STATIONARITY QUESTIONS FOR TRANSITIVE QUANTUM GROUPS

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ABSTRACT. We study the universal flat models $\pi : C(G) \to M_N(C(X))$ associated to the transitive subgroups $G \subset S_N^+$. When $\Gamma = \widehat{G}$ is amenable, one interesting question is whether π extends into a von Neumann algebraic model $\tilde{\pi} : L^\infty(G) \to M_N(L^\infty(X))$. Such models are called "stationary", and we obtain here several new results about them, regarding notably the finite quantum group case, and the Weyl matrix models.

INTRODUCTION

The quantum permutation group S_N^+ was constructed by Wang in [27]. Among the subgroups $G \subset S_N^+$, of particular interest are those which are transitive, in the sense that the standard coordinates $u_{ij} \in C(G)$, known to be projections, are all nonzero.

The basic examples here are the usual transitive subgroups $G \subset S_N$. By using various product operations, one can construct from these non-classical examples as well, $G \not\subset S_N$. Other interesting examples come as quantum automorphism groups of transitive graphs, $G = G^+(X)$ with $|X| = N$. There are as well many interesting examples coming via matrix model constructions, by using Weyl type matrices, or Hadamard matrices.

We discuss in this paper the analytic approach to such quantum groups, via matrix models $\pi : C(G) \to M_N(C(X))$. Such a model if called "flat" when the projections $P_{ij}^x = \pi(u_{ij})(x)$ with $x \in X$ are all nonzero. We are mainly interested here in the universal flat model, obtained by gluing together all the flat models for $C(G)$.

An interesting question is whether π is stationary, in the sense that it extends into a von Neumann algebraic model $\tilde{\pi} : L^\infty(G) \to M_N(L^\infty(X))$. If this is the case, the algebra $L^\infty(G)$ must be of type I, and so the discrete dual $\Gamma = \widehat{G}$ must be amenable.

Summing up, we are interested in the subgroups $G \subset S_N^+$ which are transitive and coamenable. We will perform here a systematic study of such quantum groups.

The paper is organized as follows: 1-4 contain preliminaries and some new classical group results, in 5-8 we develop some general theory, in the quantum group case, and in 9-12 we discuss the Weyl matrix models, and the Hadamard matrix models.

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1. QUANTUM PERMUTATIONS

We use Woronowicz's quantum group formalism in [30], [31], with the extra assumption $S^2 = id$. An extra source of useful information comes from the more recent papers of Maes and Van Daele [20] and Malacarne [21], which review this material, with a few simplifications. There is as well the book by Neshveyev and Tuset [23].

We recall that a magic unitary matrix is a square matrix over a C^*-algebra, $u \in M_N(A)$, whose entries are projections ($p^2 = p^* = p$), summing up to 1 on each row and each column. The following key definition is due to Wang [28]:

Definition 1.1. $C(S_N^+)$ is the universal C^*-algebra generated by the entries of a $N \times N$ magic unitary matrix $u = (u_{ij})$, with the morphisms given by

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj} \quad , \quad \varepsilon(u_{ij}) = \delta_{ij} \quad , \quad S(u_{ij}) = u_{ji}$$

as comultiplication, counit and antipode.

This algebra satisfies Woronowicz' axioms, along with the extra condition $S^2 = id$, and the underlying compact quantum group S_N^+ is called quantum permutation group. Observe that we have an inclusion $S_N \subset S_N^+$, given at the algebra level by:

$$u_{ij} = \chi\left(\sigma \in S_N \Big| \sigma(j) = i\right)$$

This inclusion is an isomorphism at $N = 2, 3$, but not at $N \geq 4$, where S_N^+ is a non-classical, infinite quantum group. In fact, we have $S_4^+ = SO_3^{-1}$, and S_N^+ with $N \geq 5$ still has the same fusion rules as SO_3, but is not coamenable. See [5], [28].

Any closed subgroup $G \subset S_N^+$ can be thought of as "acting" on the set $\{1, \ldots, N\}$, and one can talk about the orbits of this action. The theory here was developed in [14], and also recently in [11]. In what follows, we will only need the following notions:

Definition 1.2. Let $G \subset S_N^+$ be a closed subgroup, with magic unitary $u = (u_{ij})$, and consider the equivalence relation on $\{1, \ldots, N\}$ given by $i \sim j \iff u_{ij} \neq 0$.
   (1) The equivalence classes under $\sim$ are called orbits of G.
   (2) G is called transitive when the action has a single orbit.
In other words, we call a subgroup $G \subset S_N^+$ transitive when $u_{ij} \neq 0$, for any i, j.

Here the fact that $\sim$ as defined above is indeed an equivalence relation follows by applying Δ, ε, S to a formula of type $u_{ij} \neq 0$. For details, see [11].

In the classical case, $G \subset S_N$, we recover in this way the usual notions of orbits and transitivity. In the group dual case, $\widehat{\Gamma} \subset S_N^+$, we recover the notions from [14]. In general, there are many interesting examples of subgroups $G \subset S_N^+$ which are transitive, or at least quasi-transitive, in the sense that all the orbits have the same size. See [11].

To be more precise, in the transitive case, which is the one that we are interested in, in the present paper, we have the following well-known result:

Theorem 1.3. *The following are transitive subgroups $G \subset S_N^+$:*

(1) *The transitive subgroups $G \subset S_N$. These are the classical examples.*
(2) *The subgroups $\widehat{G} \subset S_{|G|}$, with G abelian. These are the group dual examples.*
(3) *The quantum groups $F \subset S_N^+$ which are finite, $|F| < \infty$, and transitive.*
(4) *The quantum automorphism groups of transitive graphs $G^+(X)$, with $|X| = N$.*
(5) *In particular, we have the hyperoctahedral quantum group $H_n^+ \subset S_N^+$, with $N = 2n$.*
(6) *We have as well the twisted orthogonal group $O_n^{-1} \subset S_N^+$, with $N = 2^n$.*
(7) *The quantum permutation groups coming from the Weyl matrix models.*
(8) *The quantum permutation groups coming from Hadamard matrix models.*

In addition, the class of transitive quantum permutation groups $\{G \subset S_N^+ | N \in \mathbb{N}\}$ is stable under direct products $\times$, wreath products $\wr$ and free wreath products $\wr_*$.

Proof. All these assertions are well-known. In what follows we briefly describe the idea of each proof, and indicate a reference. We will be back to all these examples, gradually, in the context of certain matrix modelling questions, to be formulated later on.

(1) This is something trivial. Indeed, for a classical group $G \subset S_N$, the variables $u_{ij} = \chi(\sigma \in S_N | \sigma(j) = i)$ are all nonzero precisely when G is transitive. See [11].

(2) This follows from the general results of Bichon in [14], who classified there all the group dual subgroups $\widehat{\Gamma} \subset S_N^+$. For a discussion here, we refer to [11].

(3) Here we use the convention $|F| = \dim_{\mathbb{C}} C(F)$, and the statement itself is empty, and is there just for reminding us that these examples are to be investigated.

(4) This is trivial, because X being transitive means that $G(X) \curvearrowright X$ is transitive, and by definition of $G^+(X)$, we have $G(X) \subset G^+(X)$. See [18], [24].

(5) This comes from a result from [6], stating that we have $H_n^+ = G^+(I_n)$, where I_n is the graph formed by n segments, having $N = 2n$ vertices.

(6) Once again this comes from a result from [6], stating that we have $O_n^{-1} = G^+(K_n)$, where K_n is the n-dimensional hypercube, having $N = 2^n$ vertices.

(7) The idea here is that Pauli matrices, or more generally the Weyl matrices, produce via matrix model theory certain transitive subgroups $G \subset S_N^+$. See [12].

(8) Once again, we have here a more delicate construction, coming from matrix model theory, and that we will explain more in detail later on. We refer here to [5].

Finally, the stability assertion is clear from the definition of the various products involved, from [13], [27]. This is well-known, and we will be back later on to this. $\square$

Summarizing, we have a substantial list of examples, to be investigated.

We should mention that we are in fact mainly interested in the case where G is coamenable, in the sense that its discrete dual $\Gamma = \widehat{G}$ is amenable. Here the main examples come from (1,2,3), then from (6,7,8), and from the operations $\times, \wr$.

As for the quantum groups in (4,5), and those coming from $\wr_*$, these are generically non-coamenable. We will be back later on with full details on all this.

2. Matrix models

Consider a closed subgroup $G \subset S_N^+$, and a matrix model $\pi : C(G) \to M_N(\mathbb{C})$. The elements $P_{ij} = \pi(u_{ij})$ are then projections, and they form a magic matrix $P = (P_{ij})$. We can then look at the matrix $d_{ij} = tr(P_{ij})$, which is positive and bistochastic, with sums 1. The simplest situation is when $d = (1/N)_{ij}$ is the linear algebra-theoretic flat matrix, and in this case, we call our model "flat". Observe that in this case, we have:

$$d_{ij} \neq 0 \implies P_{ij} \neq 0 \implies u_{ij} \neq 0$$

Thus, in order for the algebra $C(G)$ to admit a flat matrix model, $G \subset S_N^+$ must be transitive. More generally now, we can talk about the flatness of the parametric models of $C(G)$, and we are led in this way to the following notions, from [12]:

Definition 2.1. *Consider a transitive subgroup $G \subset S_N^+$.*

(1) *A matrix model $\pi : C(G) \to M_N(C(X))$, with X being a compact space, is called flat when the projections $P_{ij}^x = \pi(u_{ij})(x)$ have rank 1, for any i, j, x.*

(2) *The universal flat model for $C(G)$, obtained by imposing the Tannakian conditions which define G, is denoted $\pi_G : C(G) \to M_N(C(X_G))$.*

Here the notion in (1) corresponds indeed to the flatness as defined above, because $tr(P_{ij}) = \frac{1}{N} \cdot rank(P_{ij})$. As for the construction in (2), the idea here is that in order to have a morphism $\pi : C(G) \to M_N(C(X))$, the elements $P_{ij} = \pi(u_{ij})$ must satisfy the same relations as the variables $u_{ij} \in C(G)$. But these latter relations are basically those of type $T \in Hom(u^{\otimes k}, u^{\otimes l})$, coming from Tannakian duality [21], [31], and formally imposing the conditions $T \in Hom(P^{\otimes k}, P^{\otimes l})$ leads to a certain compact algebraic manifold X_G, which is the desired universal model space. For full details here, we refer to [12].

