 2.16.

Exercise 2.17.

Exercise 2.18.

Exercise 2.19.

Exercise 2.20.

Exercise 2.21.

Bonus exercise.

CHAPTER 3

Atomic theory

3a. Spectral lines

We have accumulated so far some interesting knowledge on electrons and photons, or rather on large groups of electrons and photons, let us not forget this, that our current theory and equations are most likely of statistical nature. And with an interesting conclusion coming on top of this, namely that electrons and photons, sorry, groups of electrons and photons, are most likely related to each other, by some subtle mechanisms.

What is next? Asking of course the cat, who due to some heavy rain outside, has chosen the domestic animal status today. And cat, very pleased to help, answers:

CAT 3.1. *Next comes matter, boss. More specifically hydrogen, heat it, burn it, do whatever you want, and you'll have there your magic mix of electrons and photons.*

Thanks cat, this sounds very exciting, we should use spectroscopy indeed, that we learned about in chapter 2, and for hydrogen, which looks quite basic. So leaving you now to finish my steak, busy with science, let us get to work. There is a long story with hydrogen, involving many discoveries of many people, around 1890-1900. First on our list is the following discovery, which actually came second, by Lyman in 1906:

FACT 3.2 (Lyman). *The hydrogen atom has spectral lines given by the formula*

$$\frac{1}{\lambda} = R\left(1 - \frac{1}{n^2}\right)$$

where $R \simeq 1.097 \times 10^7$ *and* $n \geq 2$*, which are as follows,*

n	Name	Wavelength	Color
	–	–	
2	α	121.567	UV
3	β	102.572	UV
4	γ	97.254	UV
$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	limit	91.175	UV

called Lyman series of the hydrogen atom.

Observe that all the Lyman series lies in UV, which is invisible to the human eye. The first discovery, which was the big one, and the breakthrough, was by Balmer, the founding father of all this, back in 1885, in the visible range, as follows:

FACT 3.3 (Balmer). *The hydrogen atom has spectral lines given by the formula*

$$\frac{1}{\lambda}=R\left(\frac{1}{4}-\frac{1}{n^2}\right)$$

where $R\simeq 1.097\times 10^7$ *and* $n\geq 3$, *which are as follows,*

n	Name	Wavelength	Color
	–	–	
3	α	656.279	red
4	β	486.135	aqua
5	γ	434.047	blue
6	δ	410.173	violet
7	ε	397.007	UV
$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	limit	346.600	UV

called Balmer series of the hydrogen atom.

So, this was Balmer's original result, which started everything, and with his original wavelength formula being in fact something equivalent to the above formula, but a bit more complicated, as follows, with $B\simeq 3.645\times 10^{-7}$ being the Balmer constant:

$$\lambda=\frac{Bn^2}{n^2-4}$$

As a third main result now, this time in IR, due to Paschen in 1908, we have:

FACT 3.4 (Paschen). *The hydrogen atom has spectral lines given by the formula*

$$\frac{1}{\lambda}=R\left(\frac{1}{9}-\frac{1}{n^2}\right)$$

where $R\simeq 1.097\times 10^7$ *and* $n\geq 4$, *which are as follows,*

n	Name	Wavelength	Color
	–	–	
4	α	1875	IR
5	β	1282	IR
6	γ	1094	IR
$\vdots$	$\vdots$	$\vdots$	$\vdots$
∞	limit	820.4	IR

called Paschen series of the hydrogen atom.

Observe the striking similarity between the above three results. In fact, we have here the following fundamental, grand result, due to Rydberg in 1888, based on the Balmer series, and with later contributions by Ritz in 1908, using the Lyman series as well:

CONCLUSION 3.5 (Rydberg, Ritz). *The spectral lines of the hydrogen atom are given by the Rydberg formula, depending on integer parameters* $n_1 < n_2$,

$$\frac{1}{\lambda_{n_1n_2}} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

with R being the Rydberg constant for hydrogen, which is as follows:

$$R \simeq 1.096\ 775\ 83 \times 10^7$$

These spectral lines combine according to the Ritz-Rydberg principle, as follows:

$$\frac{1}{\lambda_{n_1n_2}} + \frac{1}{\lambda_{n_2n_3}} = \frac{1}{\lambda_{n_1n_3}}$$

Similar formulae hold for other atoms, with suitable fine-tunings of R.

Here the first part, the Rydberg formula, generalizes the results of Lyman, Balmer, Paschen, which appear at $n_1 = 1, 2, 3$, at least retrospectively. The Rydberg formula predicts further spectral lines, appearing at $n_1 = 4, 5, 6, \ldots$, and these were discovered later, by Brackett in 1922, Pfund in 1924, Humphreys in 1953, and others aftwerwards, with all these extra lines being in far IR. The simplified complete table is as follows:

n_1	n_2	Series name	Wavelength $n_2 = \infty$	Color $n_2 = \infty$
		–	–	
1	$2 - \infty$	Lyman	91.13 nm	UV
2	$3 - \infty$	Balmer	364.51 nm	UV
3	$4 - \infty$	Paschen	820.14 nm	IR
		–	–	
4	$5 - \infty$	Brackett	1458.03 nm	far IR
5	$6 - \infty$	Pfund	2278.17 nm	far IR
6	$7 - \infty$	Humphreys	3280.56 nm	far IR
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Regarding the last assertion, concerning other elements, this is something conjectured and partly verified by Ritz, and fully verified and clarified later, via many experiments, the fine-tuning of R being basically $R \to RZ^2$, where Z is the atomic number.

But from a theoretical physics viewpoint, the main result remains the middle assertion, called Ritz-Rydberg combination principle. This is something at the same time extremely simple, and completely puzzling, the informal conclusion being as follows:

Thought 3.6. *The simplest observables of the hydrogen atom, combining via*

$$\frac{1}{\lambda_{n_1n_2}}+\frac{1}{\lambda_{n_2n_3}}=\frac{1}{\lambda_{n_1n_3}}$$

look like quite weird quantities. Why wouldn't they just sum normally.

