Introduction to subfactor theory

Teo Banica

Von Neumann algebras, Subfactors and Temperley-Lieb, Fixed point subfactors, Planar algebras, Commuting squares, Small and big index

07/20

Foreword

This is an introduction to subfactors, from a quantum group perspective, focusing on the case of integer index.

We discuss the foundational aspects of the theory, and then we discuss a number of more advanced topics.

These lecture notes consist of slides written in the Summer 2020. Presentations available at my Youtube channel.

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Von Neumann algebras and factors

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Bounded operators

Theorem. Given a Hilbert space H, the linear operators $T : H \to H$ which are bounded, in the sense that

$$||T|| = \sup_{||x|| \leq 1} ||Tx||$$

is finite, form a complex algebra $B(H)$, which:

(1) Is complete with respect to $||.||$ (Banach algebra).

(2) Has an involution $T \to T^*$, $< Tx, y >=< x, T^*y >$.

The norm and involution are related by $||TT^*|| = ||T||^2$.

Remark. Assuming $H = l^2(I)$, we have $B(H) \subset M_I(\mathbb{C})$, via $T \to (< Te_j, e_i >)_{ij}$. The adjoint is then $(M^*)_{ij} = \bar{M}_{ji}$.

Operator algebras

Definition. A von Neumann algebra is an algebra of bounded operators $A \subset B(H)$ which is:

(1) Stable under the involution: $T \in A \implies T^* \in A$.

(2) Weakly closed: $T_n \in A, T_n x \to Tx \implies T \in A$.

Examples. The simplest examples are the matrix algebras $M_N(\mathbb{C})$, and their $*$-subalgebras $A \subset M_N(\mathbb{C})$. In particular, we have

$$A =< M >$$

the $*$-algebra generated by a matrix $M \in M_N(\mathbb{C})$. In the normal case, $MM^* = M^*M$, we have $< M >= \{P(M) | P \in \mathbb{C}[X]\}$.

Commutative algebras

Theorem. Given a measured space X, we have an algebra

$$L^\infty(X) \subset B(L^2(X))$$

with the functions $f \in L^\infty(X)$ acting via $T_f : g \to fg$.

Theorem. The commutative von Neumann algebras are those of the form $L^\infty(X)$, with X being a measured space.

Proof. Basic functional analysis and operator theory. The full statement involves as well a multiplicity, in regards with H.

Random matrices

Definition. A random matrix algebra is an algebra of type:

$$A = M_N(L^\infty(X))$$

The elements of A are called random matrices.

Theorem. The matrices $M \in A$ having i.i.d. normal entries, up to the constraint $M = M^*$, follow with $N \to \infty$ the semicircle law:

$$\gamma_t = \frac{1}{2\pi t}\sqrt{4t^2 - x^2}dx$$

Proof. Moment method. The Wick formula gives with $N \to \infty$ the Catalan numbers, which are the moments of γ_t.

Free probability

Definition. Two subalgebras $B, C \subset A$ are called:

(1) Independent, if $tr(b) = tr(c) = 0$ implies $tr(bc) = 0$.

(2) Free, if $tr(b_i) = tr(c_i) = 0$ implies $tr(b_1 c_1 b_2 c_2 \ldots) = 0$.

Theorem. If $x_1, x_2, x_3, \ldots \in A$ are independent/free, i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim \mathcal{N}(0, t)/\gamma_t$$

where $\mathcal{N}(0, t)/\gamma_t$ are the normal/Wigner semicircle laws.

Proof. Linearization of the convolution, $\log F/R$ transforms.

General theory

Theorem. For a $*$-algebra of operators $A \subset B(H)$, the following conditions are equivalent:

(1) A is weakly closed, i.e. is a von Neumann algebra.

(2) A is equal to its algebraic bicommutant, $A = A''$.

This is von Neumann's "bicommutant theorem". As a consequence, the von Neumann algebras appear as commutants, $A = P'$.

Comment. Von Neumann $\implies$ norm closed. The von Neumann algebras are the norm closed $*$-algebras with separable predual.

Finite dimensions

Theorem. Let $A \subset M_N(\mathbb{C})$ be a $*$-algebra.

(1) We have $1 = p_1 + \ldots + p_k$, with $p_i \in A$ minimal projections.

(2) The spaces $A_i = p_i A p_i$ are non-unital $*$-subalgebras of A.

(3) We have a non-unital $*$-algebra sum $A = A_1 \oplus \ldots \oplus A_k$.

(4) Unital $*$-algebra isomorphisms $A_i \simeq M_{N_i}(\mathbb{C})$, $N_i = rank(p_i)$.

(5) Thus, we can decompose $A \simeq M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$.

Proof. (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5).

Reduction theory

Theorem. When writing the center of the algebra as

$$Z(A) = L^\infty(X)$$

wihh X measured space, the algebra decomposes as

$$A = \int_X A_x \, dx$$

with the summands being "factors", $Z(A_x) = \mathbb{C}$.

Example. In finite dimensions the algebra must be

$$A = M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$$

and this is its decomposition as a sum of factors.

Factors

Theorem. The factors, $Z(A) = \mathbb{C}$, fall into 3 classes:

(1) Type I. These are the usual matrix algebras $M_N(\mathbb{C})$ (type I_N), and the algebra $B(H)$, with H separable (type I_∞).

(2) Type II. These are the ∞D factors having a trace $tr : A \to \mathbb{C}$ (type II_1) and their tensor products with $B(H)$ (type II_∞).

(3) Type III. These fall into several classes, III_λ with $\lambda \in [0, 1]$, and appear from II_1 factors, via crossed product type constructions.

Proof. This is heavy, due to Murray and von Neumann, and then Connes, based on ideas of Tomita, Takesaki and others.

$\implies$ The II_1 factors are the "building blocks" of the theory.

Group duals

Definition. Consider a discrete group Γ.

(1) We endow $\mathbb{C}[\Gamma]$ with the involution $g^* = g^{-1}$.