We would like to understand the faithfulness properties of the various flat models, including those of the universal one. We use the following notions:

Definition 2.2. *A matrix model $\pi : C(G) \to M_N(C(X))$ is called:*

(1) *Inner faithful, when there is no factorization $\pi : C(G) \to C(H) \to M_N(C(X))$, with $H \subset G$ being a proper closed subgroup.*

(2) *Stationary, when the Haar integration over G appears as $\int_G = (tr \otimes \int_X)\pi$, where $\int_X$ is the integration with respect to a probability measure on X.*

These notions are both quite subtle. Regarding (1), in the group dual case, $G = \widehat{\Gamma}$, our model must come from a group representation $\rho : \Gamma \to C(X, U_N)$, and the inner faithfulness of π means precisely that ρ must be faithful. In general, what we have here is an extension of this fact. As for (2), the notion there comes from the idempotent state work on the inner faithfulness property in [3], [29]. Let us just mention here, as a basic fact regarding the stationarity, that this property implies the faithfulness.

The main theoretical result on the subject, from [29], is as follows:

Theorem 2.3. *A matrix model $\pi : C(G) \to M_N(C(X))$, with X being a compact probability space, is inner faithful if and only if*

$$\int_G = \lim_{k \to \infty} \frac{1}{k} \sum_{r=1}^{k} \psi^{*r}$$

where $\psi = (tr \otimes \int_X)\pi$, and where $\varphi * \phi = (\varphi \otimes \phi)\Delta$.

Proof. The "only if" part, which reminds Woronowicz's construction of the Haar functional in [30], as a Cesàro limit with respect to the convolution, is non-trival, and uses idempotent state theory from [19], along with some extra functional analysis ingredients. As for the "if" part, this follows by performing the Hopf image construction, $\pi : C(G) \to C(G') \to M_N(C(X))$. Indeed, since the Haar functionals of $C(G)$ and $C(G')$ are given by the same formula, we obtain $G = G'$, as claimed. See [29]. $\square$

Observe that the Cesàro convergence in Theorem 2.3 is stationary precisely when the matrix model is stationary, in the sense of Definition 2.2 (2) above.

We have as well the following useful stationarity criterion, from [3]:

Proposition 2.4. *For matrix model $\pi : C(S_N^+) \to M_N(C(X))$, given by $u_{ij} \to U_{ij}^x$, with X being a compact probability space, the following are equivalent:*

(1) *$Im(\pi)$ is a Hopf algebra, and $(tr \otimes \int_X)\pi$ is the Haar integration on it.*
(2) *$\psi = (tr \otimes \int_X)\pi$ satisfies the idempotent state property $\psi * \psi = \psi$.*
(3) *$T_p^2 = T_p, \forall p \in \mathbb{N}$, where $(T_p)_{i_1 \ldots i_p, j_1 \ldots j_p} = (tr \otimes \int_X)(U_{i_1 j_1} \ldots U_{i_p j_p})$.*

If these conditions are satisfied, we say that π is stationary on its image.

Proof. Consider the Hopf image factorization $\pi : C(S_N^+) \to C(G) \to M_K(C(X))$. Since the map on the right is inner faithful, we can use the formula in Theorem 2.3.

We will need as well the following elementary formula, where $\psi = (tr \otimes \int_X)\pi$ is as in (2), and where $(T_p)_{i_1 \ldots i_p, j_1 \ldots j_p} = (tr \otimes \int_X)(U_{i_1 j_1} \ldots U_{i_p j_p})$ is as in (3):

$$\psi^{*r}(u_{i_1 j_1} \ldots u_{i_p j_p}) = (T_p^r)_{i_1 \ldots i_p, j_1 \ldots j_p}$$

With these formulae in hand, the proof goes as follows:

(1) $\implies$ (2) This is clear from definitions, because the Haar integration on any quantum group satisfies the idempotent state equation $\psi * \psi = \psi$.

(2) $\implies$ (3) Assuming $\psi * \psi = \psi$, by using the above formula at $r = 1, 2$ we obtain that the matrices T_p and T_p^2 have the same coefficients, and so they are equal.

(3) $\implies$ (1) Assuming $T_p^2 = T_p$, by using the above formula we obtain $\psi = \psi * \psi$, and so $\psi^{*r} = \psi$ for any $r \in \mathbb{N}$. Thus the Cesàro limiting formula gives $\int_G = \psi$, and together with a standard functional analysis discussion, this finishes the proof. See [3]. $\square$

Finally, let us mention that the notion of stationarity is closely related to Thoma's theorem [26]. We refer to [9], [10], [11] for a full discussion here.

3. Classical groups

As an illustration for various notions introduced above, let us first discuss the classical case. With the convention that we identify the rank one projections in $M_N(\mathbb{C})$ with the elements of the projective space $P_{\mathbb{C}}^{N-1}$, we have the following result, from [11]:

Proposition 3.1. *Given a transitive group $G \subset S_N$, the associated universal flat model space is $X_G = E_N \times L_G$, where:*

$$E_N = \left\{ P_1, \ldots, P_N \in P_{\mathbb{C}}^{N-1} \middle| P_i \perp P_j, \forall i, j \right\}$$

$$L_G = \left\{ \sigma_1, \ldots, \sigma_N \in G \middle| \sigma_1(i), \ldots, \sigma_N(i) \text{ distinct}, \forall i \in \{1, \ldots, N\} \right\}$$

In addition, assuming that we have $L_G \neq \emptyset$, the universal flat model is stationary, with respect to the Haar measure on E_N times the discrete measure on L_G.

Proof. The point here is that two commuting rank 1 projections must be either equal, or proportional. Thus, a flat model for $C(G)$ must be of the form $u_{ij} \to P_{L_{ij}}$, with $P \in E_N$ and with $L \in M_N(1, \ldots, N)$ being a Latin square, and this gives the first assertion.

Regarding the second assertion, the idea here is that we have a natural action by translation $G \curvearrowright L_G$, which shows that the random matrix trace on $C(G)$ must be G-equivariant, and therefore equal to the Haar integration. See [11]. $\square$

Generally speaking, the condition $L_G \neq \emptyset$ can be thought of as being a "strong transitivity" condition, imposed on G. In order to discuss this property, let us introduce:

Definition 3.2. *Given a transitive subgroup $G \subset S_N$, its transitivity level is:*

$$l(G) = \min \left\{ |S| : S \subset G, \forall i, j \ \exists \sigma \in S, \sigma(j) = i \right\}$$

We say that G is strongest transitive when its level is minimal, $l(G) = N$.

Here the fact that we have $l(G) \geq N$ simply follows from the fact that we must have elements $\sigma_1, \ldots, \sigma_N \in S$ such that $\sigma_i(1) = i$, for any $i \in \{1, \ldots, N\}$.

Let us recall as well that a Latin square is a matrix $L \in M_N(1, \ldots, N)$ having the property that each of its rows and its columns is a permutation of $1, \ldots, N$.

We can slightly reformulate Proposition 3.1, as follows:

Proposition 3.3. *For a transitive subgroup $G \subset S_N$, the following are equivalent:*

(1) *The universal flat model for $C(G)$ is stationary.*
(2) *The flat model space for $C(G)$ is non-empty, $X_G \neq \emptyset$.*
(3) *There exist $\sigma_1, \ldots, \sigma_N \in G$ such that $\{\sigma_1(i), \ldots, \sigma_N(i)\} = \{1, \ldots, N\}$, $\forall i$.*
(4) *There exist $\sigma_1, \ldots, \sigma_N \in G$ and $\tau_1, \ldots, \tau_N \in S_N$ such that $\sigma_j(i) = \tau_i(j)$, $\forall i, j$.*
(5) *G has a subgroup generated by the rows of a Latin square $L \in M_N(1, \ldots, N)$.*
(6) *G is strongest transitive, $l(G) = N$, in the sense of Definition 3.2.*

Proof. We know from Proposition 3.1 that (1) $\iff$ (2) $\iff$ (3) hold indeed, and the equivalences (3) $\iff$ (4) $\iff$ (5) and (3) $\iff$ (6) are clear as well. $\square$

Among the above conditions, (5) looks probably the most appealing. As an illustration, at $N = 4$ there are four normalized Latin squares, as follows:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix} , \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 2 & 1 \\ 4 & 3 & 1 & 2 \end{pmatrix} , \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix} , \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

By using these squares and doing a case-by-case analysis, one can conclude from this that any transitive subgroup $G \subset S_4$ is automatically strongest transitive.