Getting now to quantum mechanics, and to our dreams about it, formulated before, well, good news, we have some serious data here. These spectral lines are basic and beautiful, obviously of quantized type, and in order to get started with our theory, we first need to solve the puzzle of the Ritz-Rydberg combination principle. But, how to do this? Fortunately, matrix theory comes to the rescue, as follows:

Thought 3.7. *The Ritz-Rydberg combination principle reminds the formula*

$$e_{n_1n_2}e_{n_2n_3}=e_{n_1n_3}$$

for the usual matrix units, which are the elementary matrices given by

$$e_{ij}:e_j\to e_i$$

perhaps taken in infinite dimensions, as to allow infinite-ranging indices.

This looks certainly very interesting, and following now Heisenberg, we can start dreaming of something more precise, in relation with the above, as follows:

Thought 3.8. *Observables in quantum mechanics should be some sort of infinite matrices, generalizing the Lyman, Balmer, Paschen lines of the hydrogen atom, and multiplying between them as the matrices do, as to produce further observables.*

We will talk about this in a moment, but before that, what exactly do we want to do, with our quantum mechanics theory? Well, solving of course the hydrogen atom, and heavier atoms too, with the conjecture here, by Bohr and others, being as follows:

Claim 3.9 (Bohr and others). *The atoms are formed by a core of protons and neutrons, surrounded by a cloud of electrons, basically obeying to a modified version of electromagnetism. And with a fine mechanism involved, as follows:*

(1) *The electrons are free to move only on certain specified elliptic orbits, labelled* $1,2,3,\ldots$, *situated at certain specific heights.*
(2) *The electrons can jump or fall between orbits* $n_1<n_2$, *absorbing or emitting light and heat, that is, electromagnetic waves, as accelerating charges.*
(3) *The energy of such a wave, coming from* $n_1\to n_2$ *or* $n_2\to n_1$, *is given, via the Planck viewpoint, by the Rydberg formula, applied with* $n_1<n_2$.
(4) *The simplest such jumps are those observed by Lyman, Balmer, Paschen. And multiple jumps explain the Ritz-Rydberg formula.*

And isn't this beautiful. Moreover, some further claims, also by Bohr and others, are that the theory can be further extended and fine-tuned as to explain many other phenomena, such as the findings of Einstein on the photoelectric effect, of Becquerel and Pierre and Marie Curie on radioactivity, and many more.

And the story is not over here. Following now Heisenberg, the next claim is that the underlying math in all the above can lead to a beautiful axiomatization of quantum mechanics, as a "matrix mechanics", along the lines of Thought 3.8.

3b. Quantum mechanics

We know from Heisenberg that, in order to prove Claim 3.9, we must develop some sort of "matrix mechanics". Before doing that, however, let us hear as well the point of view of Schrödinger, which came a few years later. His idea was to forget about exact things, and try to investigate the hydrogen atom statistically. We have indeed:

QUESTION 3.10. *In the context of the hydrogen atom, assuming that the proton is fixed, what is the probability density $\varphi_t(x)$ of the position of the electron e, at time t,*

$$P_t(e \in V) = \int_V \varphi_t(x)dx$$

as function of an intial probability density $\varphi_0(x)$? Moreover, can the corresponding equation be solved, and will this prove the Bohr claims for hydrogen, statistically?

In order to get familiar with this question, let us first look at examples coming from classical mechanics. In the context of linear motion, with speed v, we have:

$$\varphi_t(x) = \varphi_0(x) + vt$$

More generally, assuming that we have a particle whose position at time t is given by $x_0 + \gamma(t)$, the evolution of the probability density will be given by:

$$\varphi_t(x) = \varphi_0(x) + \gamma(t)$$

These examples are somewhat trivial, of course not in relation with the computation of γ, usually a difficult question, but in relation with our questions, and do not apply to the electron. The point indeed is that, in what regards the electron, we have:

FACT 3.11. *In respect with various simple interference experiments:*

1. *The electron is definitely not a particle in the usual sense.*
2. *But in most situations it behaves exactly like a wave.*
3. *But in other situations it behaves like a particle.*

Getting back now to the Schrödinger question, all this suggests to use, as for the waves, an amplitude function $\psi_t(x) \in \mathbb{C}$, related to the density $\varphi_t(x) > 0$ by the formula $\varphi_t(x) = |\psi_t(x)|^2$. So, let us reformulate Question 3.10, in the following way:

QUESTION 3.12. *In the context of the hydrogen atom, assuming that the proton is fixed, what is the amplitude function $\psi_t(x)$ of the position of the electron e, at time t,*

$$P_t(e \in V) = \int_V |\psi_t(x)|^2 dx$$

as function of an intial amplitude function $\psi_0(x)$? Moreover, can the corresponding equation be solved, and will this prove the Bohr claims for hydrogen, statistically?

Mathematically, what we did here is to replace the density $\varphi_t(x) > 0$ by the amplitude function $\psi_t(x) \in \mathbb{C}$. Not that a big deal, you would say, because the two are related by simple formulae as follows, with $\theta_t(x)$ being an arbitrary phase function:

$$\varphi_t(x) = |\psi_t(x)|^2 \quad , \quad \psi_t(x) = e^{i\theta_t(x)}\sqrt{\varphi_t(x)}$$

However, experience with math shows that such manipulations can be crucial, raising for instance the possibility that the amplitude function satisfies some simple equation, while the density itself, maybe not. So, let us hope for this to happen.

And this is what happens indeed. Schrödinger was led in this way to:

CLAIM 3.13 (Schrödinger). *In the context of the hydrogen atom, the amplitude function of the electron $\psi = \psi_t(x)$ is subject to the Schrödinger equation*

$$ih\dot{\psi} = -\frac{h^2}{2m}\Delta\psi + V\psi$$

m being the mass, h the modified Planck constant, and V the Coulomb potential of the proton. The same holds for movements of the electron under an arbitrary potential V.

Observe the similarity with the wave equation $\ddot{\varphi} = v^2\Delta\varphi$, and with the heat equation $\dot{\varphi} = \alpha\Delta\varphi$ too. There might be of course some speculations to be made here, but passed that, this is certainly not your easy to decipher equation. So, where does this equation come from? Is there a way of deducing it from simpler principles? And so on.

Generally speaking, however, any axiomatic explanation for the Schrödinger equation can only introduce some possible mistakes in our theory. And so we are led by precaution to the following preliminary answer, to such questions, that you might have:

COMMENT 3.14. *The Schrödinger equation comes from Schrödinger.*

And please do not take this as a joke. We are mainly interested in solving the hydrogen atom, and the Schrödinger equation can only solve it, via some calculus. So why not enjoying this, solving the hydrogen atom by using this equation, and see later what further things, beyond Schrödinger, can be said about quantum mechanics.

