

Hopf algebras and subfactors associated to vertex models

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Abstract: If H is a Hopf algebra whose square of the antipode is the identity, $v \in \mathcal{L}(V) \otimes H$ is a corepresentation, and $\pi : H \rightarrow \mathcal{L}(W)$ is a representation, then $u = (id \otimes \pi)v$ satisfies the equation $(t \otimes id)u^{-1} = ((t \otimes id)u)^{-1}$ of the vertex models for subfactors. A universal construction shows that any solution u of this equation arises in this way. A more elaborate construction shows that there exists a “minimal” triple (H, v, π) satisfying $(id \otimes \pi)v = u$. This paper is devoted to the study of this latter construction of Hopf algebras. If u is unitary we construct a \mathbf{C}^* -norm on H and we find a new description of the standard invariant of the subfactor associated to u . We discuss also the “twisted” (i.e. $S^2 \neq id$) case.

Introduction

Let V, W be finite dimensional linear spaces over a field k and consider the following condition on an invertible element $u \in \mathcal{L}(V) \otimes \mathcal{L}(W)$

$$(t \otimes id)u^{-1} = ((t \otimes id)u)^{-1} \quad (\star)$$

where $t : \mathcal{L}(V) \rightarrow \mathcal{L}(V^*)$ is the transposition. This equation appeared in the work of V. Jones, and says that a related partition function is invariant under the type II Reidemeister moves. For a first (quite disguised) appearance of (\star) see the formulas (1.3 & 1.4) in [J]. Any unitary solution of (\star) may be used for constructing a commuting square

$$\begin{array}{ccc} \mathbf{C} \otimes \mathcal{L}(W) & \subset & \mathcal{L}(V) \otimes \mathcal{L}(W) \\ \cup & & \cup \\ \mathbf{C} & \subset & u(\mathcal{L}(V) \otimes \mathbf{C})u^{-1} \end{array} \quad (\square_u)$$

hence two examples of subfactors (see [KSV]; here of course we assume that $k = \mathbf{C}$ and that V and W are Hilbert spaces). In the general case an extension of this construction is still available (see [BHJ]). For the sake of completeness we recall also that if we take bases such that $V = k^m$ and $W = k^n$ then one can associate a 2d vertex model to any $u \in M_m(k) \otimes M_n(k)$ in the following way: there are m spins per vertical edge, n spins per horizontal edge, and u is the matrix of Boltzmann weights. See the book [JS] for a global look to these facts.

This paper deals with Hopf algebras and is based on the following approach to the equation (\star) . Consider triples (H, v, π) consisting of a Hopf algebra H whose square of the antipode is the identity, a corepresentation $v \in \mathcal{L}(V) \otimes H$ and a representation $\pi : H \rightarrow \mathcal{L}(W)$. It is easy to see that $(id \otimes \pi)v$ satisfies (\star) , and a universal construction shows that any solution of (\star) arises in this way. More generally, one can describe the class of the elements of the form $(id \otimes \pi)v$, when no assumption on the square of the antipode is made - this contains for instance the solutions of some “twisted” versions of (\star) . Most of the paper is written in this generality, but for simplicity we restrict now attention to the solutions of (\star) .

Any solution of (\star) gives rise to a Hopf algebra in the following way. Let us call models for u the triples (H, v, π) such that $(id \otimes \pi)v = u$. We will show that u admits a minimal model. Here “minimality” is by definition a certain universality property, but we will find several descriptions (including a quite explicit construction) of the minimal model. It is useful to keep in mind the following heuristical interpretation: if (H, v, π) is a model for u then v and π correspond to representations $G \rightarrow \mathbf{GL}(V)$ and $\hat{G} \rightarrow \mathbf{GL}(W)$, where G is the quantum group represented by H and \hat{G} is its dual; the minimal model is then characterised by the fact that these representations are faithful.

Any unitary solution of (\star) gives rise to a Hopf \mathbf{C}^* -algebra and to a Hopf von Neumann algebra in the following way. If (H, v, π) is the minimal model for u we will construct an involution and a \mathbf{C}^* -norm on H and we will prove that the pair (\bar{H}, v) satisfies Woronowicz’ axioms from [W1],[W2] where \bar{H} is the completion of H . As the square of the antipode is the identity, the Haar measure is a trace, and by GNS construction one gets a Kac algebra of compact type in the sense of [ES]. It is useful to keep in mind the following heuristical interpretation: $\bar{H} = C(G) = \mathbf{C}^*(\Gamma)$, with G a compact quantum group and $\Gamma = \hat{G}$ a discrete quantum group. We also show that the operation $u \mapsto \bar{H}$ produces “most” of the commutative and cocommutative Hopf \mathbf{C}^* -

algebras - these come from the obvious solutions $\sum g_i \otimes e_{ii}$ and $\sum e_{ii} \otimes g_i$ of (\star) - as well as all the finite dimensional ones.

Let us call $L(\square_u)$ the standard invariant of the “vertical” subfactor associated to the commuting square \square_u . There is a description due to V. Jones of this lattice $L(\square_u)$ which uses diagrams (see for instance [JS]). We will prove that if (H, v, π) is the minimal model for

$$u' = u_{12}((id \otimes t)u^{-1})_{13} \in \mathcal{L}(V) \otimes \mathcal{L}(W) \otimes \mathcal{L}(W^*) = \mathcal{L}(V) \otimes \mathcal{L}(W \otimes W^*)$$

(which satisfies also (\star)) then $L(\square_u)$ is equal to the following lattice $L(v)$

$$\begin{array}{ccccccc} \mathbf{C} & \subset & End(v) & \subset & End(v \otimes \hat{v}) & \subset & End(v \otimes \hat{v} \otimes v) & \subset & \dots \\ & & \cup & & \cup & & \cup & & \\ & & \mathbf{C} & \subset & End(\hat{v}) & \subset & End(\hat{v} \otimes v) & \subset & \dots \\ & & & & \cup & & \cup & & \\ & & & & \mathbf{C} & \subset & End(v) & \subset & \dots \\ & & & & & & \cup & & \\ & & & & & & \dots & & \dots \end{array}$$

Comments. Summing up, our results split the operation $u \mapsto L(\square_u)$ into a composition of four disjoint operations

$$u \mapsto u' \mapsto (H, v, \pi) \mapsto (\bar{H}, v) \mapsto L(v)$$

The first operation is very explicit, the second one is the construction of the minimal model, and in the third one the involution making v unitary and the maximal \mathbf{C}^* -norm are uniquely determined. About the fourth one, we would like to mention that given a Popa system L , there are at least two reasons for trying to find a pair (A, v) satisfying Woronowicz’ axioms such that $L = L(v)$. First of all $L(v)$ and its principal graphs have simple interpretations in terms of representation theory, and in some cases (e.g. when A happens to be commutative or cocommutative) such an equality $L = L(v)$ is very close to the ultimate result in the “computation” of L . A second reason is that certain analytical notions like amenability are supposed to be better understood for Woronowicz algebras - which have a Haar measure and all the related structures - than for Popa systems or than for subfactors. See the paper [B2] for an introduction to the lattices of the form $L(v)$, from a point of view close to the one of [P].

Acknowledgements. We would like to thank Vaughan Jones for pointing out that the diagrammatic picture shows that $L(\square_u)$ satisfies the axioms for lattices of the form $L(v)$; this led us to work out the present “best” construction $u \mapsto (A, v)$ such that $L(\square_u) = L(v)$. We are also grateful to Patrick Polo for useful discussions on Hopf k -algebras.

1 Construction of the minimal model

Let k be a field. All the k -algebras will have units and the morphisms between them will be unital. The k -coalgebras will have counits and the morphisms between them will be counital. Recall that if C is a coalgebra then C^* has a canonical structure of algebra. Conversely, if A is an algebra then the subspace $A^\circ \subset A^*$ consisting of linear forms f such that $\ker(f)$ contains a finite codimensional ideal of A has a canonical structure of coalgebra. Note that if A is finite dimensional then $A^\circ = A^*$. If H is a Hopf algebra it follows that H° has a canonical structure of Hopf algebra. See [A].

Let $(H, m, 1, \Delta, \varepsilon, S)$ be a Hopf k -algebra. We call finite dimensional representations of H the morphisms of algebras $\pi : H \rightarrow \mathcal{L}(V)$, where V is a finite dimensional k -linear space. The space of coefficients of π is the following linear subspace of H° , which is easily seen to be a subcoalgebra:

$$\mathcal{C}_\pi = \{f\pi \mid f \in \mathcal{L}(V)^*\} \subset H^\circ$$

The dual notion is that of a morphism of coalgebras $\nu : \mathcal{L}(V)^* \rightarrow H$, but we prefer to work with the corresponding element in $\mathcal{L}(V) \otimes H$. That is, we call finite dimensional corepresentations of H the elements $v \in \mathcal{L}(V) \otimes H$ satisfying

$$(id \otimes \Delta)v = v_{12}v_{13}, \quad (id \otimes \varepsilon)v = 1$$

where V is a finite dimensional k -linear space. The space of coefficients of v is the following linear subspace of H , which is easily seen to be a subcoalgebra:

$$\mathcal{C}_v = \{(f \otimes id)v \mid f \in \mathcal{L}(V)^*\} \subset H$$

The following simple facts will be intensively used without reference. The element $(id \otimes S)v$ is an inverse for v in $\mathcal{L}(V) \otimes H$ - this follows by considering $(id \otimes E)v$, with $E = m(S \otimes id)\Delta = m(id \otimes S)\Delta = \varepsilon(\cdot)1$. If $t : \mathcal{L}(V) \rightarrow \mathcal{L}(V^*)$ is the transposition, then $(t \otimes S)v$ is a corepresentation of H - this follows

from the fact that t is unital and antimultiplicative, and S is counital and anticomultiplicative.