In general, however, such techniques appear to be quite weak. An alternative approach, which is more useful, uses the condition (3) above, as follows:

Theorem 3.4. *For a transitive subgroup $G \subset S_N$, the following are equivalent:*

(1) *G is strongest transitive in our sense, and so it satisfies the various equivalent conditions in Proposition 3.3 above.*

(2) *There exist $\sigma_1, \ldots, \sigma_N \in G$ such that $\sigma_i^{-1}\sigma_j \in D_G$ for any $i \neq j$, where $D_G \subset G$ is the set of derangements, i.e. of permutations without fixed points.*

(3) *There exist $\tau_1, \ldots, \tau_N \in G$ satisfying $\tau_1 \ldots \tau_N = 1$, such that any cyclic product of type $\tau_i \ldots \tau_j$ having non-trivial length, $i \neq j$, belongs to D_G.*

Proof. In this statement, (1) $\iff$ (2) is trivial. Regarding now (2) $\implies$ (3), this comes by setting $\tau_i = \sigma_i^{-1}\sigma_{i+1}$, which gives $\tau_i \ldots \tau_j = \sigma_i^{-1}\sigma_j$. Finally, regarding (3) $\implies$ (2), here we can set $\sigma_i = \tau_1 \ldots \tau_i$, and we obtain $\sigma_i^{-1}\sigma_j = \tau_i \ldots \tau_j$, which gives the result. $\square$

As a first trivial consequence, we have:

Proposition 3.5. *Assuming that $G \subset S_N$ is strongest transitive, we must have*

$$|D_G| \geq N - 1$$

where $D_G \subset G$ denotes as usual the subset of derangements.

Proof. This follows indeed from the criterion in Theorem 3.4 (2), because the elements $\sigma_i^{-1}\sigma_j \in D_G$ appearing there are distinct, when $i = 1$ and $j \in \{2, \ldots, N\}$. $\square$

The above result is quite interesting, because it makes the connection with the machinery developed in [16], and notably with the following estimate, discussed there:

$$\frac{|D_G|}{|G|} \geq \frac{1}{N}$$

Another point of interest comes from the fact that $|D_G|$, or rather $\frac{|D_G|}{|G|}$, is a spectral quantity, equal to the weight of the Dirac mass of 0 in the spectral distribution of the main character $\chi : S_N \to \mathbb{N}$. We will be back to this later on, in section 5 below.

4. DERANGING SUBGROUPS

We discuss here some concrete applications of the various criteria found above.

Let us call a permutation group $H \subset S_N$ deranging when $H = D_H \cup \{1\}$. With this notion in hand, we have the following application of Theorem 3.4 above:

Proposition 4.1. *Consider the following conditions, regarding a transitive group $G \subset S_N$:*

(1) *G has a deranging subgroup $H \subset G$ of order N.*
(2) *G has a deranging subgroup $H \subset G$ of order $\geq N$.*
(3) *G is strongest transitive, in our sense.*

We have then (1) $\implies$ (2) $\implies$ (3).

Proof. In this statement, (1) $\implies$ (2) is trivial. Regarding now (2) $\implies$ (3), let us pick a subset $\{\sigma_1, \ldots, \sigma_N\} \subset H$. We have then $\sigma_i^{-1}\sigma_j \in H - \{1\} = D_H \subset D_G$ for any $i \neq j$, and so the strongest transitivity condition in Theorem 3.4 (2) is satisfied. $\square$

The idea in what follows will be that of studying the groups having the property (1). Indeed, most of the examples of strongest transitive groups have this property.

We recall that a subgroup $G \subset S_N$ is called sharply 2-transitive if it acts simply transitively on the set of pairs of distinct elements in $\{1, \ldots, N\}$.

The above criterion has a number of straightforward applications, as follows:

Proposition 4.2. *The property for a transitive group $G \subset S_N$ to have a deranging subgroup of order N is automatic when:*

(1) *N is prime.*
(2) *$N \leq 5$.*
(3) *G is sharply 2-transitive.*

Proof. These assertions follow from a basic algebraic study, as follows:

(1) When $N = p$ is prime the order $n = |G|$ satisfies $p|n|p!$, and so G has an element of order p. Thus we obtain a copy $\mathbb{Z}_p \subset G$, which must be deranging.

(2) We are already done with $N = 2, 3, 5$, and at $N = 4$ the study is as follows:

– $|G| = 4$. Here we can take $H = G \simeq \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4$.
– $|G| = 8$. Here G contains a copy of $\mathbb{Z}_4$, and we are done again.
– $|G| = 12$. Here $G = A_{12}$, which contains a copy $\mathbb{Z}_2 \times \mathbb{Z}_2$ as desired.
– $|G| = 24$. Here $G = S_4$, which contains a copy of $\mathbb{Z}_4$, and we are done again.

(3) The finite sharply 2-transitive groups were classified in [32], and are precisely those of transformations of the form $(x \in D) \to (ax + b \in D)$ on finite near-fields D, i.e. sets equipped with a not-necessarily-abelian group structure $(D, +, 0)$ and a second group structure $(D - \{0\}, \cdot, 1)$ that is left-distributive over the first. In particular, the group $(D, +)$ of all translations $x \mapsto x + b$ is a deranging subgroup of G. $\square$

One interesting question now is about what happens at $N = 6$. Here the criterion in Proposition 4.1 does not apply any longer, as shown by the following result:

Theorem 4.3. *Consider the subgroup $PGL_2(p) \subset S_{p+1}$, with $p \geq 3$ prime.*

(1) *This group is strongest transitive, at any $p \geq 3$.*

(2) *However, $PGL_2(5)$ has no deranging subgroup of order 6.*

Proof. Consider indeed the group $G = PGL_2(p)$, acting on the projective line $\mathbb{P}_p$, which has $p+1$ elements. Since there are $(p-1)^3 + (2p-1)^2$ solutions of $ad = bc$, we have:

$$|GL_2(p)| = p^4 - (2p-1)^2 - (p-1)^3 = (p-1)^2 p(p+1)$$

Thus, we have $|PGL_2(p)| = (p-1)p(p+1)$. We will also need the fact, which is well-known and nontrivial, that at $p = 5$ we have $G \simeq S_5$.

(1) Consider an element $x \in G$ whose eigenvalues in $\mathbb{F}_{p^2}$ are generators of the multiplicative group $\mathbb{F}_{p^2}^{\times}$, i.e. are primitive roots of unity of order $p^2 - 1$.

The choice of eigenvalues ensures that the eigenvalues of $x, x^2, \ldots, x^p$ are outside $\mathbb{F}_p$. Thus, these elements of G are not diagonalizable. In conclusion, we have $p+1$ elements, namely $x_j = x^j$ with $0 \leq j \leq p$, with the property that $x_i^{-1} x_j = x^{j-i}$ is a derangement, for any $i \neq j$. But this is precisely the requirement of strongest transitivity.

(2) Assume that we have a deranging subgroup $H \subset G$ of order 6. There are two possibilities: either H is abelian, and so $H \simeq \mathbb{Z}_3 \times \mathbb{Z}_2$, or H is non-abelian, and so $H \simeq S_3$. Either way H contains an element of order 3 in the group $G \simeq S_5$. Since all elements of order 3 are pairwise conjugate in this latter group, we may as well assume that H contains the group generated by the representative in $PGL_2(5)$ of:

$$x = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

Now consider an order 2 element $y \in H$. We have either $yxy^{-1} = x$ or $yxy^{-1} = x^{-1}$ modulo scalars, but since yxy^{-1} and x must have the same eigenvalues with product 1 and sum -1, if one of these conditions holds modulo scalars then it holds literally.

The 2-dimensional algebra generated by x is maximal abelian in $M_2(\mathbb{F}_5)$, so $yxy^{-1} = x$ is impossible and we are left with $yxy^{-1} = x^{-1}$. This implies that y is one of the matrices t, tx or tx^2, where t is the permutation matrix attached to the flip, namely:

$$t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

All of these elements are easily seen to have fixed points upon acting on $\mathbb{P}_5$, hence the conclusion that some non-trivial elements of H have fixed points. $\square$

In relation with the proof of (1), observe that we cannot employ the same technique for finding $p+2$ elements x_j with the derangement property there. Indeed, the eigenvalues of x^{p+1} are $(p-1)^{th}$ roots of unity and hence belong to $\mathbb{F}_p$. In conclusion, among any $p+2$ consecutive powers x_i of x one of the elements $x_i^{-1} x_j$ will be diagonalizable.

Summarizing, we have some advances in the classical case. We will be back to these questions in the next section, after developing some more general theory.

5. General theory

In this section and in the next few ones we discuss the general case, $G \subset S_N^+$, with some inspiration from the classical case. Regarding the notion of transitivity for the arbitrary quantum subgroups $G \subset S_N^+$, we first have the following result, from [9]:

Proposition 5.1. *For a subgroup $G \subset S_N^+$, the following are equivalent:*

(1) *G is transitive.*
(2) *$Fix(u) = \mathbb{C}\xi$, where ξ is the all-one vector.*
(3) *$\int_G u_{ij} = \frac{1}{N}$, for any i, j.*

Proof. Here (1) $\implies$ (2) follows from [14], (2) $\implies$ (3) follows by using the general theory in [30], and (3) $\implies$ (1) is trivial. For details here, we refer to [9]. $\square$

In order to reach now to stationarity questions, via analytic techniques, let us first go back to the classical case. We have here the following well-known result:

Proposition 5.2. *Given a subgroup $G \subset S_N$, regarded as an algebraic group $G \subset O_N$ by using the standard permutation matrices, the law of its main character is of the form*

$$\mu = c_0 \delta_0 + c_1 \delta_1 + \ldots + c_{N-1} \delta_{N-1} + c_N \delta_N$$

with the coefficients $c_i \in [0, 1]$ satisfying $c_0 = \frac{|D_G|}{|G|}, c_{N-1} = 0, c_N = \frac{1}{|G|}$.