(2) We embed $\mathbb{C}[\Gamma] \subset B(l^2(\Gamma))$ via $T_g : h \to gh$.

(3) We let $L(\Gamma)$ be the closure of $\mathbb{C}[\Gamma]$ inside $B(l^2(\Gamma))$.

Theorem. When Γ is abelian, we have an identification

$$L(\Gamma) = L^\infty(G)$$

where $G = \{\chi : \Gamma \to \mathbb{T}\}$ is the Pontrjagin dual of Γ.

$\implies$ In general, we can define $G = \widehat{\Gamma}$, abstract "noncommutative measured space", by the formula $L^\infty(G) = L(\Gamma)$.

Noncommutative geometry

Theorem. The group algebras $L(\Gamma)$ have traces, given by

$$tr(g) = \delta_{g1}$$

and so decompose into integrals of type I_N and II_1 factors.

Theorem. A group algebra $L(\Gamma)$ is a II_1 factor precisely when Γ has infinite conjugacy classes (ICC property).

More. The group duals $G = \widehat{\Gamma}$ are the "NC tori". One can talk as well about NC spheres, quantum groups, and so on, e.g. about

$$A = L^\infty(G/H)$$

with $H \subset G$ quantum groups, with tr = uniform integration.

II_1 factors

Definition. A II_1 factor is a von Neumann algebra $A \subset B(H)$:

(1) Which is infinite dimensional, $\dim(A) = \infty$.

(2) Has trivial center, $Z(A) = \mathbb{C}$.

(3) And has a faithful positive unital trace, $tr : A \to \mathbb{C}$.

Theorem. The trace is unique.

Theorem. The trace of projections can take any value in $[0, 1]$.

$\implies$ This is very interesting, "continuous dimension".

The factor R

Theorem. The following limiting von Neumann algebra,

$$R = \lim_{k\to\infty} M_{N_k}(\mathbb{C})$$

is a II_1 factor, independent of the limiting procedure.

Theorem. R is the unique "hyperfinite" II_1 factor. In fact, R is the "building block" for the hyperfinite von Neumann algebras.

Theorem. A group algebra $L(\Gamma)$ is hyperfinite precisely when Γ is amenable. If in addition we have ICC, then $L(\Gamma) \simeq R$.

Subfactors 1/2

Definition. Consider an inclusion of II_1 factors $A \subset B$.

(1) Its index is the number $[B : A] = \dim_A B \in [1, \infty]$, defined as a Murray-von Neumann "continuous dimension" quantity.

(2) The "basic construction" is $A \subset B \subset C$, by "reflection", with $C = < B, e >$, where $e : B \to A$ is the orthogonal projection.

Theorem. Let $A_0 \subset A_1$ be a subfactor of finite index $N \in [1, \infty)$, and consider its Jones tower, obtained by basic construction:

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \ldots$$

The Jones projections $e_1, e_2, e_3, \ldots$ generate then a copy of the Temperley-Lieb algebra TL_N, inside the ambient algebra $B(H)$.

Subfactors 2/2

Theorem. The index of subfactors is "quantized",

$$N \in \left\{ 4\cos^2\left(\frac{\pi}{n}\right) \Big| n \in \mathbb{N} \right\} \cup [4, \infty]$$

and all the admissible index values are attainable.

Theorem. Consider the commutants $P_k = A_0' \cap A_k$.

(1) The graded union $P = \cup_k P_k$ contains TL_N.

(2) P is a planar algebra, "its elements behave like diagrams".

(3) In the "amenable" case, P classifies the subfactor.

$\implies$ The philosophy is that $A \subset B$ appears via an action of an underlying "quantum group", of the most general type. Of particular interest is the hyperfinite case, $A \simeq B \simeq R$.

Subfactors and the Temperley-Lieb algebra

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II_1 factors

Definition. A II_1 factor is a von Neumann algebra, i.e. a weakly closed $*$-algebra of operators $A \subset B(H)$, which:

(1) Is infinite dimensional, $\dim(A) = \infty$.

(2) Has trivial center, $Z(A) = \mathbb{C}$.

(3) And has a trace, $tr : A \to \mathbb{C}$.

– As basic example, we have the Murray-von Neumann hyperfinite factor R, obtained as inductive limit of matrix algebras.

– We have as well the group von Neumann algebras $L(\Gamma)$, with Γ being a discrete group having infinite conjugacy classes (ICC).

Comment. When Γ has the ICC property and is amenable we have $L(\Gamma) \simeq R$. In general, it is not known when $L(\Gamma) \simeq L(\Lambda)$.

Theory

Theorem 1. The trace of the II_1 factors

$$tr : A \to \mathbb{C}$$

which is by definition faithful, positive and unital, is unique.

Theorem 2. The traces of the projections $p \in A$,

$$p^2 = p^* = p$$

take as values all the numbers in $[0, 1]$.

Subfactors

Definition. A subfactor is an inclusion of II_1 factors

$$A \subset B$$

inside $B(H)$. The index of such an inclusion is the number

$$N = \dim_A B \in [1, \infty]$$

which is a Murray-von Neumann "continuous dimension" quantity.

Theorem. When the index is an integer, $N \in \mathbb{N}$, we have a basis

$$\{e_1, \ldots, e_N\} \subset B$$

of B over A, which is orthogonal with respect to tr.

Examples 1/4

Given a finite group G, acting on a von Neumann algebra P, we have the following inclusion:

$$P \subset P \rtimes G$$

When P is a II_1 factor and $G \to Aut(P)$ is minimal, this is a subfactor of index $N = |G|$, called depth 2 subfactor.

Examples 2/4

Given a compact group G, acting on a von Neumann algebra P, and given with a finite index subgroup $H \subset G$, we have:

$$P^G \subset P^H$$

When P is a II_1 factor and $G \to Aut(P)$ is minimal, this is a subfactor of index $N = [G : H]$, called subgroup subfactor.