Before doing that, let us explain however the link with the Heisenberg matrix mechanics approach. Many things can be said here, and following Heisenberg and Schrödinger, and then especially Dirac, who did the axiomatization work, we have:

Definition 3.15. *In quantum mechanics the states of the system are vectors of a Hilbert space H, and the observables of the system are linear operators*

$$T : H \to H$$

which can be densely defined, and are taken self-adjoint, $T = T^*$. The average value of such an observable T, evaluated on a state $\xi \in H$, is given by:

$$< T >=< T\xi, \xi >$$

In the context of the Schrödinger mechanics of the hydrogen atom, the Hilbert space is the space $H = L^2(\mathbb{R}^3)$ where the wave function ψ lives, and we have

$$< T >= \int_{\mathbb{R}^3} T(\psi) \cdot \bar{\psi}\, dx$$

which is called "sandwiching" formula, with the operators

$$x \quad , \quad -\frac{ih}{m}\nabla \quad , \quad -ih\nabla \quad , \quad -\frac{h^2\Delta}{2m} \quad , \quad -\frac{h^2\Delta}{2m} + V$$

representing the position, speed, momentum, kinetic energy, and total energy.

In other words, we are doing here two things. First, we are declaring by axiom that various "sandwiching" formulae found before by Heisenberg, involving the operators at the end, that we will not get into in detail here, hold true. And second, we are raising the possibility for other quantum mechanical systems, more complicated, to be described as well by the mathematics of the operators on a certain Hilbert space H, as above.

Moving ahead towards hydrogen, we are interested in the case where V is the usual quadratic Coulomb potential of the proton, given by the following formula:

$$V = -\frac{e^2}{4\pi\varepsilon_0} \cdot \frac{1}{r}$$

This potential is time-independent, and we have the following result:

Theorem 3.16. *In the case of time-independent potentials V, including the Coulomb potential of the proton, the solutions of the Schrödinger equation*

$$ih\dot{\psi} = -\frac{h^2}{2m}\Delta\psi + V\psi$$

which are of the following special form, with the time and space variables separated,

$$\psi_t(x) = w_t\phi(x)$$

are given by the following formulae, with E being a certain constant,

$$w = e^{-iEt/h}w_0 \quad , \quad E\phi = -\frac{h^2}{2m}\Delta\phi + V\phi$$

with the equation for ϕ being called time-independent Schrödinger equation.

Proof. By dividing by ψ, the Schrödinger equation becomes:

$$ih\cdot\frac{\dot{w}}{w}=-\frac{h^2}{2m}\cdot\frac{\Delta\phi}{\phi}+V$$

Now since the left-hand side depends only on time, and the right-hand side depends only on space, both quantities must equal a constant E, and this gives the result. □

Moving ahead with theory, we can further build on Theorem 3.16, with a number of key observations on the time-independent Schrödinger equation, as follows:

Theorem 3.17. *In the case of time-independent potentials V, the Schrödinger equation and its time-independent version have the following properties:*

(1) *For solutions of type $\psi=w_t\phi(x)$, the density $\varphi=|\psi|$ is time-independent, and more generally, all quantities of type $<T>$ are time-independent.*
(2) *The time-independent Schrödinger equation can be written as $\widehat{H}\phi=E\phi$, with $H=T+V$ being the total energy, of Hamiltonian.*
(3) *For solutions of type $\psi=w_t\phi(x)$ we have $<H^k>=E^k$ for any k. In particular we have $<H>=E$, and the variance is $<H^2>-<H>^2=0$.*

Proof. All the formulae are clear indeed from the fact that, when using the sandwiching formula for computing averages, the phases will cancel:

$$<T>=\int_{\mathbb{R}^3}\bar{\psi}\cdot T\cdot\psi\,dx=\int_{\mathbb{R}^3}\bar{\phi}\cdot T\cdot\phi\,dx$$

Thus, we are led to the various conclusions in the statement. □

We have as well the following result, mathematical this time:

Theorem 3.18. *The solutions of the Schrödinger equation with time-independent potential V appear as linear combinations of separated solutions*

$$\psi=\sum_n c_n e^{-iE_nt/h}\phi_h$$

with the absolute values of the coefficients being given by

$$<H>=\sum_n|c_n|^2E_n$$

$|c_n|$ being the probability for a measurement to return the energy value E_n.

Proof. This is something standard, which follows from Fourier analysis, which allows the decomposition of ψ as in the statement, and that we will not really need, in what follows next. As before, for a physical discussion here, we refer to Griffiths [43]. □

3c. Spherical harmonics

In order to solve now the hydrogen atom, by using the Schrödinger equation, the idea will be that of reformulating this equation in spherical coordinates. We have:

THEOREM 3.19. *The time-independent Schrödinger equation in spherical coordinates separates, for solutions of type $\phi=\rho(r)\alpha(s,t)$, into two equations, as follows,*

$$\frac{d}{dr}\left(r^2\cdot\frac{d\rho}{dr}\right)-\frac{2mr^2}{h^2}(V-E)\rho=K\rho$$

$$\sin s\cdot\frac{d}{ds}\left(\sin s\cdot\frac{d\alpha}{ds}\right)+\frac{d^2\alpha}{dt^2}=-K\sin^2 s\cdot\alpha$$

with K being a constant, called radial equation, and angular equation.