Definition 1.1 *Let V, W be finite dimensional k -linear spaces and let $u \in \mathcal{L}(V) \otimes \mathcal{L}(W)$. A model for u is a triple (H, v, π) consisting of a Hopf algebra H , a corepresentation $v \in \mathcal{L}(V) \otimes H$ and a representation $\pi : H \rightarrow \mathcal{L}(W)$ such that $(id \otimes \pi)v = u$.*

Comments. The identifications of the form $X \otimes Y \simeq \mathcal{L}(X^*, Y)$ show that one may use the following related definition: if C is a coalgebra and A is an algebra, a model for a linear map $\varphi : C \rightarrow A$ is a factorisation of it as

$$C \xrightarrow{\nu} H \xrightarrow{\pi} A$$

where H is a Hopf algebra, ν is a morphism of coalgebras, and π is a morphism of algebras. Another related definition may be found by extending the following interpretation of $(id \otimes \pi)v$ when H is finite dimensional: the duality $H \otimes H^* \rightarrow k$ gives by transposition a distinguished element $\xi_H \in H^* \otimes H$, and if $\rho : H^* \rightarrow \mathcal{L}(V)$ is the representation corresponding to v then $(id \otimes \pi)v$ is the image of ξ_H by the representation $\rho \otimes \pi$. For some technical reasons we prefer to use the above Def. 1.1.

If V is a linear space and $S \subset V$ is a subset, the orthogonal of S is $S^\perp = \{f \in V^* \mid f(s) = 0, \forall s \in S\}$. Also if $T \subset V^*$ is a subset we may define $T^{\perp(V)} = \{v \in V \mid t(v) = 0, \forall t \in T\}$. Given $f \in V^*$, by letting the family $\{f + a^\perp \mid a \in V\}$ be a base for a system of neighborhoods of f , V^* becomes a linear topological space. A linear subspace $T \subset V^*$ is dense in V^* with respect to this topology if and only if $T^{\perp(V)} = \{0\}$. See [A].

If $C \subset H$ is a subcoalgebra of a Hopf algebra, we denote by $\langle C \rangle$ the Hopf subalgebra of H generated by C . That is, $\langle C \rangle$ is by definition the (unital) subalgebra of H generated by the set $\cup_{k \geq 0} S^k(C)$.

Definition 1.2 *A model (H, v, π) for $u \in \mathcal{L}(V) \otimes \mathcal{L}(W)$ is said to be left-faithful if $\langle C_v \rangle = H$; right-faithful if $\langle C_\pi \rangle$ is dense in H^* ; and bi-faithful if it is both left- and right-faithful.*

Given a model (H, v, π) one can construct a left-faithful model (H', v, π') in the following way: H' is $\langle C_v \rangle$ and π' is the restriction to H' of π . Due to this simple fact, we will oftenly restrict attention to left-faithful models.

We define the morphisms $(H_1, v_1, \pi_1) \rightarrow (H_2, v_2, \pi_2)$ of left-faithful models to be the Hopf algebra morphisms $f : H_1 \rightarrow H_2$ such that $(id \otimes f)v_1 = v_2$ and $\pi_1 = \pi_2 f$. By left-faithfulness, such a morphism (if it exists) is surjective, and unique. In particular a morphism from (H, v, π) to itself has to be equal to the identity morphism. Thus given u , the category of left-faithful models for u has at most one universally repelling object, and at most one universally attracting object.

Definition 1.3 *The universally repelling (resp. attracting) object in the category of left-faithful models for u is called the maximal (resp. minimal) model for u .*

In this definition we assume of course that the category is non-empty, and that the object to be defined exists. The result below clarifies the situation.

Theorem 1.1 *Let V, W be finite dimensional k -linear spaces and let $u \in \mathcal{L}(V) \otimes \mathcal{L}(W)$. The following conditions are equivalent:*

- (i) *there exists a model for u .*
- (ii) *there exists a maximal model for u .*
- (iii) *there exists a minimal model for u .*
- (iv) *u satisfies the following sequence of conditions: $u_0 := u$ is invertible, $u_1 := (t \otimes id)u_0^{-1}$ is invertible, $u_2 := (t \otimes id)u_1^{-1}$ is invertible, $u_3 := (t \otimes id)u_2^{-1}$ is invertible, etc..*

Moreover, if these conditions are satisfied then the minimal model for u may be characterised as the unique bi-faithful model for u .

Proof of (i) \implies (iv). By recurrence, we have to prove that if there exists a model for $w \in \mathcal{L}(T) \otimes \mathcal{L}(W)$ then w is invertible and there exists a model for $(t \otimes id)w^{-1}$. Let (H, v, π) be a model for w . As $(id \otimes S)v$ is an inverse for v , we get that $(id \otimes \pi S)v$ is an inverse for w . This implies also that $(t \otimes id)w^{-1} = (t \otimes \pi S)v$. As $(t \otimes S)v$ is a corepresentation of H , this shows that $(H, (t \otimes S)v, \pi)$ is a model for $(t \otimes id)w^{-1}$.

Proof of (iv) \implies (ii). We take a basis of V which identifies $V = k^s$. Let F be the free k -algebra generated by elements $\{w_{a,b}^n\}$ with $n \geq 0$ and $a, b \in \{1, \dots, s\}$. Then F has a bialgebra structure with

$$\Delta(w_{a,b}^n) = \sum_{1 \leq c \leq s} w_{a,c}^n \otimes w_{c,b}^n, \quad \varepsilon(w_{a,b}^n) = \delta_{a,b}$$

For every n let $w^n \in M_s(F)$ be the matrix having entries $(w_{a,b}^n)_{1 \leq a,b \leq s}$. Let $J \subset F$ be the two-sided ideal generated by the relations coming from identifying the coefficients in the equalities

$$w^n(w^{n+1})^t = (w^{n+1})^t w^n = 1$$

for every n , and consider the quotient $H = F/J$. Let $v^n = (id \otimes p)w^n$, where $p : F \rightarrow H$ is the projection. Then H has the following universal property (P): given any k -algebra A and any sequence of matrices $V^n \in M_s(A)$ such that $(V^{n+1})^t = (V^n)^{-1}$ for every $n \geq 0$, there exists a (unique) morphism of algebras $f : H \rightarrow A$ such that $(id \otimes f)v^n = V^n$ for every n .

The following equality holds in $H \otimes H$

$$(v_{12}^n v_{13}^n)^{-1} = (v_{13}^n)^{-1} (v_{12}^n)^{-1} = (v_{13}^{n+1})^t (v_{12}^{n+1})^t = (v_{12}^{n+1} v_{13}^{n+1})^t$$

so by applying (P) with $A = H \otimes H$ and $V^n = v_{12}^n v_{13}^n$ we get a certain morphism $\Delta_H : H \rightarrow H \otimes H$. Also by applying (P) with $A = k$ and $V^n = 1$ we get a morphism $\varepsilon_H : H \rightarrow k$, and by applying (P) with $A = H^{op}$ and $V^n = (v^{n+1})^t$ we get a morphism $\mathcal{S}_H : H \rightarrow H^{op}$. If $j : H^{op} \rightarrow H$ is the canonical map, it is easy to see that $(H, m, 1, \Delta_H, \varepsilon_H, j\mathcal{S}_H)$ satisfies the axioms for a Hopf algebra (by verifying each of them on the generators v_{ab}^n). Once again by (P), the conditions on u in the statement allow us to define a morphism of algebras $\pi : H \rightarrow \mathcal{L}(W)$ such that $(id \otimes \pi)v^n = u_n$ for every n . This shows that (H, v^0, π) is a model for u .

Let (K, r, ν) be an arbitrary left-faithful model for u . We define a sequence of corepresentations $r^n \in M_s(K)$ by $r^{2n} = (id \otimes S^{2n})r$ and $r^{2n+1} = (t \otimes S^{2n+1})r$ for every $n \in \mathbf{N}$, where $t : M_s(k) \rightarrow M_s(k)$ is the transposition. Then $(r^n)^{-1} = (id \otimes S)r^n = (r^{n+1})^t$ for every n , so the property (P) gives a morphism of Hopf algebras $f : H \rightarrow K$ such that $(id \otimes f)v^n = r^n$ for every n . This shows that f is a morphism of left-faithful models $(H, v^0, \pi) \rightarrow (K, r, \nu)$. Thus (H, v^0, π) is the maximal model for u .