Examples 3/4

Given a compact group G, acting on a von Neumann algebra P, and given with a unitary representation $G \to PU_n$, we have:

$$P^G \subset (M_n(\mathbb{C}) \otimes P)^G$$

When P is a II_1 factor and $G \to Aut(P)$ is minimal, this is a subfactor of index $N = n^2$, called Wassermann subfactor.

Examples 4/4

Given a discrete group $\Gamma = < g_1, \ldots, g_n >$, acting on a von Neumann algebra Q, we have the following inclusion:

$$\left\{ diag(g_1(q), \ldots, g_n(q)) \middle| q \in Q \right\} \subset M_n(Q)$$

When Q is a II_1 factor and $\Gamma \subset Aut(Q)$ is outer, this is a subfactor of index $N = n^2$, called diagonal subfactor.

Basic construction

<u>Theorem</u>. Given a subfactor $A \subset B$, we consider the projection

$$e : B \to A \quad , \quad e \in B(H)$$

with respect to the trace, we construct the algebra

$$C = < B, e > \subset B(H)$$

which is a factor, and we obtain in this way a new subfactor,

$$A \subset \underline{B \subset C}$$

having the same index. This is the Jones "basic construction".

Examples

(1) Depth two, $P \subset P \rtimes G$. We obtain a crossed product by $\widehat{G}$, the dual group when G is abelian, and a quantum group in general.

(2) Subgroup, $P^G \subset P^H$. In the simplest case, where $H = \{1\}$, and the subfactor is $P^G \subset P$, we obtain $P \subset P \rtimes G$.

(3) Wassermann, $P^G \subset M_n(P)^G$. Here we obtain $M_{n^2}(P)^G$, with $G \to PU_{n^2}$ being the tensor square of $G \to PU_n$.

(4) Diagonal, $Q^{\curvearrowright\Gamma} \subset M_n(Q)$. Here we obtain $M_{n^2}(Q)$, with the embedding $M_n(Q) \subset M_{n^2}(Q)$ being twisted by Γ.

$\implies$ The basic construction $A \subset B \subset C$ is a "reflection".

The tower

Definition. Starting from the initial subfactor, relabeled

$$A_0 \subset A_1$$

we obtain by basic construction a whole tower of factors

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \ldots$$

with $e_1, e_2, e_3, \ldots$ being the Jones projections, at each step.

Temperley-Lieb

Theorem. In the context of a Jones tower as above,

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \ldots$$

the Jones projections $e_1, e_2, e_3, \ldots$ generate a copy of TL_N.

Proof. The TL_N relations follow from a careful study of

$$A \subset B \subset C$$

and also of $A \subset B \subset C \subset D$, by translation. Since we have

$$tr(\pi) = N^{loops<\pi>}$$

which is faithful on TL_N, this representation is faithful.

Index theorem

Theorem. The index of subfactors is "quantized", as follows:

$$N \in \{4\cos^2(\pi/n) | n \in \mathbb{N}\} \cup [4, \infty]$$

Moreover, all values are attained, and we have ADE at $N \leq 4$.

Proof. According to the above, we have a representation:

$$TL_N \subset B(H)$$

At $N < 4$ this is not always possible, due to positivity reasons.

Planar algebras

Recall that TL_N is a "planar" algebra, appearing as the span

$$TL_N = span(NC_2)$$

the product being vertical concatenation ($\downarrow$), with $\bigcirc = N$.

<u>Theorem</u>. Given $A_0 \subset A_1$, the planar algebra structure of

$$< e_1, e_2, e_3, \ldots >= TL_N$$

extends into a planar algebra structure of $P = (P_k)$, where

$$P_k = A_0' \cap A_k$$

are the higher relative commutants (FD complex vector spaces).

Example

Consider a Wassermann subfactor, coming from $\pi : G \to PU_n$:

$$P^G \subset M_n(P)^G$$

The Jones tower for this subfactor comes then from $\{\pi^{\otimes k}\}$:

$$P^G \subset M_n(P)^G \subset M_{n^2}(P)^G \subset M_{n^3}(P)^G \subset \ldots$$

As for the higher relative commutants, these are as follows:

$$\mathbb{C} \subset End(\pi) \subset End(\pi^{\otimes 2}) \subset End(\pi^{\otimes 3}) \subset \ldots$$

Thus, we obtain the usual planar operations on $End(\pi^{\otimes k})$.

Classification

Theorem 1. The subfactors $A \subset B$ having "finite depth" are classified by their planar algebras $P = (P_k)$.

Theorem 2. More generally, the "amenable" subfactors $A \subset B$ are classified by their planar algebras $P = (P_k)$.

Theorem 3. In general, any planar algebra produces a subfactor (complementing "any subfactor produces a planar algebra").

Group actions and fixed point subfactors

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Basic subfactors

Theorem 1. Given a finite group G, acting on a II_1 factor P, in a minimal way, $(P^G)' \cap P = \mathbb{C}$, we have a subfactor

$$P \subset P \rtimes G$$

of index $N = |G|$, called "depth 2 subfactor".

Theorem 2. Given a compact group G, acting minimally on a II_1 factor P, for any finite index subgroup $H \subset G$,

$$P^G \subset P^H$$

is a subfactor of index $N = [G : H]$, called "subgroup subfactor".

Further examples

<u>Theorem 3</u>. Given a compact group G, acting minimally on a II_1 factor P, for any projective representation $G \to PU_n$,

$$P^G \subset (M_n(\mathbb{C}) \otimes P)^G$$

is a subfactor of index $N = n^2$, called "Wassermann subfactor".

<u>Theorem 4</u>. Given a discrete group $\Gamma =< g_1, \ldots, g_n >$, acting on a II_1 factor Q, in an outer way, $Q' \cap Q \rtimes \Gamma = \mathbb{C}$,

$$\left\{ diag(g_1(q), \ldots, g_n(q)) \Big| q \in Q \right\} \subset M_n(Q)$$

is a subfactor of index $N = n^2$, called "diagonal subfactor".