PROOF. We use the following well-known formula for the Laplace operator in spherical coordinates, whose proof can be found in any geometry or calculus book:

$$\Delta=\frac{1}{r^2}\cdot\frac{d}{dr}\left(r^2\cdot\frac{d}{dr}\right)+\frac{1}{r^2\sin s}\cdot\frac{d}{ds}\left(\sin s\cdot\frac{d}{ds}\right)+\frac{1}{r^2\sin^2 s}\cdot\frac{d^2}{dt^2}$$

By using this formula, the time-independent Schrödinger equation reformulates in spherical coordinates as follows:

$$(V-E)\phi=\frac{h^2}{2m}\left[\frac{1}{r^2}\cdot\frac{d}{dr}\left(r^2\cdot\frac{d\phi}{dr}\right)+\frac{1}{r^2\sin s}\cdot\frac{d}{ds}\left(\sin s\cdot\frac{d\phi}{ds}\right)+\frac{1}{r^2\sin^2 s}\cdot\frac{d^2\phi}{dt^2}\right]$$

Let us look now for separable solutions for this latter equation, consisting of a radial part and an angular part, as in the statement, namely:

$$\phi(r,s,t)=\rho(r)\alpha(s,t)$$

By plugging this function into our equation, we obtain:

$$(V-E)\rho\alpha=\frac{h^2}{2m}\left[\frac{\alpha}{r^2}\cdot\frac{d}{dr}\left(r^2\cdot\frac{d\rho}{dr}\right)+\frac{\rho}{r^2\sin s}\cdot\frac{d}{ds}\left(\sin s\cdot\frac{d\alpha}{ds}\right)+\frac{\rho}{r^2\sin^2 s}\cdot\frac{d^2\alpha}{dt^2}\right]$$

By multiplying everything by $2mr^2/(h^2\rho\alpha)$, and then moving the radial terms to the left, and the angular terms to the right, this latter equation can be written as follows:

$$\frac{2mr^2}{h^2}(V-E)-\frac{1}{\rho}\cdot\frac{d}{dr}\left(r^2\cdot\frac{d\rho}{dr}\right)=\frac{1}{\alpha\sin^2 s}\left[\sin s\cdot\frac{d}{ds}\left(\sin s\cdot\frac{d\alpha}{ds}\right)+\frac{d^2\alpha}{dt^2}\right]$$

Since this latter equation is now separated between radial and angular variables, both sides must be equal to a certain constant $-K$, and this gives the result. □

Let us first study the angular equation, and this for reasons that will become clear later, the idea being that this equation forces the constant K to be of the form $K=l(l+1)$ with $l\in\mathbb{N}$, which can be used afterwards in the study of the radial equation.

The study will be quite long. We first have the following result:

Proposition 3.20. *The angular equation that we found before, namely*

$$\sin s\cdot\frac{d}{ds}\left(\sin s\cdot\frac{d\alpha}{ds}\right)+\frac{d^2\alpha}{dt^2}=-K\sin^2 s\cdot\alpha$$

separates, for solutions of type $\alpha=\sigma(s)\theta(t)$, *into two equations, as follows,*

$$\frac{1}{\theta}\cdot\frac{d^2\theta}{dt^2}=-m^2$$

$$\frac{\sin s}{\sigma}\cdot\frac{d}{ds}\left(\sin s\cdot\frac{d\sigma}{ds}\right)+K\sin^2 s=m^2$$

with m *being a constant, called azimuthal equation, and polar equation.*

Proof. This is something elementary, the idea being as follows:

(1) Before anything, for such questions, we need to have a better understanding of the angles s,t, and the differences between them. So, recall that these angles come from:

$$\begin{cases}x&=&r\cos s\\ y&=&r\sin s\cos t\\ z&=&r\sin s\sin t\end{cases}$$

To be more precise, here $r\in[0,\infty)$ is the radius, $s\in[0,\pi]$ is the polar angle, and $t\in[0,2\pi]$ is the azimuthal angle. Be said in passing, there are several conventions and notations here, and the above ones, that we use here, come from the general ones in N dimensions, because further coordinates can be easily added, in the obvious way.

(2) Getting back now to our question, by plugging $\alpha=\sigma(s)\theta(t)$ into the angular equation, we obtain:

$$\sin s\cdot\theta\cdot\frac{d}{ds}\left(\sin s\cdot\frac{d\sigma}{ds}\right)+\sigma\cdot\frac{d^2\theta}{dt^2}=-K\sin^2 s\cdot\sigma\theta$$

By dividing everything by $\sigma\theta$, this equation can be written as follows:

$$-\frac{1}{\theta}\cdot\frac{d^2\theta}{dt^2}=\frac{\sin s}{\sigma}\cdot\frac{d}{ds}\left(\sin s\cdot\frac{d\sigma}{ds}\right)+K\sin^2 s$$

Since the variables are separated, we must have, for a certain constant m:

$$\frac{1}{\theta}\cdot\frac{d^2\theta}{dt^2}=-m^2$$

$$\frac{\sin s}{\sigma}\cdot\frac{d}{ds}\left(\sin s\cdot\frac{d\sigma}{ds}\right)+K\sin^2 s=m^2$$

Thus, we are led to the conclusion in the statement. □

Regarding the azimuthal equation, things here are quickly settled, as follows:

Proposition 3.21. *The solutions of the azimuthal equation, namely*

$$\frac{1}{\theta}\cdot\frac{d^2\theta}{dt^2}=-m^2$$

are the functions as follows, with $a,b\in\mathbb{C}$ being parameters,

$$\theta(t)=ae^{imt}+be^{-imt}$$

and with only the case $m\in\mathbb{Z}$ being acceptable, on physical grounds.

Proof. The first assertion is clear, because we have a second order equation, and two obvious solutions for it, $e^{\pm imt}$, and then their linear combinations, and that's all. Regarding the last assertion, the point here is that by using $\theta(t)=\theta(t+2\pi)$, which is a natural physical assumption on the wave function, we are led to $m\in\mathbb{Z}$, as stated. □

We are now about to solve the angular equation, with only the polar equation remaining to be studied. However, in practice, this polar equation is 10 times more difficult that everything what we did so far, and so please be patient. We first have:

Proposition 3.22. *The polar equation that we found before, namely*

$$\frac{\sin s}{\sigma}\cdot\frac{d}{ds}\left(\sin s\cdot\frac{d\sigma}{ds}\right)+K\sin^2 s=m^2$$

with $m\in\mathbb{Z}$, translates via $\sigma(s)=f(\cos s)$ into the following equation,

$$(1-x^2)f''(x)-2xf'(x)=\left(\frac{m^2}{1-x^2}-K\right)f(x)$$

where $x=\cos s$, called Legendre equation.

