Von Neumann algebras

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Abstract. This is an introduction to the theory of von Neumann algebras. These algebras, which are related to quantum mechanics, appear as algebras of bounded linear operators $T : H \to H$ on a complex Hilbert space H, which are stable under the operator involution $T \to T^*$, and are closed under the weak operator topology, making the evaluation maps $T \to Tx$ continuous. We first discuss the basics of the theory, following Murray and von Neumann, including the reduction theorem, stating that any such algebra decomposes as $A = \int_X A_x dx$, with the factors A_x being von Neumann algebras with trivial center. Then we get into the study of factors, following again Murray and von Neumann, and Connes, and also Haagerup, Jones, Voiculescu and others.

Preface

The algebras formed by the bounded linear operators $T : H \to H$ on a complex Hilbert space H were first studied, mathematically, by von Neumann in the 1930s and 1940s, with a few papers alone, then a series of fundamental papers with Murray, laying the foundations of the theory, and then with a last paper, again alone.

Von Neumann was interested in understanding quantum mechanics, which was a new discipline at that time, developed just a bit earlier, in the 1920s, by Heisenberg, Schrödinger, Dirac and others. Indeed, one interesting thing coming from the work of Heisenberg and the others was the fact that the states of a quantum mechanical system are described by the vectors of a complex Hilbert space H, and the observables of the system are described by certain linear operators, possibly unbounded, $T : H \to H$.

Of particular interest to von Neumann were the algebras of bounded operators $A \subset B(H)$ appearing as commutants $A = \{T\}'$ of such operators T coming from quantum mechanics. His first result, the bicommutant theorem, states that the algebras $A \subset B(H)$ appearing in this way can be abstractly characterized as being the operator algebras $A \subset B(H)$ which are stable under the operator involution $T \to T^*$, and are closed under the weak operator topology, making the evaluation maps $T \to Tx$ continuous.

Many things have happened since, first with the above-mentioned fundamental work of Murray and von Neumann, done mostly in the 1930s, then with some fundamental work by Connes too, later in the early 1970s, complementing the original work of Murray and von Neumann, and then with some further key work, again by Connes, and then by Haagerup, Jones, Voiculescu and others, in the late 1970s, and all over the 1980s.

This book is an introduction to the von Neumann algebras, with the aim of introducing the reader to the above-mentioned material, all fundamental work, of rather advanced level, done in first 50 years, 1930-1980. The book is organized in 4 parts, as follows:

(1) We first discuss the basics, namely operator theory, spectral theorem, bicommutant theorem, and many other fundamental things, following Murray and von Neumann.

(2) Then we discuss the reduction theorem, stating that any von Neumann algebra decomposes as $A = \int_X A_x dx$, with the factors A_x being algebras with trivial center.

(3) We then get into the study of factors, again following Murray and von Neumann, and Connes, and also Haagerup, Jones, Voiculescu and others.

(4) Finally, we discuss the classification of hyperfinite factors, the toughest result of them all, due to Murray and von Neumann, and Connes, and Haagerup.

In the hope that you will find this book useful, and I should mention here, as a further piece of advertisement, that although all this material is from 1930-1980, things are quite difficult, to the point that there is only a handful of operator algebra experts knowing all this stuff, and with most of them being actually retired or dead. So, very modern theory that we will be explaining here, read this, and you will know things that others don't.

Many thanks to the various books and papers on the subject, and also to the above-mentioned previous generation of operator algebra experts, that I had the chance to meet and discuss with, earlier in my career, it is always good to have living proof, around you, when learning, that all these complicated things are actually understandable. More recently, thanks as well to my cats, for exactly the same reason.

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Part I

Operator algebras

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Having control, a principal obsession
Rivalry and retribution
Death: the only solution

CHAPTER 1

Linear operators

1a. Linear operators

We would like to first discuss the theory of linear operators $T : H \to H$ over a complex Hilbert space H, usually taken separable. Let us start with a basic result, as follows:

THEOREM 1.1. *Given a Hilbert space H, consider the linear operators $T : H \to H$, and for each such operator define its norm by the following formula:*

$$||T|| = \sup_{||x||=1} ||Tx||$$

The operators which are bounded, $||T|| < \infty$, form then a complex algebra $B(H)$, which is complete with respect to $||.||$. When H comes with a basis $\{e_i\}_{i\in I}$, we have

$$B(H) \subset \mathcal{L}(H) \subset M_I(\mathbb{C})$$

where $\mathcal{L}(H)$ is the algebra of all linear operators $T : H \to H$, and $\mathcal{L}(H) \subset M_I(\mathbb{C})$ is the correspondence $T \to M$ obtained via the usual linear algebra formulae, namely:

$$T(x) = Mx \quad , \quad M_{ij} =< Te_j, e_i >$$

In infinite dimensions, none of the above two inclusions is an equality.