Unification

The main examples of subfactors of integer index, namely

$$P \subset P \rtimes G \ , \ P^G \subset P^H \ , \ P^G \subset M_n(P)^G \ , \ Q^{\curvearrowright\Gamma} \subset M_n(Q)$$

can be written in a uniform way, as "fixed point subfactors",

$$(C(G) \otimes P)^G \subset (B(l^2(G)) \otimes P)^G$$

$$(\mathbb{C} \otimes P)^G \subset (C(G/H) \otimes P)^G$$

$$(\mathbb{C} \otimes P)^G \subset (M_n(\mathbb{C}) \otimes P)^G$$

$$(\mathbb{C} \otimes (Q \rtimes \Gamma))^{\widehat{\Gamma}} \subset (M_n(\mathbb{C}) \otimes (Q \rtimes \Gamma))^{\widehat{\Gamma}}$$

and so they are of the same nature, namely

$$(A_0 \otimes P)^G \subset (A_1 \otimes P)^G$$

with $A_0 \subset A_1$ being FD algebras, and G being a quantum group.

Quantum groups 1/4

Definition. A Woronowicz algebra is a C^*-algebra A, given with a unitary matrix $u \in M_N(A)$ whose entries generate A, such that:

– $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ defines a morphism $\Delta : A \to A \otimes A$.

– $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : A \to \mathbb{C}$.

– $S(u_{ij}) = u_{ji}^*$ defines a morphism $S : A \to A^{opp}$.

Notation. Given a Woronowicz algebra A we write

$$A = C(G) = C^*(\Gamma)$$

and call G, Γ compact and discrete quantum groups.

Quantum groups 2/4

Example 1. Given a compact Lie group $G \subset U_N$, we have

$$A = C(G) \quad , \quad u_{ij}(g) = g_{ij}$$

with $\Delta = m^T, \varepsilon = u^T, S = i^T$ being the transposes of m, u, i.

Example 2. Given a discrete group $\Gamma =< g_1, \ldots, g_N >$, we have

$$A = C^*(\Gamma) \quad , \quad u = diag(g_i)$$

with $\Delta(g) = g \otimes g, \varepsilon(g) = 1, S(g) = g^{-1}$ on group elements.

Quantum groups 3/4

Theorem. Any Woronowicz algebra has a Haar integration,

$$\left(\int_G \otimes id\right)\Delta = \left(id \otimes \int_G\right)\Delta = \int_G (.)1$$

constructed by starting with $\varphi \in A^*$ unital positive, and setting

$$\int_G = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \varphi^{*k}$$

with the convolution operation being $\phi * \psi = (\phi \otimes \psi)\Delta$.

Quantum groups 4/4

Definition. A corepresentation of a Woronowicz algebra A is a unitary matrix $v \in M_n(\mathcal{A})$ satisfying

$$\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj} \quad , \quad \varepsilon(v_{ij}) = \delta_{ij} \quad , \quad S(v_{ij}) = v_{ji}^*$$

where $\mathcal{A} \subset A$ is the dense $*$-subalgebra of "smooth elements".

Theorem. The following Peter-Weyl type results hold:

(1) Any corepresentation decomposes as a sum of irreducibles.

(2) The irreducibles appear inside $u^{\otimes k}$, with $k =$ colored integer.

(3) We have $\mathcal{A} = \oplus_{r \in Irr(A)} B(H_r)$, $*$-coalgebra isomorphism, $\perp$.

(4) The characters of irreps form an orthonormal basis of $\mathcal{A}_{central}$.

Fixed point subfactors

Theorem. Given a compact quantum group G, acting on a von Neumann algebra P, and acting as well on an inclusion $A_0 \subset A_1$ of finite dimensional algebras, we have the following inclusion:

$$(A_0 \otimes P)^G \subset (A_1 \otimes P)^G$$

When P is a II_1 factor, the action of G on it is minimal, and $A_0 \subset A_1$ is a Markov inclusion, with G being ergodic on both $Z(A_0), Z(A_1)$, this is a subfactor of index $N = [A_1 : A_0]$.

Examples

The main examples of subfactors of integer index, namely

$$(C(G) \otimes P)^G \subset (B(l^2(G)) \otimes P)^G$$

$$(\mathbb{C} \otimes P)^G \subset (C(G/H) \otimes P)^G$$

$$(\mathbb{C} \otimes P)^G \subset (M_n(\mathbb{C}) \otimes P)^G$$

$$(\mathbb{C} \otimes (Q \rtimes \Gamma))^{\widehat{\Gamma}} \subset (M_n(\mathbb{C}) \otimes (Q \rtimes \Gamma))^{\widehat{\Gamma}}$$

all appear as fixed point subfactors, in our sense.

Theory

Consider a fixed point subfactor $(A_0 \otimes P)^G \subset (A_1 \otimes P)^G$.

Theorem 1. The Jones tower of the subfactor is $(A_i \otimes P)^G$, where $\{A_i\}$ is the Jones tower for $A_0 \subset A_1$.

Theorem 2. The standard invariant, taken in the standard lattice sense of Popa, is the lattice $(A_i' \cap A_j)^G$.

Theorem 3. In the case $A_0 = \mathbb{C}$, which covers most of the interesting examples, the planar algebra is $P_k = A_k^G$.

Proof

Consider indeed an arbitrary fixed point subfactor:

$$(A_0 \otimes P)^G \subset (A_1 \otimes P)^G$$

The Jones tower is then obtained as follows:

$$(A_0 \otimes P)^G \subset (A_1 \otimes P)^G \subset (A_2 \otimes P)^G \subset \ldots$$

When computing the relative commutants P dissapears,

$$\left[(A_i \otimes P)^G\right]' \cap (A_j \otimes P)^G = (A_i' \cap A_j)^G$$

("invariance principle") and this gives all the results.