Proof. Let us first do a number of manipulations on our equation, before making the change of variables. By multiplying by σ, our equation becomes:

$$\sin s\cdot\frac{d}{ds}\left(\sin s\cdot\frac{d\sigma}{ds}\right)=\left(m^2-K\sin^2 s\right)\sigma$$

By differentiating at left, this equation becomes:

$$\sin s\,(\cos s\cdot\sigma'+\sin s\cdot\sigma'')=\left(m^2-K\sin^2 s\right)\sigma$$

Finally, by dividing everything by $\sin^2 s$, our equation becomes:

$$\sigma''+\frac{\cos s}{\sin s}\cdot\sigma'=\left(\frac{m^2}{\sin^2 s}-K\right)\sigma$$

Now let us set $\sigma(s)=f(\cos s)$. With this change of variables, we have:

$$\sigma=f(\cos s)$$
$$\sigma'=-\sin s\cdot f'(\cos s)$$
$$\sigma''=-\cos s\cdot f'(\cos s)+\sin^2 s\cdot f''(\cos s)$$

By plugging this data, our radial equation becomes:

$$\sin^2 s \cdot f''(\cos s) - 2\cos s \cdot f'(\cos s) = \left(\frac{m^2}{\sin^2 s} - K\right) f(\cos s)$$

Now with $x = \cos s$, which is our new variable, this equation reads:

$$(1-x^2)f''(x) - 2xf'(x) = \left(\frac{m^2}{1-x^2} - K\right) f(x)$$

But this is the Legendre equation, as stated. □

Here comes now the difficult point. We have the following non-trivial result:

THEOREM 3.23. *The solutions of the Legendre equation, namely*

$$(1-x^2)f''(x) - 2xf'(x) = \left(\frac{m^2}{1-x^2} - K\right) f(x)$$

can be explicitely computed, via complicated math, and only the case

$$K = l(l+1) \quad : \quad l \in \mathbb{N}$$

is acceptable, on physical grounds.

PROOF. The first part is something quite complicated, involving the hypergeometric functions ${}_2F_1$, that you don't want to hear about, believe me. As for the second part, analysis and physical speculations, this is something not trivial either. □

In order to construct now the solutions, we can use a piece of math, as follows:

THEOREM 3.24. *The orthonormal basis of $L^2[-1,1]$ obtained by starting with the Weierstrass basis $\{x^l\}$, and doing Gram-Schmidt, is the family of polynomials $\{P_l\}$, with each P_l being of degree l, and with positive leading coefficient, subject to:*

$$\int_{-1}^{1} P_k(x)P_l(x)\,dx = \delta_{kl}$$

These polynomials, called Legendre polynomials, satisfy the equation

$$(1-x^2)P_l''(x) - 2xP_l'(x) + l(l+1)P_l(x) = 0$$

which is the Legendre equation at $m = 0$, and with $K = l(l+1)$. Moreover,

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$$

which is called the Rodrigues formula for Legendre polynomials.

PROOF. The first assertion is clear, because Gram-Schmidt applied to the Weierstrass basis $\{x^l\}$ can only lead to a certain family of polynomials $\{P_l\}$, with each P_l being of degree l, and also unique, if we assume that it has positive leading coefficient. As for the rest, this is standard math too, by carefully looking at how Gram-Schmidt works. □

Going ahead now, we can solve in fact the Legendre equation at any m, as follows:

PROPOSITION 3.25. *The general Legendre equation, with parameters $m \in \mathbb{N}$ and $K = l(l+1)$ with $l \in \mathbb{N}$, namely*

$$(1-x^2)f''(x) - 2xf'(x) = \left(\frac{m^2}{1-x^2} - l(l+1)\right) f(x)$$

is solved by the following functions, called Legendre functions,

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \left(\frac{d}{dx}\right)^m P_l(x)$$

where P_l are as before the Legendre polynomials. Also, we have

$$P_l^m(x) = (-1)^m \frac{(1-x^2)^{m/2}}{2^l l!} \left(\frac{d}{dx}\right)^{l+m} (x^2-1)^l$$

called Rodrigues formula for Legendre functions.

PROOF. The first assertion is something elementary, coming by differentiating m times the Legendre equation, which leads to the general Legendre equation. As for the second assertion, this follows from the Rodrigues formula for Legendre polynomials. □

By putting together all the above results, we are led to the following conclusion:

THEOREM 3.26. *The separated solutions $\alpha = \sigma(s)\theta(t)$ of the angular equation,*

$$\sin s \cdot \frac{d}{ds}\left(\sin s \cdot \frac{d\alpha}{ds}\right) + \frac{d^2\alpha}{dt^2} = -K \sin^2 s \cdot \alpha$$

are given by the following formulae, where $l \in \mathbb{N}$ is such that $K = l(l+1)$,

$$\sigma(s) = P_l^m(\cos s) \quad , \quad \theta(t) = e^{imt}$$

and where $m \in \mathbb{Z}$ is a constant, and with P_l^m being the Legendre function,

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \left(\frac{d}{dx}\right)^m P_l(x)$$

where P_l are the Legendre polynomials, given by the following formula:

$$P_l(x) = \frac{1}{2^l l!}\left(\frac{d}{dx}\right)^l (x^2-1)^l$$

These solutions $\alpha = \sigma(s)\theta(t)$ are called spherical harmonics.

PROOF. This follows indeed from all the above, and with the comment that everything is taken up to linear combinations. We will normalize the wave function later. □

3d. Bohr energy

In order now to finish our study, it remains to solve the radial equation, for the Coulomb potential V of the proton. Let us begin with some generalities, valid for any time-independent potential V. As a first manipulation on the radial equation, we have:

PROPOSITION 3.27. *The radial equation, written with $K = l(l+1)$,*

$$(r^2\rho')' - \frac{2mr^2}{h^2}(V-E)\rho = l(l+1)\rho$$

takes with $\rho = u/r$ the following form, called modified radial equation,

$$Eu = -\frac{h^2}{2m} \cdot u'' + \left(V + \frac{h^2 l(l+1)}{2mr^2}\right)u$$

which is a time-independent 1D Schrödinger equation.