The construction below of the minimal model uses the maximal model and the following simple fact. Assume that \mathcal{C} is a category such that for any object a there exists an object a_1 and an arrow $a \rightarrow a_1$ such that for any object a_2 and any arrow $a \rightarrow a_2$ there exists an arrow $a_2 \rightarrow a_1$ making commutative the triangle. Then if \mathcal{C} has a universally repelling object, then it has a universally attracting object. We will show in the next Lemma that

the category of left-faithful models for u has this property. We begin by explaining what the above-mentioned construction $a \mapsto a_1$ is.

If (H, v, π) is a left-faithful model for u we define a left-faithful model (H_1, v_1, π_1) and a morphism $p_1 : (H, v, \pi) \rightarrow (H_1, v_1, \pi_1)$ in the following way. Consider the space of coefficients $\mathcal{C}_\pi \subset H^\circ$. Then \mathcal{C}_π is a subcoalgebra of H° and the orthogonal $\mathcal{C}_\pi^{\perp(H)}$ is the kernel of π . As $\langle \mathcal{C}_\pi \rangle$ is a subalgebra (resp. subcoalgebra) of H° , the orthogonal $\langle \mathcal{C}_\pi \rangle^{\perp(H)}$ is a coideal (resp. ideal) of H , cf. [A], Th. 2.3.6 (i) (resp. Th. 2.3.2 (ii)). Moreover, from the invariance of $\langle \mathcal{C}_\pi \rangle$ under the antipode S^* of H° we get the invariance of $\langle \mathcal{C}_\pi \rangle^{\perp(H)}$ under S , so the quotient

$$H_1 := H / \langle \mathcal{C}_\pi \rangle^{\perp(H)}$$

is a Hopf algebra. As $\langle \mathcal{C}_\pi \rangle^{\perp(H)}$ is contained in $\mathcal{C}_\pi^{\perp(H)}$, which is the kernel of π , we get a factorisation $\pi = \pi_1 p_1$, where $p_1 : H \rightarrow H_1$ is the projection. If we define $v_1 := (id \otimes p_1)v$, then (H_1, v_1, π_1) is a left-faithful model for u , and p_1 is a morphism of left-faithful models.

Lemma 1.1 *Let (H, v, π) be a left-faithful model and construct (H_1, v_1, π_1) and p_1 as above. Let $p_2 : (H, v, \pi) \rightarrow (H_2, v_2, \pi_2)$ be a morphism of left-faithful models for u . Then there exists a morphism of left-faithful models $f : (H_2, v_2, \pi_2) \rightarrow (H_1, v_1, \pi_1)$ such that $p_1 = f p_2$.*

Proof. Let J be the kernel of p_2 . As $\pi = \pi_2 p_2$, the ideal J is contained in the kernel $\mathcal{C}_\pi^{\perp(H)}$ of π ; we want to prove that J is contained in the kernel $\langle \mathcal{C}_\pi \rangle^{\perp(H)}$ of p_1 . By dualising we want to prove

$$\mathcal{C}_\pi \subset J^\perp \implies \langle \mathcal{C}_\pi \rangle \subset J^\perp$$

As J^\perp is stable by S^* , it contains the set $\cup_{k \geq 0} (S^*)^k(\mathcal{C}_\pi)$. Now as J^\perp is a subalgebra of H^* ([A], Th. 2.3.6 (ii)), it contains the algebra generated by $\cup_{k \geq 0} (S^*)^k(\mathcal{C}_\pi)$, which is $\langle \mathcal{C}_\pi \rangle$. Thus $J \subset \ker(p_1)$, and we get the desired Hopf algebra morphism from H_2 to H_1 . \square

Proof of (ii) \implies (iii). If (H, v, π) is the maximal model for u , then the Lemma 1.1 shows that the above construction gives the minimal model.

Proof of the last assertion. Let (H, v, π) be the minimal model for u . By applying the above construction we get a certain left-faithful model (H_1, v_1, π_1) . As there exist morphisms from (H, v, π) to (H_1, v_1, π_1) in both

senses, the unicity of morphisms between left-faithful models shows that these models are isomorphic. In particular the kernel of the projection $H \rightarrow H_1$, which is $\langle \mathcal{C}_\pi \rangle^{\perp(H)}$ by definition, is zero. Thus (H, v, π) is bi-faithful.

Now let (H', v', π') be an arbitrary bi-faithful model for u . By minimality of (H, v, π) we get a Hopf algebra morphism $p : H' \rightarrow H$ such that $v = (id \otimes p)v'$ and $\pi' = \pi p$. The left-faithfulness shows that p is surjective, and the assertion will follow from the sequence of inclusions

$$\ker(p) \subset \text{Im}(p^*)^{\perp(H')} \subset \langle \mathcal{C}_{\pi'} \rangle^{\perp(H')} \subset \{0\}$$

The first inclusion is clear: if x is in $\ker(p)$, then $p^*(f)(x) = fp(x) = 0$ for any $f \in H^*$. The second one follows by dualising $\langle \mathcal{C}_{\pi'} \rangle \subset \text{Im}(p^*)$. The third one is the definition of the right-faithfulness of (H', v', π') . \square

2 Relationship with the vertex models

The simplest way for finding elements u satisfying the conditions in the Th. 1.1 is to assume that $u_2 = u_0$; in this case the infinity of conditions (iv) in the Th. 1.1 becomes periodic and true. The condition $u_2 = u$ is nothing but the equation (\star) in the Introduction. In the sections 2,3,4 we restrict attention to this case, and we will use the following consequence of the Th. 1.1.

Theorem 2.1 *Let V, W be finite dimensional k -linear spaces. If $u \in \mathcal{L}(V) \otimes \mathcal{L}(W)$ is invertible and if the equality*

$$(t \otimes id)u^{-1} = ((t \otimes id)u)^{-1} \quad (\star)$$

holds in $\mathcal{L}(V^) \otimes \mathcal{L}(W)$, then there exists a minimal model for u . This is the unique bi-faithful model for u . \square*

The next Proposition shows that in the general case, imposing this important condition is the same as restricting attention to the Hopf algebras whose square of the antipode is the identity.

Proposition 2.1 *Let V, W be finite dimensional k -linear spaces.*

(i) Let H be a Hopf algebra whose square of the antipode is the identity, $v \in \mathcal{L}(V) \otimes H$ be a corepresentation, and $\pi : H \rightarrow \mathcal{L}(W)$ be a representation. Then $u := (id \otimes \pi)v$ is invertible and satisfies (\star) .

(ii) Assume that $u \in \mathcal{L}(V) \otimes \mathcal{L}(W)$ is invertible and satisfies (\star) . If (H, v, π) is the minimal model for u , then the square of the antipode of H is the identity.

Proof. (i) We know that $(id \otimes S)v$ is the inverse of v . Also $\hat{v} = (t \otimes S)v$ is a corepresentation, so $(id \otimes S)\hat{v}$ is an inverse for \hat{v} . By combining these results we get $(t \otimes id)v^{-1} = ((t \otimes S^2)v)^{-1}$, and as $S^2 = id$ the assertion follows by applying $id \otimes \pi$ to this equality.

(ii) This may be proved by using $(t \otimes id)v^{-1} = ((t \otimes S^2)v)^{-1}$ and the bi-faithfulness of the minimal model, but one can do better. Let \mathcal{C} be the category of left-faithful models for u such that the square of the antipode of their subjacent Hopf algebras are the identities. As the property “ $S^2 = id$ ” is preserved by surjective morphisms of Hopf algebras, it is enough to prove that \mathcal{C} is non-empty. Take a basis which identifies $V = k^s$ and let H_s be the universal k -algebra given with generators $(w_{ij}^0)_{i,j=1,\dots,s}$ and $(w_{ij}^1)_{i,j=1,\dots,s}$ and with the relations coming from the equalities

$$w^0(w^1)^t = (w^1)^t w^0 = w^1(w^0)^t = (w^0)^t w^1 = 1$$

The same arguments as in the proof of the Th. 1.1 show that there exists a Hopf algebra structure on H_s (with the antipode given by $id \otimes S : w^0 \leftrightarrow (w^1)^t$) and a representation π such that (H_s, w^0, π) is the universally repelling object in \mathcal{C} . The details are left to the reader. \square

We give now some examples of minimal models. Recall that if G is a group then the group algebra kG is the free k -module over G , with the multiplication induced by the one of G . It has a structure of (cocommutative) Hopf k -algebra with $\Delta(g) = g \otimes g$, $\varepsilon(g) = \delta_{g,e} 1_k$ and $S(g) = g^{-1}$, where $e \in G$ is the unit. If $k(G)$ is the k -algebra of all functions from G to k , then we have identifications $kG^* = k(G)$ (as algebras) and $k(G)^* = k(G)^\circ = kG$ (as coalgebras). The dual of the Hopf algebra kG is the (commutative) Hopf algebra $R_k(G) := (kG)^\circ \subset k(G)$ consisting of representative functions on G , i.e. coefficients of finite dimensional representations of G . See [A].