Quantum permutations 1/2

In order to construct new examples, we need actions

$$G \curvearrowright A$$

of compact quantum groups G on FD algebras A. But

$$A = C(X)$$

with X "finite noncommutative space". Thus, we need actions

$$G \curvearrowright X$$

of compact quantum groups G on finite NC spaces X.

$\implies$ Needs "quantum permutations", in a very general sense.

Quantum permutations 2/2

Theorem. Let X be a "finite noncommutative space", coming from a FD algebra A, which must be of the form:

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

Then X has a quantum automorphism group $G^+(X)$, as follows:

(1) Start with the quantum group U_N^+, whose coordinates form a free $N \times N$ biunitary matrix, where $N = |X| = \dim A$.

(2) Impose the conditions coming from the existence of a coaction map $A \to C(G) \otimes A$, which leaves invariant the canonical trace.

TL and FC

Theorem. Given $G = G^+(X)$ as above, the subfactor

$$P^G \subset (A \otimes P)^G$$

with $A = C(X)$ corresponds to the algebra TL_N.

Remark. This applies in particular to the quantum groups:

$$S_N^+ = G^+(1, \ldots, N) \quad , \quad PO_N^+ = PU_N^+ = G^+(M_n)$$

Theorem. With $G = G^+(X \to Y)$, the subfactor

$$(B \otimes P)^G \subset (A \otimes P)^G$$

with $A = C(X), B = C(Y)$ corresponds to the algebra FC_N.

Finite graphs

(1) The Fuss-Catalan construction is best implemented by using classical finite spaces, of type $\{1,\ldots,N\}$. We are led to

$$H_N^{s+}=\mathbb{Z}_s\wr_* S_N^+$$

called quantum reflection groups, which liberate the classical reflection groups $H_N^s=\mathbb{Z}_s\wr S_N$.

(2) These quantum groups appear as quantum automorphism groups of finite graphs. In the general graph setting

$$P=<\square>$$

is a planar algebra generated by a 2-box, in the sense of Bisch-Jones and Liu, with the Laplacian of the graph being in the box.

Planar algebras and spectral measures

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Temperley-Lieb

Definition. $TL_N(k)$ is the formal span of the noncrossing pairings between k upper points and k lower points,

$$TL_N(k) = span(NC_2(k,k))$$

with product given on generators by vertical concatenation, with the convention that "things go downwards",

$$\pi\sigma = \begin{bmatrix}\sigma \\ \pi\end{bmatrix}$$

and with the rule for the circles that might appear in the middle:

$$\bigcirc = N$$

That is, each such circle counts for a multiplicative N factor.

Properties

The algebra $TL_N(k)$ has an involution, given by:

$$A^* = \forall$$

We have embeddings of unital $*$-algebras, as follows,

$$TL_N(k) \subset TL_N(k+1)$$

obtained by adding a string at right, and the union

$$TL_N = \bigcup_{k \in \mathbb{N}} TL_N(k)$$

is a graded $*$-algebra. There is a $\otimes$ operation as well.

Subfactors

Theorem. Consider a subfactor $A_0 \subset A_1$, of index $N \in [1,\infty)$, and build by "basic construction" the associated Jones tower:

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \ldots$$

The Jones projections $e_1, e_2, e_3, \ldots$ generate then a copy of TL_N.

Proof. The TL_N relations follow from a careful study of the basic construction, by translation. Since we have

$$tr(\pi) = N^{loops<\pi>}$$

which is faithful on TL_N, this representation is faithful.

Planar algebras

Theorem. The planar algebra structure of the algebra

$$TL_N = < e_1, e_2, e_3, \ldots >$$

extends into a planar algebra structure of $P = (P_k)$, where

$$P_k = A_0' \cap A_k$$

are the higher relative commutants (FD $*$-algebras).

Definition

A planar algebra is a collection of FD complex vector spaces

$$P = (P_k)_{k \in \mathbb{N}}$$

with an action on it of the diagrams consisting of:

(1) a big box, containing s small boxes,

(2) with $2k + \sum_{i=1}^{s} 2k_i$ points on these boxes,

(3) and with NC strings connecting these points.

That is, associated to any such diagram is a linear map

$$P_{k_1} \otimes \ldots \otimes P_{k_s} \to P$$

and the gluing of diagrams corresponds to the composition of maps.

Examples

(1) The Temperley-Lieb algebra TL_N. Here the linear generators $\pi \in NC_2(k,k)$ are put into boxes in the obvious way.

(2) The Fuss-Catalan algebra $FC_{N,M}$. Same technology as for TL_N, but this time the strings are colored, with two colors.

(3) The tensor planar algebra T_N. Here $T_N(k) = M_N(\mathbb{C})^{\otimes k}$, and the operations correspond to the usual tensor calculus.

(4) The spin planar algebra S_N. Here $S_N(k) = (\mathbb{C}^N)^{\otimes k}$, and the indices are doubled, before being put into boxes.

Theory

Theorem 1. The subfactors $A_0 \subset A_1$ having "finite depth" are classified by their planar algebras $P = (P_k)$.

Theorem 2. More generally, the "amenable" subfactors $A_0 \subset A_1$ are classified by their planar algebras $P = (P_k)$.

Theorem 3. In general, any planar algebra produces a subfactor (complementing "any subfactor produces a planar algebra").

TL subfactors

Theorem. The Temperley-Lieb subfactors exist for any admissible value of the index, namely

$$N \in \left\{ 4\cos^2\left(\frac{\pi}{n}\right) \Big| n \in \mathbb{N} \right\} \cup [4, \infty]$$

and can be explicitly constructed as subfactors of $L(F_\infty)$.

Question. What about subfactors of R?

FC subfactors

Theorem. In the presence of an intermediate subfactor,

$$A_0 \subset B \subset A_1$$

the corresponding planar algebra contains the FC one:

$$FC \subset P$$

FC subfactors can be obtained by composing TL subfactors.