Proof. The permutation matrices being given by $P_\sigma(e_i) = e_{\sigma(i)}$, where $\{e_1, \ldots, e_N\}$ is the standard basis of $\mathbb{R}^N$, the main character of G is given by:

$$\chi(\sigma) = Tr(P_\sigma) = \# \left\{ i \in \{1, \ldots, N\} \Big| \sigma(i) = i \right\}$$

In other words, the main character $\chi : G \to \mathbb{R}$ counts the number of fixed points. Thus μ is given by the formula in the statement, with the coefficients c_i being:

$$c_i = \frac{1}{|G|} \# \left\{ \sigma \in G \Big| \sigma \text{ has exactly } i \text{ fixed points} \right\}$$

Finally, the formulae of c_0, c_{N-1}, c_N are all clear, by using this formula. $\square$

In relation now with our stationarity considerations, we have:

Proposition 5.3. *Assuming that $G \subset S_N$ is strongest transitive, we must have*

$$c_0 \geq (N - 1)c_N$$

where c_0, c_N are the weights of the Dirac masses at $0, N$ inside the spectral measure μ.

Proof. This is a reformulation of the condition $|D_G| \geq N - 1$ from Proposition 3.5 above, by using the formulae $c_0 = \frac{|D_G|}{|G|}$ and $c_N = \frac{1}{|G|}$ from Proposition 5.2. $\square$

In the quantum group case now, we can of course speak about the spectral measure μ of the main character $\chi = \sum_i u_{ii}$. As a basic result here, we have:

Theorem 5.4. *For a quantum permutation group $G \subset S_N^+$, the spectral measure μ of the main character $\chi = \sum_i u_{ii}$ has the following properties:*

(1) *μ is supported on $[0, N]$.*
(2) *$N \in supp(\mu)$ if and only if G is coamenable.*
(3) *G is transitive precisely when the first moment of μ is 1.*
(4) *In the finite quantum group case, the weight of δ_N inside μ is $c_N = \frac{1}{|G|}$.*

Proof. All these results are well-known, as explained for instance in [10]:

(1) This is clear from the fact that each u_{ii} is a projection.
(2) This follows from the Kesten amenability criterion, see [23].
(3) This follows from Proposition 5.1, the first moment being $\dim(Fix(u))$.
(4) This is well-known, and follows by using the Perron-Frobenius theory. $\square$

There are several other things known about μ, theorems or conjectures, and we refer here to [6], [9]. However, in relation with Proposition 5.2 and Proposition 5.3, going beyond the above results is a non-trivial question. As a first example here, the formula $c_{N-1} = 0$ from the classical case suggests that μ should have a spectral gap. However, this fails for some of the ADE quantum groups at $N = 4$, described in [6].

In relation now with our questions, Proposition 5.3 suggests that when $G \subset S_N^+$ is such that the universal flat model for $C(G)$ is stationary, we must have $c_0 \geq (N - 1)c_N$. In order to deal with the case $|G| = \infty$, we would need here an extension of the above formula $c_N = \frac{1}{|G|}$, to the case where $G \subset S_N^+$ is arbitrary. As for the finite case, $|G| < \infty$, here by stationarity we have $\mu = law(P_{11} + \ldots + P_{NN})$, for certain rank 1 projections $P_{11}, \ldots, P_{NN}$, and the problem is that of exploiting the fact that these projections are part of a magic unitary. All this looks quite plausible, but is non-trivial.

The observations that we have so far suggest the following conjecture:

Conjecture 5.5. *For a transitive closed subgroup $G \subset S_N^+$, the property of the universal flat model of $C(G)$ to be stationary is a spectral property of G, in the sense that it can be read on the distribution μ of the main character $\chi = \sum_i u_{ii}$.*

In order to comment on this statement, let us go back to the classical groups, and more specifically to the group $PGL_2(p) \subset S_{p+1}$ from section 4. We know from there that this group is strongest transitive, and this for "minimal" reasons. Thus, we can expect the associated spectral measure to be "extremal", among those allowed by Conjecture 5.5. This spectral measure is standard to compute, and is given by:

$$\mu = \frac{p}{2(p+1)}\delta_0 + \frac{1}{p}\delta_1 + \frac{p-2}{2(p-1)}\delta_2 + \frac{1}{(p-1)p(p+1)}\delta_{p+1}$$

Summarizing, we have some ingredients for approaching Conjecture 5.5, in the classical group case. Missing however, as a main ingredient, would be an example of a transitive subgroup $G \subset S_N$ which is not strongest transitive. We have no results here yet.

6. THE FINITE CASE

We keep developing here some general theory, with inspiration from the classical case. A natural framework which generalizes the classical case is that of the finite quantum groups $F \subset S_N^+$. There are many examples here, as explained for instance in [6], and some general theory is available as well, from [8]. However, the extension of the classical results appears to be a quite difficult question, and besides the various spectral theory considerations presented above, we have only some modest results on the subject.

In the group dual case, the situation is very simple, as follows:

Proposition 6.1. *The only transitive group duals $\widehat{\Gamma} \subset S_N^+$ are the subgroups*

$$\widehat{\Gamma} \subset S_{|\Gamma|}$$

with Γ being finite and abelian. The corresponding universal flat models are stationary.

Proof. Here the first assertion follows from the results of Bichon in [14], and the second assertion follows from the classical case result, applied to $G = \widehat{\Gamma} \simeq \Gamma$. $\square$

We recall now from [9], [10], [11] that a matrix model $\pi : C(G) \to M_N(C(X))$, mapping $u_{ij} \to P_{ij}^x$, is called quasi-flat when we have $rank(P_{ij}^x) \leq 1$, for any i, j, x.

With this notion in hand, we have the following result:

Proposition 6.2. *If $G \subset S_N^+$ is transitive, and $\pi : C(G) \to M_N(C(X))$ is a matrix model, mapping $u_{ij} \to P_{ij}^x$, the following are equivalent:*

(1) *π is flat.*
(2) *π is quasi-flat.*
(3) *$P_{ij}^x \neq 0$, for any i, j, x.*

Proof. All the equivalences are elementary, as follows:

(1) $\implies$ (2) This is trivial.

(2) $\implies$ (3) For any $x \in X$ the matrix d^x given by $d_{ij}^x = rank(P_{ij}^x)$ is bistochastic with sum N, and has entries $0, 1$. Thus, we must have $d^x = (1)_{ij}$, so $P_{ij}^x \neq 0$, as claimed.

(3) $\implies$ (1) Once again, the matrix d^x given by $d_{ij}^x = rank(P_{ij}^x)$ is bistochastic with sum N, and has entries ≥ 1. Thus, we must have $d^x = (1)_{ij}$, so the model is flat. $\square$

We recall that when $G \subset S_N$ is transitive, by setting $H = \{\sigma \in G | \sigma(1) = 1\}$ we have $G/H = \{1, \ldots, N\}$. Conversely, any subgroup $H \subset G$ produces an action $G \curvearrowright G/H$, given by $g(hH) = (gh)H$, and so a morphism $G \to S_N$, where $N = [G : H]$, and this latter morphism is injective when $hgh^{-1} \in H, \forall h \in G \implies g = 1$ is satisfied.

In the quantum case now, it is very unclear how to generalize this structure result. To be more precise, the various examples from [6], [8] show that we cannot expect to have an elementary generalization of the above $G/H = \{1, \ldots, N\}$ isomorphism.

However, we can at least try to extend the obvious fact that $G = N|H|$ must be a multiple of N. And here, we have the following result:

Theorem 6.3. *If $G \subset S_N^+$ is finite and transitive, then N divides $|G|$. Moreover:*

(1) *The case $|G| = N$ comes from the classical groups of order N, acting on themselves, and here the universal flat models of $C(G)$ are stationary.*

(2) *The case $|G| = 2N$ is something which is possible, in the non-classical setting, an example here being the Kac-Paljutkin quantum group, at $N = 4$.*

Proof. In order to prove the first assertion, we use the coaction of $C(G)$ on the algebra $\mathbb{C}^N = C(1, \ldots, N)$. In terms of the standard coordinates u_{ij}, the formula is:

$$\Phi : \mathbb{C}^N \to C(G) \otimes \mathbb{C}^N \quad , \quad e_i \to \sum_j u_{ij} \otimes e_j$$

For $a \in \{1, \ldots, N\}$ consider the evaluation map $\mathrm{ev}_a : \mathbb{C}^N \to \mathbb{C}$ at a. By composing Φ with $id \otimes \mathrm{ev}_a$ we obtain a $C(G)$-comodule map, as follows:

$$I_a : \mathbb{C}^N \to C(G) \quad , \quad e_i \to u_{ia}$$

Our transitivity assumption on G ensures that this map I_a is surjective, and so one-to-one. In other words, we have realized $\mathbb{C}^N$ as a coideal subalgebra of $C(G)$.

We recall now from [22] that a finite dimensional Hopf algebra is free as a module over a coideal subalgebra A provided that the latter is Frobenius, in the sense that there exists a non-degenerate bilinear form $b : A \otimes A \to \mathbb{C}$ satisfying $b(xy, z) = b(x, yz)$.

We can apply this result to the coideal subalgebra $I_a(\mathbb{C}^N) \subset C(G)$, with the remark that $\mathbb{C}^N$ is indeed Frobenius, with bilinear form $b(fg) = \frac{1}{N} \sum_{i=1}^N f(i)g(i)$. Thus $C(G)$ is a free module over the N-dimensional algebra $\mathbb{C}^N$, and this gives the result.