PROOF. This is something straightforward, the idea being as follows:

(1) The fact that we have indeed an algebra, satisfying the product condition in the statement, follows from the following estimates, which are all elementary:

$$||S + T|| \leq ||S|| + ||T|| \quad , \quad ||\lambda T|| = |\lambda| \cdot ||T|| \quad , \quad ||ST|| \leq ||S|| \cdot ||T||$$

(2) Regarding now the completness assertion, if $\{T_n\} \subset B(H)$ is Cauchy then $\{T_n x\}$ is Cauchy for any $x \in H$, so we can define the limit $T = \lim_{n\to\infty} T_n$ by setting:

$$Tx = \lim_{n\to\infty} T_n x$$

Let us first check that the application $x \to Tx$ is linear. We have:

$$\begin{aligned} T(x+y) &= \lim_{n\to\infty} T_n(x+y) \\ &= \lim_{n\to\infty} T_n(x)+T_n(y) \\ &= \lim_{n\to\infty} T_n(x)+\lim_{n\to\infty} T_n(y) \\ &= T(x)+T(y) \end{aligned}$$

Similarly, we have $T(\lambda x) = \lambda T(x)$, and we conclude that $T \in \mathcal{L}(H)$.

(3) With this done, it remains to prove now that we have $T \in B(H)$, and that $T_n \to T$ in norm. For this purpose, observe that we have:

$$\begin{aligned} ||T_n - T_m|| \leq \varepsilon\ ,\ \forall n, m \geq N &\implies ||T_n x - T_m x|| \leq \varepsilon\ ,\ \forall ||x|| = 1\ ,\ \forall n, m \geq N \\ &\implies ||T_n x - Tx|| \leq \varepsilon\ ,\ \forall ||x|| = 1\ ,\ \forall n \geq N \\ &\implies ||T_N x - Tx|| \leq \varepsilon\ ,\ \forall ||x|| = 1 \\ &\implies ||T_N - T|| \leq \varepsilon \end{aligned}$$

But this gives both $T \in B(H)$, and $T_N \to T$ in norm, and we are done.

(4) Regarding the embeddings, the correspondence $T \to M$ in the statement is indeed linear, and its kernel is $\{0\}$, so we have indeed an embedding as follows, as claimed:

$$\mathcal{L}(H) \subset M_I(\mathbb{C})$$

In finite dimensions we have an isomorphism, because any $M \in M_N(\mathbb{C})$ determines an operator $T : \mathbb{C}^N \to \mathbb{C}^N$, given by $< Te_j, e_i >= M_{ij}$. However, in infinite dimensions, we have matrices not producing operators, as for instance the all-one matrix.

(5) As for the examples of linear operators which are not bounded, these are more complicated, coming from logic, and we will not really need them in what follows. □

As a second basic result regarding the operators, we will need:

Theorem 1.2. *Each operator $T \in B(H)$ has an adjoint $T^* \in B(H)$, given by:*

$$< Tx, y >=< x, T^* y >$$

The operation $T \to T^*$ is antilinear, antimultiplicative, involutive, and satisfies:

$$||T|| = ||T^*|| \quad , \quad ||TT^*|| = ||T||^2$$

When H comes with a basis $\{e_i\}_{i \in I}$, the operation $T \to T^*$ corresponds to

$$(M^*)_{ij} = \overline{M}_{ji}$$

at the level of the associated matrices $M \in M_I(\mathbb{C})$.