PROOF. With $\rho = u/r$ as in the statement, we have:

$$\rho = \frac{u}{r} \quad , \quad \rho' = \frac{u'r - u}{r^2} \quad , \quad (r^2\rho')' = u''r$$

By plugging this data into the radial equation, this becomes:

$$u''r - \frac{2mr}{h^2}(V-E)u = \frac{l(l+1)}{r} \cdot u$$

By multiplying everything by $h^2/(2mr)$, this latter equation becomes:

$$\frac{h^2}{2m} \cdot u'' - (V-E)u = \frac{h^2 l(l+1)}{2mr^2} \cdot u$$

But this gives the formula in the statement. As for the interpretation, as time-independent 1D Schrödinger equation, this is clear as well, and with the comment here that the term added to the potential V is some sort of centrifugal term. □

Let us now, eventually, get to hydrogen. Here V is the usual quadratic Coulomb potential of the proton, given by the following formula, with e being as usual the charge of the electron, and ε_0 being the electric permittivity of free space:

$$V = -\frac{e^2}{4\pi\varepsilon_0} \cdot \frac{1}{r}$$

However, before getting into math, we must first discuss units. Remember from before in this book the story of the Coulomb constant K, which was eventually replaced by $\varepsilon_0 = 1/(4\pi K)$, due to the Gauss law, and the Maxwell equations? Well, the Maxwell equations being now obsolete, not to say wrong, in quantum mechanics, time to welcome back the Coulomb constant K. Our new conventions will be as follows:

CONVENTIONS 3.28. *We welcome back the Coulomb constant K, given by:*

$$K = 8.987\ 551\ 7923(14) \times 10^9$$

Also, we welcome as new quantity for energy the electron volt eV*,*

$$1\text{eV} = e = 1.602\ 176\ 634 \times 10^{-19}$$

with this being regarded, as per our SI philosophy, as a constant, not a unit.

As usual, lots of fun here with units. In what regards the Coulomb constant K and minus the charge of the electron e, these are given by the formulae in the statement, with the formula of e being exact, as per latest SI regulations. As for the electron volt eV, this is by definition the amount of kinetic energy gained by an electron accelerating from rest through an electric potential difference of 1 volt in vacuum.

Getting back now to the Coulomb potential of the proton, we have here:

FACT 3.29. *The Coulomb potential of the hydrogen atom proton, acting on the electron by attraction, is given according to the Coulomb law by*

$$V = -\frac{Kep}{r}$$

where p is the charge of the proton, and K is the Coulomb constant. In practice however we have $p \simeq e$ up to order 10^{-7}, and so our formula can be written as

$$V \simeq -\frac{Ke^2}{r}$$

and we will use this latter formula, and with $=$ sign, for simplifying.

Getting back now to math, it remains to solve the modified radial equation, for the above potential V. And we have here the following result, which does not exactly solve this radial equation, but provides us instead with something far better, namely the proof of the original claim by Bohr, which was at the origin of everything:

THEOREM 3.30 (Schrödinger). *In the case of the hydrogen atom, where V is the Coulomb potential of the proton, the modified radial equation, which reads*

$$Eu = -\frac{h^2}{2m} \cdot u'' + \left(-\frac{Ke^2}{r} + \frac{h^2 l(l+1)}{2mr^2}\right) u$$

leads to the Bohr formula for allowed energies,

$$E_n = -\frac{m}{2}\left(\frac{Ke^2}{h}\right)^2 \cdot \frac{1}{n^2}$$

with $n \in \mathbb{N}$, the binding energy being

$$E_1 \simeq -2.177 \times 10^{-18}$$

with means $E_1 \simeq -13.591$ eV.

Proof. This is again something non-trivial, the idea being as follows:

(1) By dividing our modified radial equation by E, this becomes:

$$-\frac{h^2}{2mE}\cdot u''=\left(1+\frac{Ke^2}{Er}-\frac{h^2l(l+1)}{2mEr^2}\right)u$$

In terms of $\alpha=\sqrt{-2mE}/h$, this equation takes the following form:

$$\frac{u''}{\alpha^2}=\left(1+\frac{Ke^2}{Er}+\frac{l(l+1)}{(\alpha r)^2}\right)u$$

In terms of the new variable $p=\alpha r$, this latter equation reads:

$$u''=\left(1+\frac{\alpha Ke^2}{Ep}+\frac{l(l+1)}{p^2}\right)u$$

Now let us introduce a new constant S for our problem, as follows:

$$S=-\frac{\alpha Ke^2}{E}$$

In terms of this new constant, our equation reads:

$$u''=\left(1-\frac{S}{p}+\frac{l(l+1)}{p^2}\right)u$$

(2) The idea will be that of looking for a solution written as a power series, but before that, we must "peel off" the asymptotic behavior. Which is something that can be done, of course, heuristically. With $p\to\infty$ we are led to $u''=u$, and ignoring the solution $u=e^p$ which blows up, our approximate asymptotic solution is:

$$u\sim e^{-p}$$

Similarly, with $p\to 0$ we are led to $u''=l(l+1)u/p^2$, and ignoring the solution $u=p^{-l}$ which blows up, our approximate asymptotic solution is:

$$u\sim p^{l+1}$$

(3) The above heuristic considerations suggest writing our function u as follows:

$$u=p^{l+1}e^{-p}v$$

So, let us do this. In terms of v, we have the following formula:

$$u'=p^le^{-p}\left[(l+1-p)v+pv'\right]$$

Differentiating a second time gives the following formula:

$$u''=p^le^{-p}\left[\left(\frac{l(l+1)}{p}-2l-2+p\right)v+2(l+1-p)v'+pv''\right]$$

Thus the radial equation, as modified in (1) above, reads:

$$pv''+2(l+1-p)v'+(S-2(l+1))v=0$$

(4) We will be looking for a solution v appearing as a power series:

$$v=\sum_{j=0}^\infty c_jp^j$$

But our equation leads to the following recurrence formula for the coefficients:

$$c_{j+1}=\frac{2(j+l+1)-S}{(j+1)(j+2l+2)}\cdot c_j$$

(5) We are in principle done, but we still must check that, with this choice for the coefficients c_j, our solution v, or rather our solution u, does not blow up. And the whole point is here. Indeed, at $j>>0$ our recurrence formula reads, approximately:

$$c_{j+1}\simeq\frac{2c_j}{j}$$

But, surprisingly, this leads to $v\simeq c_0e^{2p}$, and so to $u\simeq c_0p^{l+1}e^p$, which blows up.