Regarding now the remaining assertions, the proof here goes as follows:

(1) Since $C(G) = < u_{ij} >$ is of dimension N, and its commutative subalgebra $< u_{1j} >$ is of dimension N already, $C(G)$ must be commutative. Thus G must be classical, and by transitivity, the inclusion $G \subset S_N$ must come from the action of G on itself.

Consider now the regular representation of $C(G)$, constructed as follows:

$$\lambda : C(G) \to B(l^2(G)) \simeq M_{|G|}(\mathbb{C}) \quad , \quad \lambda(x) = (y \to xy)$$

Since G is transitive and λ is faithful, we have $\lambda(u_{ij}) \neq 0$ for any i, j. Now by using our assumption $|G| = N$, Proposition 6.2 above tells us that λ is flat. On the other hand, λ is well-known to be stationary. Thus, if we take as mesure on the model space X_G the Dirac mass at the regular represention, we have the stationarity property.

(2) The closed subgroups $G \subset S_4^+$ are fully classified, and among them we have indeed the Kac-Paljutkin quantum group, which satisfies $|G| = 8$, and is transitive. See [6]. $\square$

The above result suggests further looking into the case where $|G| = 2N$, with algebraic and analytic methods. However, we have no further results here, so far.

7. Product operations

According to the list of examples in Theorem 1.3 above, we have as well a number of product operations to be investigated, including the usual product operation [27], and the classical and free wreath product operations, discussed in [13].

The first operation that we discuss is the usual tensor product. Let us recall indeed from Wang's paper [27] that we have:

Definition 7.1. *Given two closed subgroups $G \subset S_M^+, H \subset S_N^+$, with associated magic unitary matrices $u = (u_{ij})$, $v = (v_{ab})$, we construct the direct product $G \times H \subset S_{MN}^+$, with associated magic unitary matrix $w_{ia,jb} = u_{ij} \otimes v_{ab}$.*

In the classical case we recover the usual product operation for permutation groups. Observe that if G, H are transitive, then so is $G \times H$. In addition, as required by stationarity, if both G, H are coamenable, then so is $G \times H$. See [27], [28].

Quite surprisingly, this trivial notion leads to some non-trivial questions, going back to the study in [5]. Let us begin with an elementary result, as follows:

Proposition 7.2. *Given $G \subset S_M^+, H \subset S_N^+$, both transitive, consider their direct product $G \times H \subset S_{MN}^+$, and consider the associated universal flat models $\pi_G, \pi_H, \pi_{G \times H}$.*
  (1) *We have an embedding $X_G \times X_H \subset X_{G \times H}$.*
  (2) *If both π_G, π_H are faithful, then so is $\pi_{G \times H}$.*
  (3) *If both π_G, π_H are stationary, then so is $\pi_{G \times H}$.*

Proof. All these results are elementary, and follow from definitions, as follows:
  (1) Assume indeed that we have two flat models, as follows:

$$\pi : C(G) \to M_M(C(X_G)) \quad , \quad u_{ij} \to P_{ij}^x$$
$$\rho : C(H) \to M_N(C(X_H)) \quad , \quad v_{ab} \to Q_{ab}^y$$

We can form the tensor product of these models, as follows:

$$\pi \otimes \rho : C(G \times H) \to M_{MN}(C(X_G \times X_H)) \quad , \quad u_{ij} \otimes v_{ab} \to P_{ij}^x \otimes Q_{ab}^y$$

Thus we have an inclusion of model spaces $X_G \times X_H \subset X_{G \times H}$, as claimed.
  (2) This is now clear, by using the above subspace $X_G \times X_H \subset X_{G \times H}$, because the restriction of the universal representation to this parameter space is faithful.
  (3) This is clear as well, because we can take here as measure on $X_{G \times H}$ the product measure on $X_G \times X_H$, and the null measure elsewhere. $\qquad\square$

Regarding now the non-trivial questions concerning the tensor products, a first one concerns the exact computation of $X_{G \times H}$. Our conjecture here would be that $X_{G \times H}$ can be recaptured from the knowledge of X_G, X_H, and that in addition, when both X_G, X_H are assumed to be homogeneous, $X_{G \times H}$ follows to be homogeneous as well.

Here is the best result on the subject that we have, so far:

Theorem 7.3. *Assume that a closed subgroup $G \subset S_M^+$ is transitive, and consider the quantum group $G \times \mathbb{Z}_N \subset S_{NM}^+$, which is transitive as well.*

(1) *$X_{G \times \mathbb{Z}_N}$ is the total space of a bundle with fiber X_G^N, and whose base E is the space of N-tuples of pairwise orthogonal M-dimensional subspaces of $\mathbb{C}^{NM}$.*

(2) *Assuming that the model space X_G for the quantum group G is homogeneous, so is the model space $X_{G \times \mathbb{Z}_N}$, for the quantum group $G \times \mathbb{Z}_N$.*

(3) *Assuming that π_G is stationary with respect to an integration functional $\int_X$, then so is $\pi_{G \times \mathbb{Z}_2}$, with respect to the integration functional $(\int_X)^N \times \int_E$.*

Proof. The fact that $G \times \mathbb{Z}_N \subset S_{NM}^+$ is indeed transitive follows from definitions, as explained above. Regarding now the various claims, the proof goes as follows:

(1) The universal flat model space $X_{G \times \mathbb{Z}_N}$ consists by definition of the flat representations π of the algebra $C(G) \otimes C(\mathbb{Z}_N)$ on the space $\mathbb{C}^{NM} \simeq \mathbb{C}^M \otimes \mathbb{C}^N$.

Now observe that having a representation of the tensorand $C(\mathbb{Z}_N)$ amounts in choosing N pairwise orthogonal projections $P_1, \dots, P_N$, with P_j being the w^j-eigenspace of a fixed generator of the group $\mathbb{Z}_N \subset C(\mathbb{Z}_N) \simeq C^*(\mathbb{Z}_N)$, where $w = e^{2\pi i/N}$.

The range of each projection P_j is invariant under $C(X)$, because the latter commutes with $C(\mathbb{Z}_N)$, and moreover the flatness assumption ensures that the restriction of $\pi|_{C(G)}$ to each space $Im(P_j)$ is flat. It follows in particular that all the spaces $Im(P_j)$ are M-dimensional, and hence $(P_1, \dots, P_N)$ is indeed as described in the statement.

Conversely, having chosen a decomposition of $\mathbb{C}^{NM}$ as a direct sum of summands $Im(P_j)$ for pairwise orthogonal rank M projections P_j, with $1 \leq j \leq N$ and a flat representation of $C(G)$ on each $Im(P_j)$, working out the above decomposition backwards gives us a flat representation of $C(G) \otimes C(\mathbb{Z}_N)$. Thus, we have proved our claim.

(2) This is a consequence of (1), given that the unitary group U_{NM} acts as an automorphism group of the map $X_{G \times \mathbb{Z}_N} \to E$, the action being transitive on the base E.

(3) The U_M-invariance of $\int_X$ ensures that we can indeed make sense of the product $(\int_X)^N \times \int_E$ on the total space $X_{G \times \mathbb{Z}_N}$ of the fibration from (1). The conclusion is now a simple application of the Fubini theorem on integration against product measures. $\square$

Finally, some interesting questions concern the usual wreath products $\wr$, and the free wreath products $\wr_*$. For details regarding these operations, we refer to [13].

Regarding the usual wreath products $\wr$, the simplest example here is the hyperoctahedral group $H_n = \mathbb{Z}_2 \wr S_n$, which has two standard realizations as a quantum permutation group, namely $H_n \subset S_{2n}$ and $H_n \subset S_{2^n}$. These realizations are both strongest transitive, as explained in Proposition 8.4 below, because they have suitable deranging subgroups.

As for the free wreath products $\wr_*$, the situation here is less interesting, because these quantum groups are generically not coamenable. The basic example here is the hyperoctahedral quantum group H_n^+, and we will discuss it in the next section.

8. Finite graphs

Let us go back to the list in Theorem 1.3. We will investigate now the examples (4) there, namely the quantum automorphism groups of the transitive graphs:

Definition 8.1. *Let X be a graph, having $N < \infty$ vertices.*

(1) *The automorphism group of X is the subgroup $G(X) \subset S_N$ consisting of the permutations of the vertices of X, which preserve the edges.*
(2) *The quantum automorphism group of X is the subgroup $G^+(X) \subset S_N^+$ obtained via the relation $du = ud$, where $d \in M_N(0, 1)$ is the adjacency matrix of X.*

Observe that we have $G(X) \subset G^+(X)$, because any permutation $\sigma \in S_N$ which preserves the edges of X must commute with the adjacency matrix. In fact, the converse of this fact holds as well, and we therefore have $G(X) = G^+(X)_{class}$. See [18], [24].

As a first word of warning, the theory here is much harder to develop than the one in the classical case, and we have for instance the following well-known conjecture:

Conjecture 8.2. *For a finite graph X, the following are equivalent:*

(1) *X is vertex-transitive, in the sense that $G(X)$ is transitive.*
(2) *X is quantum vertex-transitive, in the sense that $G^+(X)$ is transitive.*

Observe that the implication (1) $\implies$ (2) holds, because of the above-mentioned inclusion $G(X) \subset G^+(X)$. Indeed, if we denote by v_{ij}, u_{ij} the standard coordinates on $G(X), G^+(X)$, at the algebraic level we have a morphism which sends $u_{ij} \to v_{ij}$, and so from $v_{ij} \neq 0$ we obtain $u_{ij} \neq 0$, as desired. As for the converse, (2) $\implies$ (1), this is known to hold in all the cases where $G^+(X)$ has been explicitly computed. See [18].