Question. Same as before, what about R?

Tensor subfactors

<u>Theorem</u>. The planar algebra of a Wassermann type subfactor

$$A^G \subset (M_N(\mathbb{C}) \otimes A)^G$$

is a subalgebra of the corresponding tensor planar algebra

$$\left(End(u^{\otimes k}) \right)_{k \in \mathbb{N}} \subset T_N$$

and any subalgebra of T_N appears in this way.

<u>Comment</u>. This follows from Tannaka, and the correspondence is not bijective, because we have to lift the projective version.

Spin subfactors

Theorem. The planar algebra of subfactor of type

$$A^G \subset (\mathbb{C}^N \otimes A)^G$$

is a subalgebra of the corresponding spin planar algebra

$$\left(Fix(u^{\otimes k})\right)_{k\in\mathbb{N}} \subset S_N$$

and any subalgebra of S_N appears in this way.

Comment. Once again follows from Tannaka. The correspondence is now bijective, because $G \subset S_N^+$ implies $1 \in u$.

Invariants

The good. The spectral measure of a planar algebra $P = (P_k)$ is the real probability measure μ having as moments:

$$M_k = \dim(P_k)$$

The bad. The Poincaré series of P is the following series, with $z \in \mathbb{C}$, which is the Stieltjes transform of μ:

$$f(z) = \sum_{k=0}^{\infty} \dim(P_k) z^k$$

The ugly. The principal graph of P is the Bratelli diagram of

$$P_0 \subset P_1 \subset P_2 \subset \ldots$$

with the reflections coming from basic constructions removed.

Examples 1/2

(1) TL. Here we obtain the Marchenko-Pastur law

$$\pi = \frac{1}{2\pi}\sqrt{4x^{-1} - 1}dx$$

also known as free Poisson law, or squared semicircle law.

(2) FC. Here we obtain the real free Bessel law

$$\beta = \pi_{\varepsilon_2}$$

which appears as a compound free Poisson measure.

Examples 2/2

(3) Tensor subfactors. Here we obtain the character law

$$\mu = law(\chi\chi^*)$$

with $\chi = Tr(u)$, assuming that $G \to PU_n$ comes via $ad(u)$.

(4) Spin subfactors. Here we obtain the character law

$$\mu = law(\chi)$$

with $\chi = Tr(u)$, where u corresponds to the action $G \curvearrowright \mathbb{C}^N$.

Questions

1. In the tensor and spin algebra context, we can truncate,

$$\chi_t = \sum_{i=1}^{[tN]} u_{ii}$$

with respect to a parameter $t > 0$. What about in general?

2. In index 4, Jones' manipulation on the Poincaré series,

$$\Theta(q) = q + \frac{1-q}{1+q} f\left(\frac{q}{(1+q)^2}\right)$$

blows up the spectral measure on $\mathbb{T}$. What about in general?

Commuting squares, vertex and spin models

Teo Banica

"Introduction to subfactor theory", 5/6

07/20

Orthogonal MASA

Definition. A pair of orthogonal MASA is a pair of MASA

$$B, C \subset A$$

which are orthogonal: $tr(bc) = tr(b)tr(c)$, for any $b \in B, c \in C$.

Theorem. Up to a unitary, the pairs of orthogonal MASA in the simplest von Neumann factor, namely $M_N(\mathbb{C})$, are

$$A = \Delta \quad , \quad B = H\Delta H^*$$

with $\Delta =$ diagonal matrices, and $H \in M_N(\mathbb{C})$ Hadamard.

Hadamard matrices

The Fourier matrix, $F_N = (w^{ij})$ with $w = e^{2\pi i/N}$,

$$F_N = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & w & w^2 & \dots & w^{N-1} \\ 1 & w^2 & w^4 & \dots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{N-1} & w^{(2N-1)} & \dots & w^{(N-1)^2} \end{pmatrix}$$

is Hadamard. More generally, given a finite abelian group G,

$$(F_G)_{i\chi} = \chi(i)$$

regarded via $G \simeq \widehat{G}$ as a matrix $F_G \in M_G(\mathbb{C})$, is Hadamard.

Expectations

<u>Theorem</u>. Given $H \in M_N(\mathbb{C})$ Hadamard, the associated pair of MASA form a subfactor-theoretic "commuting square"

$$\begin{array}{ccc} \Delta & \longrightarrow & M_N(\mathbb{C}) \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & H\Delta H^* \end{array}$$

in the sense that the expectations to the middle algebras commute, and their product is the expectation to the small algebra.

$\Longrightarrow$ What is the subfactor associated to H?

Commuting squares

Definition. A commuting square in the sense of subfactor theory is a commuting diagram of FD algebras with traces,

$$\begin{array}{ccc} A & \longrightarrow & B \\ \uparrow & & \uparrow \\ C & \longrightarrow & D \end{array}$$

such that the expectations to the middle algebras commute, and their product is the expectation to the small algebra.

Basic construction

<u>Theorem</u>. Given a commuting square, by basic construction we obtain a Jones tower of commuting squares,

$$\begin{array}{ccccccc}
A_0 & \longrightarrow & A_1 & \longrightarrow & A_2 & \cdots\cdots\rightarrow & A_\infty \\
\uparrow & & \uparrow & & \uparrow & & \\
C_0 & \longrightarrow & C_1 & \longrightarrow & C_2 & \cdots\cdots\rightarrow & C_\infty
\end{array}$$

with the limiting algebras being hyperfinite II_1 factors,

$$A_\infty \simeq C_\infty \simeq R$$

and so with $C_\infty \subset A_\infty$ being a subfactor of R.