(6) As a conclusion, the only possibility for u not to blow up is that where the series defining v terminates at some point. Thus, we must have for a certain index j:

$$2(j+l+1)=S$$

In other words, we must have, for a certain integer $n>l$:

$$S=2n$$

(7) We are almost there. Recall from (1) above that S was defined as follows:

$$S=-\frac{\alpha Ke^2}{E}\quad:\quad\alpha=\frac{\sqrt{-2mE}}{h}$$

Thus, we have the following formula for the square of S:

$$S^2=\frac{\alpha^2K^2e^4}{E^2}=-\frac{2mE}{h^2}\cdot\frac{K^2e^4}{E^2}=-\frac{2mK^2e^4}{h^2E}$$

Now by using the formula $S=2n$ from (6), the energy E must be of the form:

$$E=-\frac{2mK^2e^4}{h^2S^2}=-\frac{mK^2e^4}{2h^2n^2}$$

Calling this energy E_n, depending on $n\in\mathbb N$, we have, as claimed:

$$E_n=-\frac{m}{2}\left(\frac{Ke^2}{h}\right)^2\cdot\frac{1}{n^2}$$

(8) Thus, we proved the Bohr formula. Regarding now the numerics, the data is as follows, with all formulae being of course approximative:

$$K=8.988\times10^9\quad,\quad e=1.602\times10^{-19}$$

$$h=1.055\times10^{-34}\quad,\quad m=9.109\times10^{-31}$$

We obtain succesively that we have the following formulae:

$$\frac{Ke^2}{h} = \frac{8.988 \times 1.602^2}{1.055} \times \frac{10^9 \times 10^{-38}}{10^{-34}} = 2.186 \times 10^6$$

$$\left(\frac{Ke^2}{h}\right)^2 = 2.186^2 \times 10^{12} = 4.779 \times 10^{12}$$

$$\frac{m}{2}\left(\frac{Ke^2}{h}\right)^2 = \frac{9.109 \times 4.779}{2} \times 10^{12-31} = 2.177 \times 10^{-18}$$

Thus E_1 is as in the statement. In electron volts now, the figure is:

$$\frac{E_1}{e} = \frac{2.177 \times 10^{-18}}{1.602 \times 10^{-19}} = 13.591$$

Thus, we are led to the conclusion in the statement. □

As a first remark, all this agrees with the Rydberg formula, due to:

THEOREM 3.31. *The Rydberg constant for hydrogen is given by*

$$R = -\frac{E_1}{h_0 c}$$

where E_1 is the Bohr binding energy, and the Rydberg formula itself, namely

$$\frac{1}{\lambda_{n_1 n_2}} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right)$$

simply reads, via the energy formula in Theorem 3.30,

$$\frac{1}{\lambda_{n_1 n_2}} = \frac{E_{n_2} - E_{n_1}}{h_0 c}$$

which is in agreement with the Planck formula $E = h_0 c/\lambda$.

PROOF. Here the first assertion is something numeric, coming from the fact that the formula in the statement gives, when evaluated, the Rydberg constant:

$$R = \frac{-E_1}{h_0 c} = \frac{2.177 \times 10^{-18}}{6.626 \times 10^{-34} \times 2.998 \times 10^8} = 1.096 \times 10^7$$

Regarding now the second assertion, by dividing $R = -E_1/(h_0 c)$ by any number of type n^2 we obtain, according to the energy convention in Theorem 3.30:

$$\frac{R}{n^2} = -\frac{E_n}{h_0 c}$$

But these are exactly the numbers which are subject to substraction in the Rydberg formula, and so we are led to the conclusion in the statement. □

With these spectacular applications explained, let us go back now to our study of the Schrödinger equation, done throughout this chapter. Our conclusions are as follows:

THEOREM 3.32. *The wave functions of the hydrogen atom are the following functions, labelled by three quantum numbers, n, l, m,*

$$\phi_{nlm}(r,s,t) = \rho_{nl}(r)\alpha_l^m(s,t)$$

where $\rho_{nl}(r) = p^{l+1}e^{-p}v(p)/r$ with $p = \alpha r$ as before, with the coefficients of v subject to

$$c_{j+1} = \frac{2(j+l+1-n)}{(j+1)(j+2l+2)} \cdot c_j$$

and $\alpha_l^m(s,t)$ being the spherical harmonics found before.

PROOF. This follows indeed by putting together all the results obtained so far, and with the remark that everything is up to the normalization of the wave function. ☐

In what regards the main wave function, that of the ground state, we have:

THEOREM 3.33. *With the hydrogen atom in its ground state, the wave function is*

$$\phi_{100}(r,s,t) = \frac{1}{\sqrt{\pi a^3}}\, e^{-r/a}$$

where $a = 1/\alpha$ is the inverse of the parameter appearing in our computations above,

$$\alpha = \frac{\sqrt{-2mE}}{h}$$

called Bohr radius of the hydrogen atom. This Bohr radius is the mean distance between the electron and the proton, in the ground state, and is given by the formula

$$a = \frac{h^2}{mKe^2}$$

which numerically means $a \simeq 5.291 \times 10^{-11}$.

PROOF. There are several things going on here, as follows:

(1) According to the various formulae in the proof of Theorem 3.30, taken at $n = 1$, the parameter α appearing in the computations there is given by:

$$\alpha = \frac{\sqrt{-2mE}}{h} = \frac{1}{h} \cdot m \cdot \frac{Ke^2}{h} = \frac{mKe^2}{h^2}$$

Thus, the inverse $\alpha = 1/a$ is indeed given by the formula in the statement.

(2) Regarding the wave function, according to Theorem 3.32 this consists of:

$$\rho_{10}(r) = \frac{2e^{-r/a}}{\sqrt{a^3}} \quad , \quad \alpha_0^0(s,t) = \frac{1}{2\sqrt{\pi}}$$

By making the product, we obtain the formula of ϕ_{100} in the statement.