In relation now with our matrix model questions, we will restrict the attention to the case where X is vertex-transitive. Generally speaking, the questions here are quite difficult, because for the simplest example of a finite graph, namely the graph X_N having N vertices and no edges, we have $G^+(X_N) = S_N^+$. And, regarding the universal flat model for $C(S_N^+)$, which is conjecturally inner faithful, nothing much is known. See [12].

We can, however, formulate at least a theoretical result, as follows:

Proposition 8.3. *For a vertex-transitive graph X, the universal affine flat model space $\tilde{X}_G$ for the associated quantum group $G = G^+(X)$ appears by imposing the relation*

$$span\left(\xi_{kj} \Big| k - i\right) = span\left(\xi_{ik} \Big| k - j\right) \quad \forall i, j$$

where $i - j$ means that i, j are connected by an edge of X, to the arrays $\xi = (\xi_{ij})$ of norm one vectors in $\mathbb{C}^N$, which are pairwise orthogonal on the rows and columns.

Proof. This is something trivial, which comes from the explicit Tannakian construction of X_G, as explained in section 2 above. Indeed, we have:

$$dP = Pd \quad \Longleftrightarrow \quad \sum_k d_{ik}P_{kj} = \sum_k P_{ik}d_{kj}, \forall i, j$$

$$\Longleftrightarrow \quad \sum_{k-i} P_{kj} = \sum_{k-j} P_{ik}, \forall i, kj$$

Now by assuming that we are in the rank 1 case, $P_{ij} = Proj(\xi_{ij})$, for a certain array of norm one vectors $\xi = (\xi_{ij})$, we obtain the condition in the statement. $\square$

At the level of basic examples now, we have:

Proposition 8.4. *Consider the n-segment graph I_n, having $2n$ vertices and n edges, and the n-cube graph K_n, having 2^n vertices and $2^{n-1}n$ edges. We have then*

$$G(I_n) = H_n \subset S_{2n} \quad , \quad G(K_n) = H_n \subset S_{2^n}$$

and in both cases, the symmetry group has the property in Proposition 4.2.

Proof. The first assertion is well-known. Regarding the second assertion, here we have subgroups as in Proposition 4.2 above, constructed as follows:

(1) For $H_n \subset S_{2n}$ we can use a copy of $\mathbb{Z}_n \rtimes \mathbb{Z}_2$ coming from the cyclic rotations of the segments, and from a joint switch on all the segments.

(2) For $H_n \subset S_{2^n}$ we can proceed by recurrence on $n \in \mathbb{N}$, by taking the group constructed at step $n-1$, crossed product with the $\mathbb{Z}_2$ coming from the middle symmetry. $\square$

In the quantum group setting now, we have:

Theorem 8.5. *For the quantum automorphism groups of the transitive graphs having 4 vertices, the universal flat models for the associated Hopf algebras are stationary.*

Proof. There are 4 such graphs, and for the empty and complete graphs, the result follows by using the Pauli matrix representation, as explained in section 10 below.

For the remaining 2 graphs, which are I_2 and K_2, we can use:

$$G^+(I_2) = G^+(K_2) = H_2^+ = O_2^{-1}$$

Indeed, the isomorphisms are all well-known, and we refer here to [6], and the stationarity property for $C(O_2^{-1})$ was proved in our previous paper [10]. $\square$

In general now, as explained in [6], the graphs K_n, I_n have different quantum symmetry groups. Regarding $G^+(K_n) = O_n^{-1}$, our conjecture here is that the universal flat model for $C(O_n^{-1})$ is stationary, at any $n \in \mathbb{N}$. As for $G^+(I_n) = H_n^+$, our conjecture here is that the universal flat model for $C(H_n^+)$ is inner faithful, once again at any $n \in \mathbb{N}$.

Observe the similarity with the conjecture for $C(S_n^+)$, discussed in [12]. In particular, we recall from there that such conjectures would solve the corresponding Connes embedding questions, therefore substantially improving the results in [17].

9. Weyl matrix models

In this section and in the next one we discuss the Weyl matrix models, following some previous work from [3], [12], that we will extend here. We will need:

Definition 9.1. *A 2-cocycle on a group G is a function $\sigma : G \times G \to \mathbb{T}$ satisfying:*

$$\sigma(gh, k)\sigma(g, h) = \sigma(g, hk)\sigma(h, k) \quad , \quad \sigma(g, 1) = \sigma(1, g) = 1$$

The algebra $C^*(G)$, with multiplication given by $g \cdot h = \sigma(g, h)gh$, and with the involution making the standard generators $g \in C^*_\sigma(G)$ unitaries, is denoted $C^*_\sigma(G)$.

As explained in [12], we have the following general construction:

Proposition 9.2. *Given a finite group $G = \{g_1, \ldots, g_N\}$ and a 2-cocycle $\sigma : G \times G \to \mathbb{T}$ we have a matrix model as follows,*

$$\pi : C(S_N^+) \to M_N(C(E)) \quad : \quad w_{ij} \to [x \to Proj(g_i x g_j^*)]$$

for any closed subgroup $E \subset U_A$, where $A = C^*_\sigma(G)$.

Proof. This is indeed clear from definitions, because the standard generators $\{g_1, \ldots, g_N\}$ are pairwise orthogonal with respect to the canonical trace of A. See [12]. $\square$

Im order to investigate the stationarity of π, we use Proposition 2.4. We have:

Proposition 9.3. *We have the formula*

$$\begin{aligned}
(T_p)_{i_1 \ldots i_p, j_1 \ldots j_p} &= \overline{\sigma(i_1, i_1^{-1} i_2)} \ldots \overline{\sigma(i_p, i_p^{-1} i_1)} \cdot \overline{\sigma(j_2, j_2^{-1} j_1)} \ldots \overline{\sigma(j_1, j_1^{-1} j_p)} \\
&\quad \frac{1}{N} \int_E tr(g_{i_1^{-1} i_2} x g_{j_2^{-1} j_1} x^*) \ldots \ldots tr(g_{i_p^{-1} i_1} x g_{j_1^{-1} j_p} x^*) dx
\end{aligned}$$

with all the indices varying in a cyclic way.

Proof. According to the definition of T_p, we have the following formula:

$$\begin{aligned}
(T_p)_{i_1 \ldots i_p, j_1 \ldots j_p} &= \left(tr \otimes \int_E \right) \left(Proj(g_{i_1} x g_{j_1}^*) \ldots Proj(g_{i_p} x g_{j_p}^*) \right) dx \\
&= \frac{1}{N} \int_E < g_{i_1} x g_{j_1}^*, g_{i_2} x g_{j_2}^* > \ldots < g_{i_p} x g_{j_p}^*, g_{i_1} x g_{j_1}^* > dx
\end{aligned}$$

Since we have $g_i g_{i^{-1} k} = \sigma(i, i^{-1} k) g_k$, and so $g_i^* g_k = \overline{\sigma(i, i^{-1} k)} g_{i^{-1} k}$, we obtain:

$$\begin{aligned}
< g_i x g_j^*, g_k x g_l^* > &= tr(g_j x^* g_i^* g_k x g_l^*) \\
&= tr(g_i^* g_k x g_l^* g_j x^*) \\
&= \overline{\sigma(i, i^{-1} k)} \cdot \overline{\sigma(l, l^{-1} j)} \cdot tr(g_{i^{-1} k} x g_{l^{-1} j} x^*)
\end{aligned}$$

By plugging these quantities into the formula of T_p, we obtain the result. $\square$

We have the following result, which generalizes some previous computations in [3]:

Theorem 9.4. *For any intermediate closed subgroup $G \subset E \subset U_A$, the matrix model $\pi : C(S_N^+) \to M_N(C(E))$ constructed above is stationary on its image.*

Proof. We use the formula in Proposition 9.3. Let us write $(T_p)_{ij} = \rho(i,j)(T_p^\circ)_{ij}$, where $\rho(i,j)$ is the product of σ terms appearing there. We have:

$$(T_p^2)_{ij} = \sum_k (T_p)_{ik}(T_p)_{kj} = \sum_k \rho(i,k)\rho(k,j)(T_p^\circ)_{ik}(T_p^\circ)_{kj}$$

Let us first compute the ρ term. We have:

$$\rho(i,k)\rho(k,j) = \overline{\sigma(i_1, i_1^{-1}i_2)} \ldots \overline{\sigma(i_p, i_p^{-1}i_1)} \cdot \overline{\sigma(k_2, k_2^{-1}k_1)} \ldots \overline{\sigma(k_1, k_1^{-1}k_p)}$$
$$\overline{\sigma(k_1, k_1^{-1}k_2)} \ldots \overline{\sigma(k_p, k_p^{-1}k_1)} \cdot \overline{\sigma(j_2, j_2^{-1}j_1)} \ldots \overline{\sigma(j_1, j_1^{-1}j_p)}$$
$$= \sigma(i,j) \cdot \overline{\sigma(k_2, k_2^{-1}k_1)} \cdot \overline{\sigma(k_1, k_1^{-1}k_2)} \ldots \ldots \overline{\sigma(k_1, k_1^{-1}k_p)} \cdot \overline{\sigma(k_p, k_p^{-1}k_1)}$$