Proof. This is standard too, and can be proved in 3 steps, as follows:

(1) The existence of the adjoint operator T^*, given by the formula in the statement, comes from the fact that the function $\varphi(x) =< Tx, y >$ being a linear map $H \to \mathbb{C}$, we must have a formula as follows, for a certain vector $T^*y \in H$:

$$\varphi(x) =< x, T^*y >$$

Moreover, since this vector is unique, T^* is unique too, and we have as well:

$$(S+T)^* = S^* + T^* \quad , \quad (\lambda T)^* = \bar{\lambda} T^* \quad , \quad (ST)^* = T^*S^* \quad , \quad (T^*)^* = T$$

Observe also that we have indeed $T^* \in B(H)$, because:

$$\begin{aligned} ||T|| &= \sup_{||x||=1} \sup_{||y||=1} < Tx, y > \\ &= \sup_{||y||=1} \sup_{||x||=1} < x, T^*y > \\ &= ||T^*|| \end{aligned}$$

(2) Regarding now $||TT^*|| = ||T||^2$, which is a key formula, observe that we have:

$$||TT^*|| \leq ||T|| \cdot ||T^*|| = ||T||^2$$

On the other hand, we have as well the following estimate:

$$\begin{aligned} ||T||^2 &= \sup_{||x||=1} | < Tx, Tx > | \\ &= \sup_{||x||=1} | < x, T^*Tx > | \\ &\leq ||T^*T|| \end{aligned}$$

By replacing $T \to T^*$ we obtain from this $||T||^2 \leq ||TT^*||$, as desired.

(3) Finally, when H comes with a basis, the formula $< Tx, y >=< x, T^*y >$ applied with $x = e_i$, $y = e_j$ translates into the formula $(M^*)_{ij} = \overline{M}_{ji}$, as desired. □

Let us discuss now the diagonalization problem for the operators $T \in B(H)$, in analogy with the diagonalization problem for the usual matrices $A \in M_N(\mathbb{C})$. As a first observation, we can talk about eigenvalues and eigenvectors, as follows:

Definition 1.3. *Given an operator $T \in B(H)$, assuming that we have*

$$Tx = \lambda x$$

we say that $x \in H$ is an eigenvector of T, with eigenvalue $\lambda \in \mathbb{C}$.

We know many things about eigenvalues and eigenvectors, in the finite dimensional case. However, most of these will not extend to the infinite dimensional case, or at least not extend in a straightforward way, due to a number of reasons:

(1) Most of basic linear algebra is based on the fact that $Tx = \lambda x$ is equivalent to $(T-\lambda)x = 0$, so that λ is an eigenvalue when $T - \lambda$ is not invertible. In the infinite dimensional setting $T - \lambda$ might be injective and not surjective, or vice versa, or invertible with $(T-\lambda)^{-1}$ not bounded, and so on.

(2) Also, in linear algebra $T - \lambda$ is not invertible when $\det(T-\lambda) = 0$, and with this leading to most of the advanced results about eigenvalues and eigenvectors. In infinite dimensions, however, it is impossible to construct a determinant function $\det : B(H) \to \mathbb{C}$, and this even for the diagonal operators on $l^2(\mathbb{N})$.

Summarizing, we are in trouble. Forgetting about (2), which obviously leads nowhere, let us focus on the difficulties in (1). In order to cut short the discussion there, regarding the various properties of $T-\lambda$, we can just say that $T-\lambda$ is either invertible with bounded inverse, the "good case", or not. We are led in this way to the following definition:

Definition 1.4. *The spectrum of an operator $T \in B(H)$ is the set*

$$\sigma(T) = \left\{\lambda \in \mathbb{C}\Big| T - \lambda \notin B(H)^{-1}\right\}$$

where $B(H)^{-1} \subset B(H)$ is the set of invertible operators.

As a basic example, in the finite dimensional case, $H = \mathbb{C}^N$, the spectrum of a usual matrix $A \in M_N(\mathbb{C})$ is the collection of its eigenvalues, taken without multiplicities. We will see many other examples. In general, the spectrum has the following properties:

Proposition 1.5. *The spectrum of $T \in B(H)$ contains the eigenvalue set*

$$\varepsilon(T) = \left\{\lambda \in \mathbb{C}\Big| \ker(T - \lambda) \neq \{0\}\right\}$$

and $\varepsilon(T) \subset \sigma(T)$ is an equality in finite dimensions, but not in infinite dimensions.

Proof. We have several assertions here, the idea being as follows:

(1) First of all, the eigenvalue set is indeed the one in the statement, because $Tx = \lambda x$ tells us precisely that $T - \lambda$ must be not injective. The fact that we have $\varepsilon(T) \subset \sigma(T)$ is clear as well, because if $T - \lambda$ is not injective, it is not bijective.