Let $G \subset \mathbf{GL}(V)$ be a subgroup, with V finite dimensional. The linear map $\nu : kG \rightarrow \mathcal{L}(V)$ which maps $g \in kG$ into $g \in G \subset \mathbf{GL}(V) \subset \mathcal{L}(V)$ is a representation of kG , called the fundamental one. We denote by $k[G] \subset R_k(G)$ the Hopf subalgebra generated by the coefficients of the representation $G \rightarrow \mathcal{L}(V)$. The space of these coefficients being the image of the transpose

$\nu^* : \mathcal{L}(V)^* \rightarrow kG^*$, we get by restriction a coalgebra morphism from $\mathcal{L}(V)^*$ to $k[G]$, hence a corepresentation $v \in \mathcal{L}(V) \otimes k[G]$, called the fundamental one. These constructions depend of course on the embedding $G \subset \mathbf{GL}(V)$.

Proposition 2.2 *Let V be a finite dimensional space and let g_1, \dots, g_n be elements of $\mathbf{GL}(V)$. Let $G \subset \mathbf{GL}(V)$ be the group generated by g_1, \dots, g_n .*

(i) *The element $u = \sum e_{ii} \otimes g_i \in \mathcal{L}(k^n) \otimes \mathcal{L}(V)$ satisfies (\star) . If $w = \sum e_{ii} \otimes g_i \in \mathcal{L}(k^n) \otimes kG$ and ν is the fundamental representation of kG , then (kG, w, ν) is the minimal model for u .*

(ii) *The element $s = \sum g_i \otimes e_{ii} \in \mathcal{L}(V) \otimes \mathcal{L}(k^n)$ satisfies (\star) . If v is the fundamental corepresentation of $k[G]$ and $\pi : k[G] \rightarrow \mathcal{L}(k^n)$ is the linear map $f \mapsto \sum e_{pp} f(g_p)$, then $(k[G], v, \pi)$ is the minimal model for s .*

Proof. It is clear from definitions that $(k[G], v, \pi)$ and (kG, w, ν) are left-faithful models for s , respectively u .

(i) Let (H, w_1, ν_1) be the minimal model for u and $p : kG \rightarrow H$ be the corresponding projection. If K denotes the image of $G \subset (kG)^\times$ by the underlying group morphism $p^\times : (kG)^\times \rightarrow H^\times$, then the elements of K are group-like elements of H , so they are linearly independent ([A], Th. 2.1.2). On the other hand these elements generate H as a linear space, and it follows that H may be identified with the group algebra kK . Now the equality $\nu = \nu_1 p$ reads $\nu^\times|_G = \nu_1^\times p^\times|_G$, and as $\nu^\times|_G$ is injective we get that $p^\times|_G$ is injective. Thus the surjection $G \rightarrow K$ is an isomorphism, so p is an isomorphism.

(ii) By transposing the inclusions $k[G] \subset \mathcal{R}_k(G) \subset k(G)$ we get surjections $k(G)^* \rightarrow \mathcal{R}_k(G)^* \rightarrow k[G]^*$. If q denotes the projection from $kG = k(G)^*$ to $k[G]^*$, then $\pi^* = qw$. As $\langle \mathcal{C}_w \rangle = kG$, it follows that $\langle \mathcal{C}_\pi \rangle = k[G]^*$. In particular $(k[G], v, \pi)$ is right-faithful. \square

We end with a simple Lemma which gives some more examples.

Lemma 2.1 *Assume that $u \in \mathcal{L}(V) \otimes \mathcal{L}(W)$ satisfies (\star) .*

(i) *$\bar{u} = (id \otimes t)u^{-1}$ is equal to $((id \otimes t)u)^{-1}$ and satisfies (\star) .*

(ii) *$\hat{u} = (t \otimes id)u^{-1}$ satisfies (\star) .*

(iii) *$u_{12}\bar{u}_{13}u_{14}\dots \in \mathcal{L}(V) \otimes \mathcal{L}(W \otimes W^* \otimes W\dots)$ (i terms) satisfies (\star) .*

(iv) *$u_{1,i+1}\hat{u}_{2,i+1}u_{3,i+1}\dots \in \mathcal{L}(V \otimes V^* \otimes V\dots) \otimes \mathcal{L}(W)$ (i terms) satisfies (\star) .*

Proof. The first equality follows by applying the antimorphism $t \otimes t$ to the equation (\star) . The fact that the four elements in the statement satisfy

(\star) is elementary, but we will give a nice proof which will be used later on. By the Prop. 2.1 (i) it is enough to construct models for them such that the square of the antipode is the identity. Let (H, v, π) be the minimal model for u ; by the Prop. 2.1 (ii) the square of the antipode of H is the identity. If $\hat{v} = (t \otimes S)v$ and $\hat{\pi} = t\pi S$ then $(H, v, \hat{\pi})$ is a model for \bar{u} , (H, \hat{v}, π) is a model for \hat{u} , and

$$(H, v, (\pi \otimes \hat{\pi} \otimes \pi \otimes \dots) \Delta^{(i-1)}), \quad (H, v_{1,i+1} \hat{v}_{2,i+1} v_{3,i+1} \dots, \pi)$$

(i terms) are models for $u_{12} \bar{u}_{13} u_{14} \dots$ and for $u_{1,i+1} \hat{u}_{2,i+1} u_{3,i+1} \dots$. \square

3 Tensor products and intertwiners

If (\mathcal{C}, \otimes) is a k -linear tensor category and X, \hat{X} are objects of \mathcal{C} we may define the following lattice of k -algebras

$$\begin{array}{ccccccc} k & \subset & \text{End}(X) & \subset & \text{End}(X \otimes \hat{X}) & \subset & \text{End}(X \otimes \hat{X} \otimes X) & \subset & \dots \\ & & \cup & & \cup & & \cup & & \\ & & k & \subset & \text{End}(\hat{X}) & \subset & \text{End}(\hat{X} \otimes X) & \subset & \dots \\ & & & & \cup & & \cup & & \\ & & & & k & \subset & \text{End}(X) & \subset & \dots \\ & & & & & & \cup & & \\ & & & & & & \dots & & \dots \end{array}$$

where the inclusions are the obvious ones. In our examples the object \hat{X} will be always the “dual” of X in some canonical sense, so this lattice will be denoted simply by $L(X)$. See the Prop. 1.1 in [B2] for some precise “duality conditions” to be imposed on (X, \hat{X}) as to get Jones projections, traces, etc..

Example. Let H be a Hopf k -algebra whose square of the antipode is the identity and consider the tensor category of finite dimensional corepresentations of H . That is, the objects are the finite dimensional corepresentations of H , and if $v \in \mathcal{L}(V) \otimes H$ and $w \in \mathcal{L}(W) \otimes H$ are two corepresentations, then the arrows between them are the intertwining operators

$$\text{Hom}(v, w) = \{T \in \mathcal{L}(V, W) \mid (T \otimes 1_H)v = w(T \otimes 1_H)\}$$

and their tensor product is $v \otimes w := v_{13} w_{23}$. For any corepresentation v we define \hat{v} to be the contragradient corepresentation $(t \otimes S)v = (t \otimes id)v^{-1}$, and we use the above notation $L(v)$.

Example. Let A be a k -algebra. We define a tensor category in the following way. The objects are elements of $\mathcal{L}(V) \otimes A$, with V ranging over all finite dimensional linear spaces. If $v \in \mathcal{L}(V) \otimes A$ and $w \in \mathcal{L}(W) \otimes A$ are two objects the space of arrows between them is

$$\text{Hom}(v, w) = \{T \in \mathcal{L}(V, W) \mid (T \otimes 1_A)v = w(T \otimes 1_A)\}$$

and their tensor product is $v \otimes w := v_{13}w_{23}$. If $u \in \mathcal{L}(V) \otimes A$ is an invertible element satisfying the condition

$$(t \otimes id)u^{-1} = ((t \otimes id)u)^{-1}$$

we define $\hat{u} \in \mathcal{L}(V^*) \otimes A$ to be $(t \otimes id)u^{-1}$ and we use the notation $L(u)$.

Note that if A is a Hopf algebra whose square of the antipode is the identity then the category defined in the first example is a subcategory of the one in the second example, and that if v is a corepresentation of A then the two \hat{v} 's (hence the two $L(v)$'s) defined in these ways are equal. Note also that if $A = \mathcal{L}(W)$ then the above condition on u is exactly the condition (\star) .

Proposition 3.1 *Let V, W be finite dimensional k -linear spaces and assume that $u \in \mathcal{L}(V) \otimes \mathcal{L}(W)$ satisfies (\star) . Then the element*

$$u' = u_{12}((id \otimes t)u^{-1})_{13} \in \mathcal{L}(V) \otimes \mathcal{L}(W) \otimes \mathcal{L}(W^*) = \mathcal{L}(V) \otimes \mathcal{L}(W \otimes W^*)$$

satisfies (\star) . If (H, v, π) is a minimal model for u' then $L(u') = L(v)$.