Now observe that by multiplying $\sigma(i, i^{-1}k)g_i^* g_k = g_{i^{-1}k}$ and $\sigma(k, k^{-1}i)g_k^* g_i = g_{k^{-1}i}$ we obtain $\sigma(i, i^{-1}k)\sigma(k, k^{-1}i) = \sigma(i^{-1}k, k^{-1}i)$. Thus, our expression further simplifies:

$$\rho(i,k)\rho(k,j) = \sigma(i,j) \cdot \overline{\sigma(k_2^{-1}k_1, k_1^{-1}k_2)} \ldots \ldots \overline{\sigma(k_1^{-1}k_p, k_p^{-1}k_1)}$$

On the other hand, the T° term can be written as follows:

$$(T_p^\circ)_{ik}(T_p^\circ)_{kj} = \frac{1}{N^2} \int_E \int_E tr(g_{i_1^{-1}i_2} x g_{k_2^{-1}k_1} x^*) tr(g_{k_1^{-1}k_2} y g_{j_2^{-1}j_1} y^*)$$
$$\ldots \ldots$$
$$tr(g_{i_p^{-1}i_1} x g_{k_1^{-1}k_p} x^*) tr(g_{k_p^{-1}k_1} y g_{j_1^{-1}j_p} y^*) dx dy$$

We therefore conclude that we have the following formula:

$$(T_p^2)_{ij} = \frac{\sigma(i,j)}{N^2} \int_E \int_E \sum_{k_1 \ldots k_p} \overline{\sigma(k_2^{-1}k_1, k_1^{-1}k_2)} tr(g_{i_1^{-1}i_2} x g_{k_2^{-1}k_1} x^*) tr(g_{k_1^{-1}k_2} y g_{j_2^{-1}j_1} y^*)$$
$$\ldots \ldots$$
$$\overline{\sigma(k_1^{-1}k_p, k_p^{-1}k_1)} tr(g_{i_p^{-1}i_1} x g_{k_1^{-1}k_p} x^*) tr(g_{k_p^{-1}k_1} y g_{j_1^{-1}j_p} y^*) dx dy$$

By using now $g_i^* = \overline{\sigma(i, i^{-1})}g_{i^{-1}}$, and moving as well the x^* variables at left, we obtain:

$$(T_p^2)_{ij} = \frac{\sigma(i,j)}{N^2} \int_E \int_E \sum_{k_1 \ldots k_p} tr(x^* g_{i_1^{-1}i_2} x g_{k_2^{-1}k_1}) tr(g_{k_2^{-1}k_1}^* y g_{j_2^{-1}j_1} y^*)$$
$$\ldots \ldots$$
$$tr(x^* g_{i_p^{-1}i_1} x g_{k_1^{-1}k_p}) tr(g_{k_1^{-1}k_p}^* y g_{j_1^{-1}j_p} y^*) dx dy$$

We can compute the products of traces by using the following formula:

$$tr(Ag_k)tr(g_k^* B) = \sum_{qs} <g_q, Ag_k><g_s, g_k^* B> = \sum_{qs} tr(g_q^* Ag_k)tr(g_s^* g_k^* B)$$

Thus are left with an integral involving the variable $z = xy$, which gives T_p°. $\qquad\square$

10. Algebraic aspects

In this section we build on the analytic approach from the previous section, by investigating this time algebraic aspects. We would like to explicitly compute the quantum group, with the aim of making the link with the twisting material in [1], [7].

Let us first discuss the precise relationship with the Weyl matrices, and with the Pauli matrices, where a number of things are already known, regarding the corresponding quantum groups. We recall that the Pauli matrices are, up to some scalars:

$$W_{00} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \ W_{10} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \ W_{11} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \ W_{01} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The first statement here, coming from [12], is as follows:

Proposition 10.1. *Given a finite abelian group H, consider the product $G = H \times \widehat{H}$, and endow it with its standard Fourier cocycle.*

(1) *With $E = U_n$, where $n = |H|$, the model $\pi : C(S_N^+) \to M_N(C(U_n))$ constructed above, where $N = n^2$, is the Weyl matrix model associated to H.*

(2) *When assuming in addition that H is cyclic, $H = \mathbb{Z}_n$, we obtain in this way the matrix model for $C(S_N^+)$ coming from the usual Weyl matrices.*

(3) *In the particular case $H = \mathbb{Z}_2$, the model $\pi : C(S_4^+) \to M_N(C(U_2))$ constructed above is the matrix model for $C(S_4^+)$ coming from the Pauli matrices.*

Proof. All this is well-known. The general construction in Proposition 9.2 above came in fact by succesively generalizing $(3) \to (2) \to (1)$, and then by performing one more generalization, with $G = H \times \widehat{H}$ with its standard Fourier cocycle being replaced by an arbitary finite group G, with a 2-cocycle on it. For full details here, see [12]. $\square$

Regarding now the associated quantum groups, the first result is that the Pauli matrix representation, from (3) above, is stationary, and so the quantum group is S_4^+ itself. Moreover, we have an identification $S_4^+ = SO_3^{-1}$. All this is explained in [6].

In the context of (2) now, or more generally in the context of (1), it was shown in [12] that the law of the main character of the corresponding quantum group coincides with the law of the main character of PU_n. Observe that this is in agreement with the Pauli matrix result at $n = 2$, because of the canonical identification $PU_2 = SO_3$, and of the standard fact that the law of the main character is invariant under twisting. See [6].

In the general context of Proposition 9.2 now, we have the following result:

Theorem 10.2. *For a generalized Weyl matrix model, as in Proposition 9.2 above, the moments of the main character of the associated quantum group are*

$$c_p = \frac{1}{N} \sum_{j_1 \ldots j_p}^{\circ} \int_E tr(g_{j_1} x g_{j_1}^* x^*) \ldots tr(g_{j_p} x g_{j_p}^* x^*) dx$$

where $\circ$ means that the indices are subject to the condition $j_1 \ldots j_p = 1$.

Proof. According to Proposition 9.2 and to Theorem 9.3 above, the moments of the main character are the following numbers:

$$c_p \;=\; \frac{1}{N} \sum_{i_1 \ldots i_p} \overline{\sigma(i_1, i_1^{-1} i_2)} \ldots \overline{\sigma(i_p, i_p^{-1} i_1)} \cdot \overline{\sigma(i_2, i_2^{-1} i_1)} \ldots \overline{\sigma(i_1, i_1^{-1} i_p)}$$

$$\int_E tr(g_{i_1^{-1} i_2} x g_{i_2^{-1} i_1} x^*) \ldots \ldots tr(g_{i_p^{-1} i_1} x g_{i_1^{-1} i_p} x^*) dx$$

We can compact the cocycle part by using the following formulae:

$$
\begin{aligned}
\sigma(i_p, i_p^{-1} i_{p+1}) \sigma(i_{p+1}, i_{p+1}^{-1} i_p)
&= \sigma(i_{p+1}, i_{p+1}^{-1} i_p \cdot i_p^{-1} i_{p+1}) \sigma(i_{p+1}^{-1} i_p, i_p^{-1} i_{p+1}) \\
&= \sigma(i_{p+1}, 1) \sigma(i_{p+1}^{-1} i_p, i_p^{-1} i_{p+1}) \\
&= \sigma(i_{p+1}^{-1} i_p, i_p^{-1} i_{p+1})
\end{aligned}
$$

Thus, in terms of the indices $j_1 = i_1^{-1} i_2, \ldots, j_p = i_p^{-1} i_1$, which are subject to the condition $j_1 \ldots j_p = 1$, we have the following formula:

$$c_p = \frac{1}{N} \sum_{j_1 \ldots j_p}^{\circ} \overline{\sigma(j_1^{-1}, j_1)} \ldots \overline{\sigma(j_p^{-1}, j_p)} \int_E tr(g_{j_1} x g_{j_1^{-1}} x^*) \ldots tr(g_{j_p} x g_{j_p^{-1}} x^*) dx$$

Here the $\circ$ symbol above the sum is there for reminding us that the indices are subject to the condition $j_1 \ldots j_p = 1$. By using now $g_j^* = \overline{\sigma(j^{-1}, j)} g_{j^{-1}}$, we obtain:

$$c_p = \frac{1}{N} \sum_{j_1 \ldots j_p}^{\circ} \int_E tr(g_{j_1} x g_{j_1}^* x^*) \ldots tr(g_{j_p} x g_{j_p}^* x^*) dx$$

Thus, we have obtained the formula in the statement.   □

It is quite unclear whether the above formula further simplifies, in general. In the context of the Fourier cocycles, as in Proposition 10.1, it is possible to pass to a plain sum, by inserting a certain product of multiplicative factors $c(j_1) \ldots c(j_p)$, which equals 1 when $j_1 \ldots j_p = 1$, and the computation can be finished as follows:

$$c_p = \frac{1}{N} \int_E \left(\sum_j c(j) tr(g_j x g_j^* x^*) \right)^p dx = \frac{1}{N} \int_E tr(xx^*) dx$$

Thus, the law of the main character of the corresponding quantum group coincides with the law of the main character of PE. All this suggests the following conjecture:

Conjecture 10.3. *The quantum group associated to a Weyl matrix model, as above, should appear as a suitable twist of PE.*

In addition, we believe that in the case where E is easy these examples should be covered by a suitable projective extension of the twisting procedure in [1].