(2) In finite dimensions we have $\varepsilon(T) = \sigma(T)$, because $T - \lambda$ is injective if and only if it is bijective, with the boundedness of the inverse being automatic.

(3) In infinite dimensions we can assume $H = l^2(\mathbb{N})$, and the shift operator $S(e_i) = e_{i+1}$ is injective but not surjective. Thus $0 \in \sigma(T) - \varepsilon(T)$. □

Philosophically, the best way of thinking at this is as follows: the numbers $\lambda \notin \sigma(T)$ are good, because we can invert $T - \lambda$, the numbers $\lambda \in \sigma(T) - \varepsilon(T)$ are bad, because so they are, and the eigenvalues $\lambda \in \varepsilon(T)$ are evil. Welcome to operator theory.

Let us develop now some general theory. As a first goal, we would like to prove that the spectra are non-empty. This is something quite tricky, the result being as follows:

THEOREM 1.6. *The spectrum of a bounded operator $T \in B(H)$ is:*

(1) *Compact.*
(2) *Contained in the disc $D_0(||T||)$.*
(3) *Non-empty.*

PROOF. This can be proved by using some complex analysis, as follows:

(1) In view of (2) below, it is enough to prove that $\sigma(T)$ is closed. But this follows from the following computation, with $|\varepsilon|$ being small:

$$\begin{aligned}\lambda \notin \sigma(T) &\implies T - \lambda \in B(H)^{-1} \\ &\implies T - \lambda - \varepsilon \in B(H)^{-1} \\ &\implies \lambda + \varepsilon \notin \sigma(T)\end{aligned}$$

(2) This follows indeed from the following computation:

$$\begin{aligned}\lambda > ||T|| &\implies \left|\left|\frac{T}{\lambda}\right|\right| < 1 \\ &\implies 1 - \frac{T}{\lambda} \in B(H)^{-1} \\ &\implies \lambda - T \in B(H)^{-1} \\ &\implies \lambda \notin \sigma(T)\end{aligned}$$

(3) Assume by contradiction $\sigma(T) = \emptyset$. Given a linear form $f \in B(H)^*$, consider the following map, which is well-defined, due to our assumption $\sigma(T) = \emptyset$:

$$\varphi : \mathbb{C} \to \mathbb{C} \quad , \quad \lambda \to f((T - \lambda)^{-1})$$

By using the fact that $T \to T^{-1}$ is differentiable, which is something elementary, we conclude that this map is differentiable, and so holomorphic. Also, we have:

$$\begin{aligned}\lambda \to \infty &\implies T - \lambda \to \infty \\ &\implies (T - \lambda)^{-1} \to 0 \\ &\implies f((T - \lambda))^{-1} \to 0\end{aligned}$$

Thus by the Liouville theorem we obtain $\varphi = 0$. But, in view of the definition of φ, this gives $(T - \lambda)^{-1} = 0$, which is a contradiction, as desired. □

Here is now a second basic result regarding the spectra, inspired from what happens in finite dimensions, for the usual complex matrices, and which shows that things do not necessarily extend without troubles to the infinite dimensional setting:

THEOREM 1.7. *We have the following formula, valid for any operators S, T:*

$$\sigma(ST) \cup \{0\} = \sigma(TS) \cup \{0\}$$

In finite dimensions we have $\sigma(ST) = \sigma(TS)$, but this fails in infinite dimensions.

PROOF. There are several assertions here, the idea being as follows:

(1) This is something that we know in finite dimensions, coming from the fact that the characteristic polynomials of the associated matrices A, B coincide:

$$P_{AB} = P_{BA}$$

Thus we obtain $\sigma(ST) = \sigma(TS)$ in this case, as claimed. Observe that this improves twice the general formula in the statement, first because we have no issues at 0, and second because what we obtain is actually an equality of sets with mutiplicities.