Note. The notion of equality for lattices in this statement is the obvious one: there exists a family of inclusion-preserving isomorphisms of algebras between the algebras of the first lattice and the algebras of the second lattice. In fact we will prove a little more finer statement: the lattices $L(u')$ and $L(v)$ are equal as sublattices of $L(V)$, where $L(V)$ is the lattice associated to the linear space V in the sense of [B2], i.e. $L(V)$ is the lattice obtained by applying the above construction with $(\mathcal{C}, \otimes) =$ the tensor category of finite dimensional linear spaces, and with $\hat{V} := V^*$.

Proof. The fact that u' satisfies (\star) follows from the Lemma 2.1. Recall that the argument in there was that if (K, w, ν) is the minimal model for u , then

$$(K, w, (\nu \otimes t\nu S)\Delta)$$

is a model for u' . Let $p : K \rightarrow H$ be the Hopf algebra surjection constructed by the universal property of the minimal model, which is such that $\pi p = (\nu \otimes t\nu S)\Delta$. By transposing this equality we get

$$p^* \pi^* = m_{K^*}(\nu^* \otimes S^* \nu^* t^*)$$

The image of the right term is invariant under S^* , so it follows that the image of $p^* \pi^*$ is invariant under the antipode of K° . As p^* is injective, we get that $\mathcal{C}_\pi = \text{Im}(\pi^*)$ is invariant under the antipode of H° . By right-faithfulness we obtain that \mathcal{C}_π generates a dense subalgebra of H^* .

We have to prove that $L(u') = L(v)$. Let $0 \leq j \leq i$ and consider the algebra of $L(v)$ sitting on the $(i, -j)$ position. This algebra is of the form $\text{End}(r)$, with r a certain tensor product between v 's and \hat{v} 's. As $u' = (\text{id} \otimes \pi)v$, it follows that the algebra of $L(u')$ sitting on the $(i, -j)$ position is $\text{End}((\text{id} \otimes \pi)r)$. Thus it is enough to prove the following general result. \square

Lemma 3.1 *Let H be a Hopf algebra and $\pi : H \rightarrow \mathcal{L}(W)$ be a finite dimensional representation such that \mathcal{C}_π generates a dense subalgebra of H^* . If $r \in \mathcal{L}(T) \otimes H$ is a finite dimensional corepresentation then $\text{End}((\text{id} \otimes \pi)r) = \text{End}(r)$.*

Proof. The inclusion $\text{End}(r) \subset \text{End}((\text{id} \otimes \pi)r)$ is clear. For the converse let $x \in \text{End}((\text{id} \otimes \pi)r)$ be an arbitrary element and consider the set

$$S_x = \{f \in H^* \mid [x, (\text{id} \otimes f)r] = 0\}$$

We have to prove that $x \in \text{End}(r)$, which is equivalent to $S_x = H^*$. First of all, if f, g are two elements of the algebra H^* then $(\text{id} \otimes fg)r$ is equal to

$$(\text{id} \otimes f \otimes g)(\text{id} \otimes \Delta)r = (\text{id} \otimes f \otimes g)(r_{12}r_{13}) = ((\text{id} \otimes f)r)((\text{id} \otimes g)r)$$

and this shows that S_x is a subalgebra of H^* . On the other hand we have $\mathcal{C}_\pi \subset S_x$, so S_x contains the subalgebra of H^* generated by \mathcal{C}_π ; thus S_x is dense in H^* . Let now e_1, \dots, e_s be a basis of T and identify $\mathcal{L}(T)$ with $M_s(k)$. By writing $r = \sum e_{ij} \otimes r_{ij}$ and $x = \sum x_{ij} e_{ij}$ we see that an element $f \in H^*$ is in S_x if and only if

$$\sum_k x_{ik} f(r_{kj}) = \sum_k f(r_{ik}) x_{kj}$$

for any i, j . By denoting $a_{ij} = \sum_k x_{ik}r_{kj} - x_{kj}r_{ik}$ we have shown that

$$S_x = \bigcap_{i,j} a_{ij}^\perp$$

This shows that S_x is an open subspace of H^* , so $S_x = H^*$. \square

4 Relationship with the commuting squares

In this section k is a field of characteristic zero. Consider a Markov commuting square of multimatrix k -algebras (see [BHJ])

$$\begin{array}{ccc} A_{01} & \subset & A_{11} \\ \cup & & \cup \\ A_{00} & \subset & A_{10} \end{array} \quad (\square)$$

By performing the basic constructions in all the possible directions we obtain an infinite lattice of multimatrix algebras $(A_{ij})_{i,j \geq 0}$, where the labeling is the obvious one. For every $i \geq 0$ denote by $A_{i\infty}$ the inductive limit $\lim_{j \rightarrow \infty} (A_{ij})$. The inclusion $A_{0\infty} \subset A_{1\infty}$ is called the vertical inclusion associated to the initial commuting square. The Jones tower of this inclusion is

$$A_{0\infty} \subset A_{1\infty} \subset A_{2\infty} \subset A_{3\infty} \subset \dots$$

Also for every $0 \leq j \leq i$ the canonical inclusion $A_{i0} \subset A_{i\infty}$ induces an isomorphism $A'_{j1} \cap A_{i0} \simeq A'_{j\infty} \cap A_{i\infty}$, so the lattice $L(\square)$ of higher relative commutants of the vertical inclusion is

$$\begin{array}{ccccccc} A'_{01} \cap A_{00} & \subset & A'_{01} \cap A_{10} & \subset & A'_{01} \cap A_{20} & \subset & A'_{01} \cap A_{30} & \subset & \dots \\ & & \cup & & \cup & & \cup & & \\ & & A'_{11} \cap A_{10} & \subset & A'_{11} \cap A_{20} & \subset & A'_{11} \cap A_{30} & \subset & \dots \\ & & & & \cup & & \cup & & \\ & & & & A'_{21} \cap A_{20} & \subset & A'_{21} \cap A_{30} & \subset & \dots \\ & & & & & & \cup & & \\ & & & & & & \dots & & \dots \end{array}$$

These are analogues of well-known results from the theory of commuting squares of finite dimensional von Neumann algebras, including the Compactness Theorem of Ocneanu. See [GHJ] for commuting squares, [O],[JS] for

Oceanu's theorem, and [BHJ] for the above analogues. For the purposes of this paper, one may take the above description of $L(\square)$ as a definition for it.

An example of commuting squares is the following one. Let V, W be finite dimensional k -linear spaces and let $u \in \mathcal{L}(V) \otimes \mathcal{L}(W)$ be an invertible element. Then the diagram

$$\begin{array}{ccc} k \otimes \mathcal{L}(W) & \subset & \mathcal{L}(V) \otimes \mathcal{L}(W) \\ \cup & & \cup \\ k & \subset & u(\mathcal{L}(V) \otimes k)u^{-1} \end{array} \quad (\square_u)$$

is a Markov commuting square if and only if u satisfies the condition (\star) (see [BHJ]). This is an analogue of the well-known result when V, W are finite dimensional Hilbert spaces and u is unitary, which will be discussed in detail in the next section.

Theorem 4.1 *Let $u \in \mathcal{L}(V) \otimes \mathcal{L}(W)$ be an element satisfying (\star) . If (H, v, π) is the minimal model for the element*

$$u' = u_{12}((id \otimes t)u^{-1})_{13} \in \mathcal{L}(V) \otimes \mathcal{L}(W) \otimes \mathcal{L}(W^*) = \mathcal{L}(V) \otimes \mathcal{L}(W \otimes W^*)$$

then the lattice $L(\square_u)$ is equal to the lattice $L(v)$.

Proof. This will follow from $L(\square_u) = L(u')$ and from the Prop. 3.1. Most of the partial results that we need claim no originality and may be deduced from [KSV],[JS] by changing the ground field, the notations etc.. We prefer to give a short self-contained proof of $L(\square_u) = L(u')$, using our notations.