Ocneanu compactness

<u>Theorem</u>. In the context of a subfactor commuting square,

$$\begin{array}{ccccccc}
A_{\infty 0} & & A_{\infty 1} & & A_{\infty 2} & & \\
\uparrow & & \uparrow & & \uparrow & & \\
A_{20} & \rightarrow & A_{21} & \rightarrow & A_{22} & \cdots\rightarrow & A_{2\infty} \\
\uparrow & & \uparrow & & \uparrow & & \\
A_{10} & \rightarrow & A_{11} & \rightarrow & A_{12} & \cdots\rightarrow & A_{1\infty} \\
\uparrow & & \uparrow & & \uparrow & & \\
A_{00} & \rightarrow & A_{01} & \rightarrow & A_{02} & \cdots\rightarrow & A_{0\infty}
\end{array}$$

the horizontal planar algebra is $A'_{\infty 0} \cap A_{\infty k} = A'_{10} \cap A_{0k}$.

Further results

There are many examples, including those coming from Hadamard matrices, or from more general "vertex and spin models", or even more generally, from squares having $\mathbb{C}$ in the lower left corner:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & D \end{array}$$

All these subfactors have integer index. In non-integer index there are many interesting examples as well. In fact, any finite depth subfactor can be shown to appear from a commuting square.

Basic examples

Consider a vertex model or spin model commuting square, or more generally a commuting square having $\mathbb{C}$ in the lower left corner:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & D \end{array}$$

Our claim is that this produces a fixed point subfactor, coming from an action of a certain compact quantum group G.

Hopf images

Theorem. Any commuting square having $\mathbb{C}$ in the lower left corner, as above, can be written in the following way,

$$\begin{array}{ccc} A\otimes_G\mathbb{C} & \longrightarrow & A\otimes_G D\\ \uparrow & & \uparrow\\ \mathbb{C}\otimes_G\mathbb{C} & \longrightarrow & \mathbb{C}\otimes_G D \end{array}$$

with G being a compact quantum group, obtained from

$$B = A\otimes_G D$$

by performing a certain Hopf image factorization procedure.

Jones tower

Theorem. The horizontal Jones tower is given by

$$\begin{array}{ccccccc}
A \otimes_G \mathbb{C} & \longrightarrow & A \otimes_G D & \longrightarrow & A \otimes_G E & \cdots\cdots\to & A \otimes_G R \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathbb{C} \otimes_G \mathbb{C} & \longrightarrow & \mathbb{C} \otimes_G D & \longrightarrow & \mathbb{C} \otimes_G E & \cdots\cdots\to & \mathbb{C} \otimes_G R
\end{array}$$

and a similar result holds for the vertical Jones tower.

Proof. The computations here are very similar to those needed when computing the Jones tower of fixed point subfactors.

Conclusion

Theorem. The subfactor associated to a commuting square of the following form, having $\mathbb{C}$ in the lower left corner,

$$\begin{array}{ccc} A & \longrightarrow & B \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & D \end{array}$$

is a fixed point subfactor, coming from an action of a certain compact quantum group G, obtained by solving

$$B = A \otimes_G D$$

via a Hopf image factorization construction. The corresponding planar algebra can be computed as well in terms of G.

Hadamard models

<u>Definition</u>. Given an Hadamard matrix $H \in M_N(\mathbb{C})$, we associate to it the smallest subgroup $G \subset S_N^+$ producing a factorization

$$\begin{array}{ccc} C(S_N^+) & \xrightarrow{\pi} & M_N(\mathbb{C}) \\ & \searrow \quad \nearrow & \\ & C(G) & \end{array}$$

of the representation mapping $u_{ij} \in C(S_N^+)$ to the projections

$$P_{ij} = Proj\left(\frac{H_i}{H_j}\right)$$

where $H_1, \ldots, H_N \in \mathbb{C}^N$ are the rows of H, regarded inside $\mathbb{T}^N$.

General theory

Theorem. The above quantum group $G \subset S_N^+$ is the one wich produces the subfactor, and computes the planar algebra.

Theorem. For a Fourier matrix F_G of a finite abelian group G, the above construction produces G itself, acting on itself.

Theorem. For a product of Hadamard matrices, $H = H' \otimes H''$, we obtain a product of quantum groups, $G = G' \times G$".

Diță deformations

Theorem. Given two finite abelian groups G, H, with $|G| = M$, $|H| = N$, consider the main character χ of the quantum group associated to the Diță deformation $\mathcal{F}_{G\times H}$. We have then

$$law\left(\frac{\chi}{N}\right) = \left(1 - \frac{1}{M}\right)\delta_0 + \frac{1}{M}\,\pi_t$$

in moments, with $M = tN \to \infty$, where π_t is the free Poisson law of parameter $t > 0$. In addition, this holds for any generic fiber.

Proof. Long story here (B, BB, B, B).

Further results

All this fits into the matrix modelling theory for quantum groups. The central object here is the stationary matrix model

$$\pi : C(S_4^+) \to M_4(C(SU_2))$$

given on the standard coordinates by the formula

$$\pi(u_{ij}) = [x \to Proj(c_i x c_j)]$$

where $x \in SU_2$, and c_1, c_2, c_3, c_4 are the Pauli matrices. There is a natural generalization of this, involving the Weyl matrices.

Subfactors of small index and big index

Teo Banica

"Introduction to subfactor theory", 6/6

07/20

Subfactors

Theorem. Consider a subfactor $A_0 \subset A_1$, of finite index $N \in [1, \infty)$. Build by "basic construction" the associated Jones tower:

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \ldots$$

The Jones projections generate then a copy of TL_N:

$$TL_N = < e_1, e_2, e_3, \ldots >$$

The planar algebra structure of TL_N extends into a planar algebra structure of the graded algebra $P = (P_k)$, where

$$P_k = A_0' \cap A_k$$

are the higher relative commutants, which are FD algebras.

Theory

Theorem 1. The subfactors $A_0 \subset A_1$ having "finite depth" are classified by their planar algebras $P = (P_k)$.

Theorem 2. More generally, the "amenable" subfactors $A_0 \subset A_1$ are classified by their planar algebras $P = (P_k)$.

Theorem 3. In general, any planar algebra produces a subfactor (complementing "any subfactor produces a planar algebra").