(3) But this formula of ϕ_{100} shows in particular that the Bohr radius a is indeed the mean distance between the electron and the proton, in the ground state. ☐

In order to advance, we can use more mathematics, as follows:

PROPOSITION 3.34. *We have $v(p) = L_{n-l-1}^{2l+1}(p)$, with the polynomials on the right, called associated Laguerre polynomials, being given by*

$$L_q^p(x) = (-1)^p \left(\frac{d}{dx}\right)^p L_{p+q}(x)$$

with L_{p+q} being the Laguerre polynomials, given by the following formula:

$$L_q(x) = \frac{e^x}{q!} \left(\frac{d}{dx}\right)^q (e^{-x} x^q)$$

PROOF. The story here is very similar to that of the Legendre polynomials. Consider the Hilbert space $H = L^2[0, \infty)$, with the following scalar product on it:

$$< f, g >= \int_0^\infty f(x)g(x)e^{-x}\, dx$$

The orthogonal basis obtained by applying Gram-Schmidt to the basis $\{x^q\}$ is then the basis formed by the Laguerre polynomials $\{L_q\}$, and this gives the results. □

With the above result in hand, we can now improve Theorem 3.32, as follows:

THEOREM 3.35. *The wave functions of the hydrogen atom are given by*

$$\phi_{nlm}(r, s, t) = \sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n(n+l)!}}\, e^{-r/na} \left(\frac{2r}{na}\right)^l L_{n-l-1}^{2l+1}\left(\frac{2r}{na}\right) \alpha_l^m(s, t)$$

with $\alpha_l^m(s, t)$ being the spherical harmonics found before.

PROOF. This follows indeed by putting together the various results above. □

3e. Exercises

Exercises:

EXERCISE 3.36.

EXERCISE 3.37.

EXERCISE 3.38.

EXERCISE 3.39.

EXERCISE 3.40.

EXERCISE 3.41.

EXERCISE 3.42.

EXERCISE 3.43.

Bonus exercise.

CHAPTER 4

Fine structure

4a. Fine structure

Fine structure.

4b. Angular momentum

Angular momentum.

4c. Electron spin

Electron spin.

4d. Hydrogen, again

Hydrogen, again.

4e. Exercises

Exercises:

Exercise 4.1.

Exercise 4.2.

Exercise 4.3.

Exercise 4.4.

Exercise 4.5.

Exercise 4.6.

Exercise 4.7.

Exercise 4.8.

Bonus exercise.

Part II

Protons, neutrons

The cars crawl past all stuffed with eyes
Street lights share their hollow glow
Your brain seems bruised with numb surprise
Still one place to go

CHAPTER 5

Atomic nucleus

5a.

5b.

5c.

5d.

5e. Exercises

Exercises:

Exercise 5.1.

Exercise 5.2.

Exercise 5.3.

Exercise 5.4.

Exercise 5.5.

Exercise 5.6.

Exercise 5.7.

Exercise 5.8.

Bonus exercise.

CHAPTER 6

Radioactivity

6a.

6b.

6c.

6d.

6e. Exercises

Exercises:

Exercise 6.1.

Exercise 6.2.

Exercise 6.3.

Exercise 6.4.

Exercise 6.5.

Exercise 6.6.

Exercise 6.7.

Exercise 6.8.

Bonus exercise.

CHAPTER 7

Atomic bombs

7a.

7b.

7c.

7d.

7e. Exercises

Exercises:

Exercise 7.1.

Exercise 7.2.

Exercise 7.3.

Exercise 7.4.

Exercise 7.5.

Exercise 7.6.

Exercise 7.7.

Exercise 7.8.

Bonus exercise.

CHAPTER 8

Stars and fusion

8a.

8b.

8c.

8d.

8e. Exercises

Exercises:

EXERCISE 8.1.

EXERCISE 8.2.

EXERCISE 8.3.

EXERCISE 8.4.

EXERCISE 8.5.

EXERCISE 8.6.

EXERCISE 8.7.

EXERCISE 8.8.

Bonus exercise.

Part III

Further particles

They got an alligator stew and a crawfish pie
A Gulf storm blowing into town tonight
Living on the Delta's quite a show
They got hurricane parties every time it blows

CHAPTER 9

Antiparticles

9a.

9b.

9c.

9d.

9e. Exercises

Exercises:

EXERCISE 9.1.

EXERCISE 9.2.

EXERCISE 9.3.

EXERCISE 9.4.

EXERCISE 9.5.

EXERCISE 9.6.

EXERCISE 9.7.

EXERCISE 9.8.

Bonus exercise.

CHAPTER 10

Mesons

10a.

10b.

10c.

10d.

10e. Exercises

Exercises:

Exercise 10.1.

Exercise 10.2.

Exercise 10.3.

Exercise 10.4.

Exercise 10.5.

Exercise 10.6.

Exercise 10.7.

Exercise 10.8.

Bonus exercise.

CHAPTER 11

Neutrinos

11a.

11b.

11c.

11d.

11e. Exercises

Exercises:

EXERCISE 11.1.

EXERCISE 11.2.

EXERCISE 11.3.

EXERCISE 11.4.

EXERCISE 11.5.

EXERCISE 11.6.

EXERCISE 11.7.

EXERCISE 11.8.

Bonus exercise.

CHAPTER 12

Strange particles

12a.

12b.

12c.

12d.

12e. Exercises

Exercises:

EXERCISE 12.1.

EXERCISE 12.2.

EXERCISE 12.3.

EXERCISE 12.4.

EXERCISE 12.5.

EXERCISE 12.6.

EXERCISE 12.7.

EXERCISE 12.8.

Bonus exercise.

Part IV

Quarks and gluons

We're heading for something
Somewhere I've never been
Sometimes I am frightened but I'm ready to learn
Of the power of love

CHAPTER 13

Quarks, gluons

13a.

13b.

13c.

13d.

13e. Exercises

Exercises:

Exercise 13.1.

Exercise 13.2.

Exercise 13.3.

Exercise 13.4.

Exercise 13.5.

Exercise 13.6.

Exercise 13.7.

Exercise 13.8.

Bonus exercise.

CHAPTER 14

The Standard Model

14a.

14b.

14c.

14d.

14e. Exercises

Exercises:

Exercise 14.1.

Exercise 14.2.

Exercise 14.3.

Exercise 14.4.

Exercise 14.5.

Exercise 14.6.

Exercise 14.7.

Exercise 14.8.

Bonus exercise.

CHAPTER 15

The Big Bang

15a.

15b.

15c.

15d.

15e. Exercises

Exercises:

Exercise 15.1.

Exercise 15.2.

Exercise 15.3.

Exercise 15.4.

Exercise 15.5.

Exercise 15.6.

Exercise 15.7.

Exercise 15.8.

Bonus exercise.

CHAPTER 16

Dark matter

16a.

16b.

16c.

16d.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter.

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