Step I. If T is a finite dimensional linear space we denote by e^T the Jones projection in $\mathcal{L}(T \otimes T^*)$, i.e. the element $n^{-1} \sum e_{ij} \otimes e_{ij}^*$ which does not depend on the basis e_1, \dots, e_n of T . For any $i \geq 0$ we define an algebra

$$A^i = \mathcal{L}(V \otimes V^* \otimes V \otimes V^* \otimes \dots)$$

(i terms). Let us prove that the following diagram is a sequence of basic constructions for commuting squares (note that the first one is \square_u)

$$\begin{array}{ccccccc} \mathcal{L}(W) & \subset & A^1 \otimes \mathcal{L}(W) & \subset & A^2 \otimes \mathcal{L}(W) & \subset & A^3 \otimes \mathcal{L}(W) & \subset & \dots \\ \cup & & \cup & & \cup & & \cup & & \\ k & \subset & ad(U^1)A^1 & \subset & ad(U^2)A^2 & \subset & ad(U^3)A^3 & \subset & \dots \end{array}$$

where $U^i \in A^i \otimes \mathcal{L}(W) = \mathcal{L}(V) \otimes \mathcal{L}(V^*) \otimes \dots \otimes \mathcal{L}(W)$ is the element

$$U^i = u_{1,i+1} \hat{u}_{2,i+1} u_{3,i+1} \hat{u}_{4,i+1} \dots$$

(i terms). Here of course the inclusions in the upper line are the obvious ones, and the Jones projection for the i -th inclusion of this line is by definition $id_{A_{i-2}} \otimes e^T \otimes id_W$, where $T = V$ if i is odd and $T = V^*$ if i is even. By recurrence, we have to prove that in the diagram below, if the square on the left is a commuting square, then the square on the right is obtained by basic construction from the one on the left.

$$\begin{array}{ccccc} A^i \otimes \mathcal{L}(W) & \subset & A^i \otimes \mathcal{L}(T) \otimes \mathcal{L}(W) & \subset & A^i \otimes \mathcal{L}(T) \otimes \mathcal{L}(T^*) \otimes \mathcal{L}(W) \\ \cup & & \cup & & \cup \\ ad(U^i)A^i & \subset & ad(U^{i+1})(A^i \otimes \mathcal{L}(T)) & \subset & ad(U^{i+2})(A^i \otimes \mathcal{L}(T) \otimes \mathcal{L}(T^*)) \end{array}$$

Here $T = V$ or V^* depending on the parity of i . Three of the algebras in the square on the right are the good ones, and the remaining one is the subalgebra X of $A^i \otimes \mathcal{L}(T) \otimes \mathcal{L}(T^*) \otimes \mathcal{L}(W)$ generated by the image of $ad(U^{i+1})(A^i \otimes \mathcal{L}(T))$ and by the Jones projection e_{23}^T . Let v be equal to u if i is even and equal to $\hat{u} = (t \otimes id)u^{-1}$ otherwise. Then $U^{i+1} = U_{12}^i v_{23}$ and $U^{i+2} = U_{14}^i v_{24} \hat{v}_{34}$. By the Lemma 2.1 v satisfies (\star) , and it is easy to see that this implies that $ad(v_{24} \hat{v}_{34})e_{23}^T = e_{23}^T$. By applying $ad(U_{14}^i)$ we get

$$ad(U^{i+2})e_{23}^T = ad(U_{14}^i v_{24} \hat{v}_{34})e_{23}^T = e_{23}^T$$

On the other hand the image of the embedding

$$ad(U_{13}^i v_{23})(A^i \otimes \mathcal{L}(T)) \subset A^i \otimes \mathcal{L}(T) \otimes \mathcal{L}(T^*) \otimes \mathcal{L}(W)$$

is $ad(U_{14}^i v_{24})(A^i \otimes \mathcal{L}(T))$, which is equal to $ad(U^{i+2})(A^i \otimes \mathcal{L}(T))$. By combining these two results we get that $ad((U^{i+2})^{-1})X$ is the algebra generated by $A^i \otimes \mathcal{L}(T)$ and by e_{23}^T . But this algebra is the basic construction in the upper line, so it is equal to $A^i \otimes \mathcal{L}(T) \otimes \mathcal{L}(T^*)$ as desired.

Step II. Fix $0 \leq j \leq i$ and consider the algebra $A = \mathcal{L}(\dots \otimes V \otimes V^* \otimes V \otimes \dots)$ such that $A^j \otimes A = A^i$. That is, the product has $i - j$ terms and begins with V if j is even and with V^* otherwise. Let also $U \in A \otimes \mathcal{L}(W)$ be such that $U^j \otimes U = U^i$. That is,

$$U = v_{1,i-j+1} \hat{v}_{2,i-j+1} v_{3,i-j+1} \hat{v}_{4,i-j+1} \dots$$

where the tensor product has $j - i$ terms and $v = u$ if j is even and $v = \hat{u}$ if j is odd. The result in the first step tells us that the algebra of $L(\square_u)$ which sits on the $(i, -j)$ position is

$$D = (A^j \otimes 1_A \otimes \mathcal{L}(W))' \cap ad(U^i)(A^j \otimes 1_{\mathcal{L}(W)})$$

The commutant of $A^j \otimes 1_A \otimes \mathcal{L}(W)$ is $1_{A^j} \otimes A \otimes 1_{\mathcal{L}(W)}$. By using this and by applying $ad(U^j)^{-1}$ we get an isomorphism

$$D \simeq D' := (A \otimes 1_{\mathcal{L}(W)}) \cap ad(U)(A \otimes 1_{\mathcal{L}(W)})$$

We prove that $x \mapsto x \otimes 1_{\mathcal{L}(W^*)}$ induces an isomorphism

$$D' \simeq (A \otimes 1_{\mathcal{L}(W)} \otimes 1_{\mathcal{L}(W^*)}) \cap (U_{12}((id \otimes t)U^{-1})_{13})'$$

Indeed, if $T \in A$ then $T \otimes 1_{\mathcal{L}(W)} \otimes 1_{\mathcal{L}(W^*)}$ is in algebra on the right iff

$$(T \otimes 1_{\mathcal{L}(W)} \otimes 1_{\mathcal{L}(W^*)})U_{12}((id \otimes t)U^{-1})_{13} = U_{12}((id \otimes t)U^{-1})_{13}(T \otimes 1_{\mathcal{L}(W)} \otimes 1_{\mathcal{L}(W^*)})$$

By the Lemma 2.1 $(id \otimes t)(U^{-1}) = ((id \otimes t)U)^{-1}$, so this is equivalent to

$$(U^{-1}(T \otimes 1_{\mathcal{L}(W)})U)_{12} = (((id \otimes t)U^{-1})(T \otimes 1_{\mathcal{L}(W)}((id \otimes t)U))_{13}$$

It is easy to see that the equality $X_{12} = Y_{13}$ in $A \otimes \mathcal{L}(W) \otimes \mathcal{L}(W^*)$ is equivalent to the existence of $S \in A$ such that $X = S \otimes 1_{\mathcal{L}(W)}$ and $Y = S \otimes 1_{\mathcal{L}(W^*)}$. Thus our condition on T is equivalent to the existence of $S \in A$ such that:

$$(T \otimes 1_{\mathcal{L}(W)})U = U(S \otimes 1_{\mathcal{L}(W)}), (T \otimes 1_{\mathcal{L}(W^*)})((id \otimes t)U) = ((id \otimes t)U)(S \otimes 1_{\mathcal{L}(W^*)})$$

The second condition could be obtained by applying $(id \otimes t)$ to the first one. Thus the only condition on T remains the first one, which is exactly $T \in D'$.

Step III. If S, T are linear spaces and $w \in \mathcal{L}(S) \otimes \mathcal{L}(T)$ is invertible we use the notation $\bar{w} = (id \otimes t)w^{-1}$. The Step II gives an isomorphism

$$D' \simeq End(U_{12}((id \otimes t)U^{-1})_{13})$$

where the *End* sign is in the sense of the second section. The element $U_{12}((id \otimes t)U^{-1})_{13}$ is equal to the product

$$(v_{1,i-j+1}\hat{v}_{2,i-j+1}v_{3,i-j+1}\hat{v}_{4,i-j+1}\dots)(\bar{v}_{1,i-j+2}\bar{\hat{v}}_{2,i-j+2}\bar{v}_{3,i-j+2}\bar{\hat{v}}_{4,i-j+2}\dots)$$

By rearranging the terms we get that this is equal to

$$(v_{1,i-j+1}\bar{v}_{1,i-j+2})(\hat{v}_{2,i-j+1}\widehat{\bar{v}}_{2,i-j+2})(v_{3,i-j+1}\bar{v}_{3,i-j+2})\dots$$

If $v' := v_{12}\bar{v}_{13}$ then $\widehat{v}' = \hat{v}_{12}\widehat{\bar{v}}_{13}$, so we get that

$$U_{12}((id \otimes t)U^{-1})_{13} = v' \otimes \widehat{v}' \otimes v' \otimes \widehat{v}' \otimes \dots$$

($i - j$ terms, and where the sign \otimes is in the sense of the second section). Summing up, we get that D is isomorphic to the $(i, -j)$ algebra of the lattice $L(u')$, where $u' = u_{12}\bar{u}_{13} = u_{12}((id \otimes t)u^{-1})_{13}$. It is easy to see that these isomorphisms commute with the inclusions, so $L(\square_u) = L(u')$. The Prop. 3.1 applies and gives the result. \square

Examples. Let V be a finite dimensional linear space and let g_1, \dots, g_n be elements of $\mathbf{GL}(V)$.

(i) The element $u = \sum e_{ii} \otimes g_i$ satisfies (\star) . Let G be the subgroup of $\mathbf{PGL}(V)$ generated by $\tilde{g}_1, \dots, \tilde{g}_n$, where $g \mapsto \tilde{g}$ is the projection $\mathbf{GL}(V) \rightarrow \mathbf{PGL}(V)$, and consider the corepresentation

$$v = \sum e_{ii} \otimes g_i \in M_n(k) \otimes kG$$

Then $L(\square_u) = L(v)$. This follows from the Th. 4.1, from the equality $u_{12}((id \otimes t)u^{-1})_{13} = \sum e_{ii} \otimes g_i \otimes (g_i^{-1})^t$, from the Prop. 2.2 (i), and by identifying the subgroup of $\mathbf{GL}(V \otimes V^*)$ generated by the $g_i \otimes (g_i^{-1})^t$'s with the subgroup of $\mathbf{PGL}(V)$ generated by the \tilde{g}_i 's.