Question. What are the planar algebras of the subfactors of the Murray-von Neumann hyperfinite factor R?

Invariants

The good. The spectral measure μ, having as moments:

$$M_k = \dim(P_k)$$

The bad. The Poincaré series, Stieltjes transform of μ:

$$f(z) = \sum_{k=0}^{\infty} \dim(P_k) z^k$$

The ugly. The principal graph, Bratelli diagram of

$$P_0 \subset P_1 \subset P_2 \subset \dots$$

with the reflections coming from basic constructions removed.

Small index 1/4

Theorem. The subfactors of index $N \leq 4$, which must satisfy

$$N \in \left\{ 4\cos^2\left(\frac{\pi}{n}\right) \Big| n \in \mathbb{N} \right\}$$

are subject to an ADE classification result.

For spin model subfactors, and at $N = 4$, this is related to:

Theorem. The quantum groups $G \subset S_4^+$ appear via

$$S_4^+ = SO_3^{-1}$$

as twists of the usual ADE subgroups of SO_3.

Small index 2/4

Invariants. In general, $N \leq 4$, these can be computed as follows:

(1) The principal graphs are ADE.

(2) The Poincaré series coefficients count loops on these graphs.

(3) The spectral measures can be recovered by Stieltjes.

At $N = 4$ we can simply compute laws of characters.

Advanced. The Jones manipulation on the Poincaré series,

$$\Theta(q) = q + \frac{1-q}{1+q} f\left(\frac{q}{(1+q)^2}\right)$$

blows up the spectral measure on $\mathbb{T}$. Very simple formulae.

Small index 3/4

Theorem. The subfactors of index $N \leq 5$ and a bit higher can be fully classified, by using advanced planar algebra techniques.

For spin model subfactors, and at $N = 5$, this is related to:

Theorem. The quantum groups $G \subset S_5^+$ can be fully classified, by using the above subfactor classification result.

Small index 4/4

Question 1. What is the correct blowup of the spectral measure, in index 5, and more generally, in the "understood" index range?

Question 2. As a consequence, the inclusion $S_N \subset S_N^+$ follows to be maximal at $N = 4, 5$. What about $N = 6$, and in general?

Question 3. What is the natural extra assumption to be added, as for the subfactors of index 6 to become classifiable?

Big index 1/6

Motivation. The various mathematical "objects", once classified by classification theorems, fall into two classes:

(1) Serial.

(2) Exceptional.

This happens for instance for the simple Lie algebras, or for the complex reflection groups. There are many other examples.

Big index 2/6

There are many "uniform" constructions of subfactors. In the quantum group context, the uniformity comes via:

Definition. A closed subgroup $G \subset U_N^+$ is called easy when

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \middle| \pi \in D(k,l)\right)$$

for a certain category of partitions $D \subset P$, where

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with $\delta_\pi \in \{0,1\}$ depending on whether the indices fit or not.

Big index 3/6

<u>Theorem</u>. The basic unitary quantum groups, namely

$$\begin{array}{ccc} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{array}$$

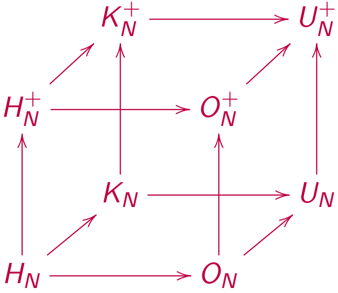
are all easy, coming from the following categories of pairings:

$$\begin{array}{ccc} NC_2 & \longleftarrow & \mathcal{NC}_2 \\ \downarrow & & \downarrow \\ P_2 & \longleftarrow & \mathcal{P}_2 \end{array}$$

The spectral measures are normal $\mathbb{R}/\mathbb{C}$ and $\cap/\bigcirc$ with $N \to \infty$.

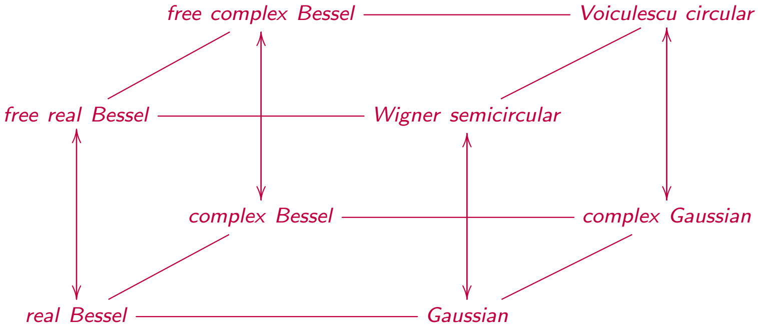
Big index 4/6

Theorem. The main unitary and reflection quantum groups



are all easy, coming from various basic categories of partitions.

Theorem. The asymptotic laws of truncated characters are



with the vertical arrows standing for the Bercovici-Pata bijection.

Big index 6/6

Various questions, which are under current investigation:

(1) Classification of the easy quantum groups.

(2) Various extensions of the easiness theory.

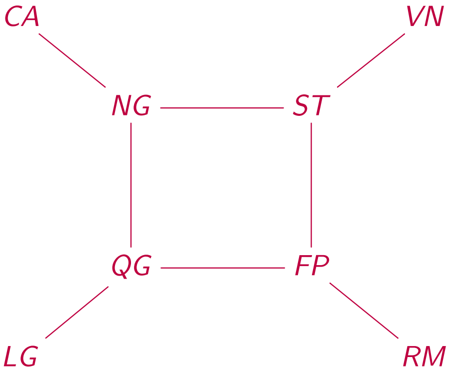
(3) In particular, the super-quizziness problem.

(4) Extensions covering the noncommutative tori.

All this is a mixture of QG, NG, FP, of interest for ST.

Conclusion

When looking for "serial subfactors", we are led to the scheme



for operator algebras in general, with the hot stuff in the middle.

Question

What about R?

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