11. HAMADARD MATRICES

In this section and in the next one we discuss the Hadamard matrix models. Let us first recall that we have the following definition:

Definition 11.1. *A complex Hadamard matrix is a square matrix $H \in M_N(\mathbb{C})$ whose entries are on the unit circle, $|H_{ij}| = 1$, and whose rows are pairwise orthogonal.*

Observe that the orthogonality condition between the rows tells us that we must have $H \in \sqrt{N}U_N$, and so the columns must be pairwise orthogonal as well. In fact, the $N \times N$ complex Hadamard matrices are the points of the following algebraic manifold:

$$X_N = M_N(\mathbb{T}) \cap \sqrt{N}U_N$$

As basic examples, we have the Fourier matrices F_G of the finite abelian groups G. In the cyclic group case, $G = \mathbb{Z}_N$, this matrix is $F_N = (w^{ij})_{ij}$ with $w = e^{2\pi i/N}$. In general, with $G = \mathbb{Z}_{N_1} \times \ldots \times \mathbb{Z}_{N_s}$ we have $F_G = F_{N_1} \otimes \ldots \otimes F_{N_s}$, and this provides us with an explicit formula for F_G. There are of course many other examples. See [25].

In relation now with the quantum permutation groups, we have:

Proposition 11.2. *If $H \in M_N(\mathbb{C})$ is Hadamard, the rank one projections*

$$P_{ij} = Proj\left(\frac{H_i}{H_j}\right)$$

where $H_1, \ldots, H_N \in \mathbb{T}^N$ are the rows of H, form a magic unitary.

Proof. This is clear, the verification for the rows being as follows:

$$\left\langle \frac{H_i}{H_j}, \frac{H_i}{H_k} \right\rangle = \sum_l \frac{H_{il}}{H_{jl}} \cdot \frac{H_{kl}}{H_{il}} = \sum_l \frac{H_{kl}}{H_{jl}} = N\delta_{jk}$$

The verification for the columns is similar. See [5]. $\qquad\square$

We can proceed now in the same way as we did with the Weyl matrices, namely by constructing a model of $C(S_N^+)$, and performing the Hopf image construction.

The basic known results regarding this construction are as follows:

Proposition 11.3. *Assume that $H \in M_N(\mathbb{C})$ is Hadamard, and consider the model $\pi : C(S_N^+) \to C(G) \to M_N(\mathbb{C})$ given by $u_{ij} \to P_{ij}$, factorized via its Hopf image.*

    (1) *For a Fourier matrix $H = F_G$ we obtain the group G itself, acting on itself.*
    (2) *For $H \notin \{F_G\}$, the quantum group G is not classical, nor a group dual.*
    (3) *For a tensor product $H = H' \otimes H''$ we obtain a product, $G = G' \times G''$.*

Proof. All this material is standard, and for details here, along with a number of supplementary facts on this construction, we refer to [5]. $\qquad\square$

Generally speaking, going beyond Proposition 11.3 is a quite delicate task. The only known results so far concern the "simplest" complex Hadamard matrices which are not of Fourier type, namely certain affine deformations of F_G, and the situation here is already quite complicated, and far from being understood. We refer here to [2], [5], [15].

One interesting question is that of abstractly characterizing the flat magic matrices coming from the complex Hadamard matrices. As a first observation here, we have:

Proposition 11.4. *Given an Hadamard matrix $H \in M_N(\mathbb{C})$, the vectors $\xi_{ij} = H_i/H_j$, on which the magic unitary entries P_{ij} project, have the following properties:*

  (1) $\xi_{ii} = \xi$ *is the all-one vector.*
  (2) $\xi_{ij}\xi_{jk} = \xi_{ik}$, *for any i, j, k.*
  (3) $\xi_{ij}\xi_{kl} = \xi_{il}\xi_{kj}$, *for any i, j, k, l.*

Proof. All these assertions are trivial, using the formula $\xi_{ij} = H_i/H_j$. $\qquad\square$

These observations lead to the following result, at the magic basis level:

Theorem 11.5. *The magic bases $\xi \in M_N(S_{\mathbb{C}}^{N-1})$ coming from the complex Hadamard matrices are those having the following properties:*

  (1) *We have $\xi_{ij} \in \mathbb{T}^N$, after a suitable rescaling.*
  (2) *The conditions in Proposition 11.4 are satisfied.*

Proof. By using the multiplicativity conditions (1,2,3) in Proposition 11.4, we conclude that, up to a rescaling, we must have $\xi_{ij} = \xi_i/\xi_j$, where $\xi_1, \ldots, \xi_N$ is the first row of the magic basis. Together with our assumption $\xi_{ij} \in \mathbb{T}^N$, this gives the result. $\qquad\square$

Regarding now the corresponding flat magic unitaries $P \in M_N(M_N(\mathbb{C}))$, by using the well-known rank 1 projection formula $Proj(\xi) = \frac{1}{||\xi||^2}(\xi_i\bar{\xi}_j)_{ij}$, we obtain:

$$(P_{ij})_{kl} = Proj\left(\frac{H_i}{H_j}\right)_{kl} = \left(\frac{H_i}{H_j}\right)_k \left(\frac{H_j}{H_i}\right)_l = \frac{H_{ik}H_{jl}}{H_{jk}H_{il}}$$

Thus, for an Hadamard matrix, the corresponding magic unitary is made up of quantities of the following type, having symmetry properties as those in Theorem 11.5:

$$(P_{ij})_{kl} \in \mathbb{T}$$

It would be very interesting to find a common framework for the Weyl matrix models and for the Hadamard matrix models, inside the general flat matrix model framework. In fact, a perhaps more reasonable question would be that of finding a common framework for the Weyl matrix models and for certain special Hadamard matrix models, such as the Fourier ones, and their affine deformations, considered in [2], [5], [15].

12. Open problems

We discuss here stationarity questions for the Hadamard matrix models. The subject is quite difficult, and has already been explored, in a quite systematic way. We basically have no new results here, and we will just discuss some open questions.

The various findings in [2], [5], [15] suggest the following conjecture:

Conjecture 12.1. *The only Hadamard matrix models which are stationary on their images are the Fourier matrix models.*

In order to discuss this statement, we recall from Proposition 2.4 that a matrix model $u_{ij} \to U_{ij}$ is stationary on its image precisely when $T_p^2 = T_p$ for any $p \in \mathbb{N}$, where:

$$(T_p)_{i_1 \ldots i_p, j_1 \ldots j_p} = tr(U_{i_1 j_1} \ldots U_{i_p j_p})$$

In the case of a flat model, given by $U_{ij} = Proj(\xi_{ij})$, we can use the well-known formula for a trace of a product of rank 1 projections, and we obtain:

$$(T_p)_{i_1 \ldots i_p, j_1 \ldots j_p} = \frac{1}{N} < \xi_{i_1 j_1}, \xi_{i_2 j_2} > \ldots < \xi_{i_p j_p}, \xi_{i_1 j_1} >$$

Now in the case of an Hadamard matrix model, where we have $\xi_{ij} = H_i / H_j$, with $H_1, \ldots, H_N \in \mathbb{T}^N$ being the rows of $H \in M_N(\mathbb{C})$, this formula becomes:

$$(T_p)_{i_1 \ldots i_p, j_1 \ldots j_p} = \frac{1}{N} \sum_{s_1 \ldots s_p} \frac{H_{i_1 s_1} H_{j_2 s_1}}{H_{j_1 s_1} H_{i_2 s_1}} \cdots \cdots \frac{H_{i_p s_p} H_{j_1 s_p}}{H_{j_p s_p} H_{i_1 s_p}}$$

We must impose now the stationarity condition $T_p^2 = T_p$, for any $p \in \mathbb{N}$. By using the usual matrix multiplication rule, the square of the above matrix is given by:

$$
\begin{aligned}
(T_p^2)_{i_1 \ldots i_p, j_1 \ldots j_p} = \quad & \frac{1}{N} \sum_{k_1 \ldots k_p} \sum_{s_1 \ldots s_p} \sum_{t_1 \ldots t_p} \frac{H_{i_1 s_1} H_{k_2 s_1}}{H_{k_1 s_1} H_{i_2 s_1}} \cdots \cdots \frac{H_{i_p s_p} H_{k_1 s_p}}{H_{k_p s_p} H_{i_1 s_p}} \\
& \frac{H_{k_1 t_1} H_{j_2 t_1}}{H_{j_1 t_1} H_{k_2 t_1}} \cdots \cdots \frac{H_{k_p t_p} H_{j_1 t_p}}{H_{j_p t_p} H_{k_1 t_p}}
\end{aligned}
$$

We can see that there is no miracle here, and unless we have some extra information regarding H, we cannot perform the sum, and solve the problem.

As observed in [5], and then in [15], and then more systematically explained in [2], the Hadamard matrix model setting is not exactly the ideal one, and the parametric Hadamard matrix model setting seems to be more appropriate, in order to deal with various probabilistic aspects. To be more precise, the work there suggests looking at parametric Hadamard models, with the parameter space being a torus T.

This is quite similar to what happens for the Weyl matrix models, and we are led once again to the unification question raised at the end of the previous section. In fact, based on the results in [2], [5], [15], we have now the following more precise problem: is

there anything intermediate between stationarity and inner faithfulness, which covers the deformed Fourier models, with full parameter space?

Finally, some interesting questions appear in connection with the recent work in [4], on the Hadamard type matrices $U \in M_N(A)$, with A being an arbitrary C^*-algebra. We have the following hierarchy, with (1,2) corresponding to the work in [2], [5], [15]:

(1) Hadamard models, $A = \mathbb{C}$.
(2) Parametric Hadamard models, $A = C(T)$.
(3) Hadamard models with random matrix entries, $A = M_K(C(T))$.

The question is whether in the setting of (3), the matrix model defining the quantum group can be "flattened". To be more precise, the model defining the quantum group is not flat at $K > 1$, and the question is whether the quantum group can be recovered, however, from a flat model, obtained for instance by applying the trace on the matrix part. There are probably some computations to be done here, at $K = 2$.

Summarizing, we have many interesting questions here. We intend to come back to the various questions raised in this paper, in some future work.

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