(ii) The element $u = \sum g_i \otimes e_{ii}$ satisfies (\star) . Let $G \subset \mathbf{GL}(V)$ be the group generated by $g_2g_1^{-1}, \dots, g_n g_1^{-1}$ and let v be the fundamental corepresentation of $k[G]$. Then $L(\square_u) = L(v)$. This follows from the Th. 4.1, from the equality $u_{12}((id \otimes t)u^{-1})_{13} = \sum g_i g_j^{-1} \otimes e_{ii} \otimes e_{jj}$, and from the Prop. 2.2 (ii).

These results were obtained in [BHJ]; the reader may find in there the interpretation of the lattice $L(v)$ and of its principal graphs for these special kinds of corepresentations v .

5 Hopf \mathbf{C}^* -algebras and (twisted) biunitaries

In this section $k = \mathbf{C}$. Recall that an involution of a \mathbf{C} -algebra A is a unital antilinear antimultiplicative map $*$: $A \rightarrow A$ such that $*^2 = id$. A \mathbf{C}^* -norm

on A is a norm making A into a normed algebra, and such that $\|a^*a\| = \|a\|^2$ for any $a \in A$. The completion of A with respect to the biggest \mathbf{C}^* -norm (if such a norm exists) is a \mathbf{C}^* -algebra called enveloping \mathbf{C}^* -algebra and denoted here \bar{A} . An involution of a Hopf \mathbf{C} -algebra H is an involution $*$ of the subjacent algebra such that for any $a \in H$ the following formulas hold

$$\Delta(a^*) = \Delta(a)^*, \varepsilon(a^*) = \overline{\varepsilon(a)}, S(S(a^*)^*) = a$$

If V is a Hilbert space, a corepresentation $v \in \mathcal{L}(V) \otimes H$ which is a unitary element of the $*$ -algebra $\mathcal{L}(V) \otimes H$ is said to be unitary corepresentation.

In [W1], [W2] it was developed a theory for pairs (A, v) consisting of a unital \mathbf{C}^* -algebra and a unitary $v \in \mathcal{L}(V) \otimes A$, where V is some finite dimensional Hilbert space, subject to the following conditions, to be referred as “Woronowicz’ axioms”.

- the coefficients of v generate A as a dense $*$ -subalgebra (called A_s).
- there exists a \mathbf{C}^* -morphism $\delta : A \rightarrow A \otimes_{\min} A$ such that $(id \otimes \delta)v = v_{12}v_{13}$.
- there exists a linear antimultiplicative map $\kappa : A_s \rightarrow A_s$ such that $(id \otimes \kappa)v = v^{-1}$ and such that $\kappa(\kappa(a^*)^*) = a$ for any $a \in A_s$.

These conditions imply that A_s has a canonical structure of Hopf $*$ -algebra, and that v is a unitary corepresentation (see [W1]). While we are interested only in such pairs (A, v) we recall that with a suitable choice of arrows, the category \mathcal{C} of inductive limits of such objects is called the category of “Woronowicz algebras”, or “unital Hopf \mathbf{C}^* -algebras”. By reversing the arrows of \mathcal{C} we get the category of “compact quantum groups” and by reversing them once again (sic!) we get the category of “discrete quantum groups”.

Definition 5.1 *A twisted biunitary is a unitary $u \in \mathcal{L}(V) \otimes \mathcal{L}(W)$, where V and W are finite dimensional Hilbert spaces, which satisfies the equation*

$$((Q^t)^{-2} \otimes id)((t \otimes id)u^{-1})((Q^t)^2 \otimes id) = ((t \otimes id)u)^{-1} \quad (\star_Q)$$

for some positive operator $Q \in \mathcal{L}(V)$.

For $Q = id$ the condition (\star_Q) is exactly (\star) , and is equivalent to the fact that \square_u is a commuting square in the sense of subfactor theory; in this case u is said to be a biunitary (see [KSV]). By taking suitable bases the twisted biunitarity condition has the following equivalent formulation. A unitary

$u = (u_{by}^{ax}) \in M_m(\mathbf{C}) \otimes M_n(\mathbf{C})$ is said to be a twisted biunitary if there exist positive real numbers q_1, \dots, q_m such that

$$\sum_{b,x} q_b \bar{u}_{by}^{ax} u_{cx}^{bz} = q_a \delta_{a,c} \delta_{y,z}, \quad \forall a, c, y, z$$

Proposition 5.1 *If (A, v) satisfies Woronowicz' axioms and $\pi : A \rightarrow \mathcal{L}(W)$ is a $*$ -representation then $(id \otimes \pi)v$ is a twisted biunitary. Any twisted biunitary arises in this way.*

We will need the following easy lemma, which gives in particular some more equivalent formulations of the twisted biunitarity condition.

Lemma 5.1 *If V is a finite dimensional Hilbert space, $Q \in \mathcal{L}(V)$ is a positive operator, A is a \mathbf{C}^* -algebra, and $u \in \mathcal{L}(V) \otimes A$ is a unitary then the following conditions are equivalent:*

- (i) $(Q^t \otimes id)((t \otimes id)u^{-1})((Q^t)^{-1} \otimes id)$ is unitary.
- (ii) $((Q^t)^{-1} \otimes id)((t \otimes id)u)(Q^t \otimes id)$ is unitary.
- (iii) $((Q^t)^{-2} \otimes id)((t \otimes id)u^{-1})((Q^t)^2 \otimes id) = ((t \otimes id)u)^{-1}$. \square

Proof of the Prop. 5.1. Consider the family of characters $(f_z)_{z \in \mathbf{C}} : A_s \rightarrow \mathbf{C}$ introduced in [W1] and let $Q = (id \otimes f_{\frac{1}{2}})v$. Then $Q > 0$ and

$$(Q^t \otimes id)((t \otimes id)v^{-1})((Q^t)^{-1} \otimes id) = (t \otimes j)v$$

where $j : A_s \rightarrow A_s$ is the linear map $x \mapsto f_{\frac{1}{2}} * \kappa(x) * f_{-\frac{1}{2}}$ and $*$ is the convolution over A_s (cf. the formulas in the 5th section of [W1]). These formulas show also that j is an antimorphism of $*$ -algebras, so $t \otimes j$ is also an antimorphism of $*$ -algebras, so it maps unitaries to unitaries. By applying $id \otimes \pi$ to $(t \otimes j)v$ we get the result. See the proof of the Lemma 1.5 in [B2] for more details.

Conversely, let $u \in \mathcal{L}(V) \otimes \mathcal{L}(W)$ be a twisted biunitary. Choose an orthonormal basis in V consisting of eigenvectors of Q , so that $V = \mathbf{C}^n$ and $Q \in M_n(\mathbf{C})$ is diagonal and positive. Consider the universal \mathbf{C}^* -algebra $A_u(Q)$ generated by the coefficients of a unitary $n \times n$ matrix w such that $Q\bar{w}Q^{-1}$ (or, equivalently, $Q^{-1}w^tQ$) is also unitary. Then $(A_u(Q), w)$ satisfies Woronowicz' axioms (see [B1]) and as u and $Q^{-1}u^tQ$ are unitaries (cf. the

Lemma 5.1) we get a \mathbf{C}^* -morphism $f : A_u(Q) \rightarrow \mathcal{L}(W)$ such that $(id \otimes f)w = u$. \square

With the above notations $(A_u(F)_s, w, f)$ is a model for u . Thus the Th. 1.1 applies to any twisted biunitary. In fact the conditions (iv) of the Th. 1.1 are also easy to verify - one gets by recurrence that for every $n \geq 0$:

$$u_{2n} = (Q^{2n} \otimes id)u(Q^{-2n} \otimes id), \quad u_{2n+1} = ((Q^{-2n})^t \otimes id)u_1((Q^{2n})^t \otimes id)$$

Theorem 5.1 *Let $u \in \mathcal{L}(V) \otimes \mathcal{L}(W)$ be a twisted biunitary. If (H, v, π) is the minimal model for u then there exists an involution on H such that:*

- (i) v is a unitary representation and π is a $*$ -representation.
- (ii) H has a (biggest) \mathbf{C}^* -norm.
- (iii) (\bar{H}, v) satisfies Woronowicz' axioms.

Moreover, this involution of H is the unique one such that v is unitary.

Proof. Choose a basis such that $V = \mathbf{C}^n$ and $Q \in M_n(\mathbf{C})$ is positive and diagonal. Consider the model $(A_u(Q)_s, w, f)$ constructed in the proof of the Prop. 5.1, and let $q : A_u(Q)_s \rightarrow H$ be the corresponding Hopf algebra morphism. The following equality holds in $M_n(\mathbf{C}) \otimes A_u(Q)$

$$(id \otimes S^2)w = (t \otimes id)((t \otimes id)w^{-1})^{-1} = (Q^2 \otimes id)w(Q^{-2} \otimes id)$$

By applying $id \otimes q$ to this equality we get

$$(id \otimes S^2)v = (Q^2 \otimes id)v(Q^{-2} \otimes id)$$

In particular \mathcal{C}_v is stable under S^2 , and by left-faithfulness we obtain that H is generated as an algebra by \mathcal{C}_v and $S(\mathcal{C}_v)$.

Let us prove firstly the unicity part. Such an involution $*$ has to satisfy $v^* = v^{-1} = (id \otimes S)v$, so it is uniquely determined on \mathcal{C}_v . As $*^2 = id$ the restriction of $*$ to $S(\mathcal{C}_v)$ is the inverse of the restriction of $*$ to \mathcal{C}_v , so it is also uniquely determined. As \mathcal{C}_v and $S(\mathcal{C}_v)$ generate H as an algebra, $*$ extends uniquely by antimultiplicativity.

We denote as usual by $*$ the involutions of $\mathcal{L}(V)$ and $\mathcal{L}(W)$. Let K be complex conjugate of H and denote by $j : H \rightarrow K$ the canonical antilinear isomorphism. Then $*\pi j^{-1}$ is a representation of K and $(* \otimes jS)v$ is a corepresentation of K . As $u = (id \otimes \pi)v$ and $v^{-1} = (id \otimes S)v$ we get that

$$u = (u^{-1})^* = (* \otimes *) (id \otimes \pi S)v = (id \otimes *\pi j^{-1})(* \otimes jS)v$$

so $(K, (* \otimes jS)v, * \pi j^{-1})$ is a model for u . We have $S^2(\mathcal{C}_v) = \mathcal{C}_v$, so this model is left-faithful and by the universality property of the minimal model we get a Hopf algebra morphism $p : K \rightarrow H$ such that $\pi p = * \pi j^{-1}$ and $(* \otimes pjS)v = v$. With $r = pj$ these formulas are

$$\pi r = * \pi, \quad (* \otimes rS)v = v \quad (\dagger)$$

We prove that r is an involution of H . The facts that r is unital, antilinear, antimultiplicative, anticounital and comultiplicative follow from the corresponding properties of p and j . From $(id \otimes S)v = v^{-1}$ and (\dagger) we get that $(* \otimes r)v^{-1} = v$. As $* \otimes r$ is an antiautomorphism of the algebra $M_n(\mathbf{C}) \otimes H$, this shows also that $(* \otimes r)v = v^{-1}$. By applying $* \otimes r$ to these two formulas we get that $(id \otimes r^2)v = v$ and $(id \otimes r^2)v^{-1} = v^{-1}$. Thus $r^2 = id$ on both \mathcal{C}_v and $S(\mathcal{C}_v)$, and as these spaces generate H as an algebra we get that $r^2 = id$. The proof of $(rS)^2 = id$ is similar: we have $(* \otimes rS)v = v$, and as $* \otimes rS$ is an antiautomorphism we get also that $(* \otimes rS)v^{-1} = v^{-1}$. By applying $* \otimes rS$ we get that $(rS)^2 = id$ on both \mathcal{C}_v and $S(\mathcal{C}_v)$, and this implies that $(rS)^2 = id$.

Thus r is an involution of H ; from now on we denote it by $*$. The point (i) is clear from the above formulas (\dagger) . As v is unitary, the norms of its coefficients are less than one for every \mathbf{C}^* -seminorm on H . By left-faithfulness these coefficients generate H as a $*$ -algebra, so there exists a maximal \mathbf{C}^* -seminorm $\| \cdot \|$ on H . The point (ii) is equivalent to the fact that this is a norm. If A denotes the completion of the separation of H by $\| \cdot \|$, this is the same as proving that the canonical map $i : H \rightarrow A$ is injective.

Consider the $*$ -morphism $(i \otimes i)\Delta : H \rightarrow A \otimes_{min} A$. It has values in a \mathbf{C}^* -algebra, so it extends to a \mathbf{C}^* -morphism $\delta : A \rightarrow A \otimes_{min} A$ which satisfies $(id \otimes \delta)V = V_{12}V_{13}$, where $V = (id \otimes i)v$. On the other hand from the fact that w^t is invertible in $M_n(A_u(Q))$ we get that v^t is invertible in $M_n(H)$, so V^t is invertible in $M_n(A)$. Summing up, the pair (A, V) satisfies the first two axioms of Woronowicz and is such that V^t is invertible; by [W3] we get that (A, V) satisfies all Woronowicz' axioms. In particular we get a Hopf algebra structure on $i(H) = A_s$ such that V is a corepresentation of it.

Consider the $*$ -morphism $\pi : H \rightarrow \mathcal{L}(W)$. It has values in a \mathbf{C}^* -algebra, so it extends to a \mathbf{C}^* -morphism $\bar{\pi} : A \rightarrow \mathcal{L}(W)$. Thus $(A_s, V, \bar{\pi}|_{A_s})$ is a model for u . By the universality property of (H, v, π) we get a section for i , so i is injective. This finishes the proof of (ii) and (iii). \square

Example. Any finite dimensional Hopf \mathbf{C}^* -algebra is subadjacent to a minimal model for a biunitary. Indeed, if A is finite dimensional, then the square of its antipode is the identity (see [W1]), and (A, v, π) is a bi-faithful model for $u = (id \otimes \pi)v$, where π is the regular representation, and v is the coregular corepresentation.

Example. Let V be a finite dimensional Hilbert space and let g_1, \dots, g_n be elements of $\mathbf{U}(V)$. Let $G \subset \mathbf{U}(V)$ be the group generated by g_1, \dots, g_n .

(i) The element $u = \sum e_{ii} \otimes g_i$ is biunitary. Consider the minimal model (kG, w, ν) for u (cf. Prop 2.2 (i)). Then the involution constructed in the Th. 5.1 is given by $\sum c_i g_i \mapsto \sum \bar{c}_i g_i$, and the \mathbf{C}^* -algebra \overline{kG} is the (full) group \mathbf{C}^* -algebra $\mathbf{C}^*(G)$.

(ii) The element $u = \sum g_i \otimes e_{ii}$ is a biunitary. Consider the minimal model $(k[G], v, \pi)$ for u (cf. Prop 2.2 (ii)). Then the involution constructed in the Th. 5.1 is given by $f \mapsto (g \mapsto \bar{f}(g))$, and the classical Peter-Weyl theory shows that the \mathbf{C}^* -algebra $\overline{k[G]}$ is the algebra $C(\bar{G})$ of continuous functions on the closure $\bar{G} \subset \mathbf{U}(V)$ of G .

By combining these results with the Th. 4.1 one gets the descriptions from [KSV] of the standard invariants and of the principal graphs of the corresponding subfactors.

References

- [A] E. Abe, “Hopf algebras”, Cambridge University Press (1977)
- [BHJ] R. Bacher, P. de la Harpe and V.F.R. Jones, Carrés commutatifs et invariants de structures combinatoires, *C.R. Acad. Sci. Paris* **320** (1995), 1049-1054.
- [B1] T. Banica, Le groupe quantique compact libre $U(n)$, *Commun. Math. Phys.* **190** (1997), 143-172.
- [B2] T. Banica, Representations of compact quantum groups and subfactors, <http://www.math.jussieu.fr/~banica/subf.dvi> (1997)
- [ES] M. Enock and J.M. Schwartz, “Kac algebras and duality of locally compact groups”, Springer-Verlag, Berlin (1992)

- [GHJ] F.M. Goodman, P. de la Harpe and V.F.R. Jones, “Coxeter graphs and towers of algebras”, Publ. M.S.R.I. 62, Springer (1989)
- [J] V.F.R. Jones, On knot invariants related to some statistical mechanical models, *Pacific J. of Math.* **137** (1989), 311-334.
- [JS] V.F.R. Jones and V.S. Sunder, “Introduction to subfactors”, LMS Lecture Notes 234, Cambridge University Press (1997)
- [KSV] U. Krishnan, V.S. Sunder and C. Varughese, On some subfactors of integer index arising from vertex models, *J. Funct. Analysis* **140** (1996), 449-471.
- [O] A. Ocneanu, “Quantum symmetry, differential geometry of finite graphs, and classification of subfactors”, Lecture Notes recorded by Y. Kawahigashi, Univ. of Tokyo Seminary Notes (1990)
- [P] S. Popa, An axiomatization of the lattice of higher relative commutants of a subfactor, *Invent. Math.* **120** (1995), 427-445.
- [W1] S.L. Woronowicz, Compact matrix pseudogroups. *Commun. Math. Phys.* **111** (1987), 613-665.
- [W2] S.L. Woronowicz, Tannaka-Krein duality for compact matrix pseudogroups. Twisted $SU(n)$ groups, *Invent. Math.* **93** (1988), 35-76.
- [W3] S.L. Woronowicz, A remark on compact matrix quantum groups, *Lett. Math. Phys.* **21** (1991), 35-39.