(2) In general now, let us first prove the main assertion, stating that $\sigma(ST), \sigma(TS)$ coincide outside 0. We first prove that we have the following implication:

$$1 \notin \sigma(ST) \implies 1 \notin \sigma(TS)$$

Assume indeed that $1 - ST$ is invertible, with inverse denoted R:

$$R = (1 - ST)^{-1}$$

We have then the following formulae, relating our variables R, S, T:

$$RST = STR = R - 1$$

By using $RST = R - 1$, we have the following computation:

$$\begin{aligned} (1 + TRS)(1 - TS) &= 1 + TRS - TS - TRSTS \\ &= 1 + TRS - TS - TRS + TS \\ &= 1 \end{aligned}$$

A similar computation, using $STR = R - 1$, shows that we have:

$$(1 - TS)(1 + TRS) = 1$$

Thus $1 - TS$ is invertible, with inverse $1 + TRS$, which proves our claim. Now by multiplying by scalars, we deduce from this that for any $\lambda \in \mathbb{C} - \{0\}$ we have:

$$\lambda \notin \sigma(ST) \implies \lambda \notin \sigma(TS)$$

But this leads to the conclusion in the statement.

(3) Regarding now the counterexample to the formula $\sigma(ST) = \sigma(TS)$, in general, let us take S to be the shift on $H = L^2(\mathbb{N})$, given by the following formula:

$$S(e_i) = e_{i+1}$$

As for T, we can take it to be the adjoint of S, and we have:

$$S^*S = 1 \implies 0 \notin \sigma(SS^*)$$

$$SS^* = Proj(e_0^\perp) \implies 0 \in \sigma(SS^*)$$

Thus, the spectra do not match on 0, and so we have our counterexample. □

1b. Spectral radius

Let us develop now some systematic theory for the computation of the spectra, based on what we know about the eigenvalues of the usual complex matrices. As a first result, which is well-known for the usual matrices, and extends well, we have:

THEOREM 1.8. *We have the "polynomial functional calculus" formula*

$$\sigma(P(T)) = P(\sigma(T))$$

valid for any polynomial $P \in \mathbb{C}[X]$, and any operator $T \in B(H)$.

PROOF. We pick a scalar $\lambda \in \mathbb{C}$, and we decompose the polynomial $P - \lambda$:

$$P(X) - \lambda = c(X - r_1) \ldots (X - r_n)$$

We have then the following equivalences:

$$\begin{aligned}
\lambda \notin \sigma(P(T)) &\iff P(T) - \lambda \in B(H)^{-1} \\
&\iff c(T - r_1) \ldots (T - r_n) \in B(H)^{-1} \\
&\iff T - r_1, \ldots, T - r_n \in B(H)^{-1} \\
&\iff r_1, \ldots, r_n \notin \sigma(T) \\
&\iff \lambda \notin P(\sigma(T))
\end{aligned}$$

Thus, we are led to the formula in the statement. □

The above result is something very useful, and generalizing it will be our next task. As a first ingredient here, assuming that $A \in M_N(\mathbb{C})$ is invertible, we have:

$$\sigma(A^{-1}) = \sigma(A)^{-1}$$

It is possible to extend this formula to the arbitrary operators, and we will do this in a moment. Before starting, however, we have to find a class of functions generalizing both the polynomials $P \in \mathbb{C}[X]$ and the inverse function $x \to x^{-1}$. The answer to this question is provided by the rational functions, which are as follows:

DEFINITION 1.9. *A rational function $f \in \mathbb{C}(X)$ is a quotient of polynomials:*

$$f = \frac{P}{Q}$$

Assuming that P, Q are prime to each other, we can regard f as a usual function,

$$f : \mathbb{C} - X \to \mathbb{C}$$

with X being the set of zeros of Q, also called poles of f.

Now that we have our class of functions, the next step consists in applying them to operators. Here we cannot expect $f(T)$ to make sense for any f and any T, for instance because T^{-1} is defined only when T is invertible. We are led in this way to:

Definition 1.10. *Given an operator $T\in B(H)$, and a rational function $f=P/Q$ having poles outside $\sigma(T)$, we can construct the following operator,*

$$f(T)=P(T)Q(T)^{-1}$$

that we can denote as a usual fraction, as follows,

$$f(T)=\frac{P(T)}{Q(T)}$$

due to the fact that $P(T),Q(T)$ commute, so that the order is irrelevant.

To be more precise, $f(T)$ is indeed well-defined, and the fraction notation is justified too. In more formal terms, we can say that we have a morphism of complex algebras as follows, with $\mathbb{C}(X)^T$ standing for the rational functions having poles outside $\sigma(T)$:

$$\mathbb{C}(X)^T\to B(H)\quad,\quad f\to f(T)$$

Summarizing, we have now a good class of functions, generalizing both the polynomials and the inverse map $x\to x^{-1}$. We can now extend Theorem 1.8, as follows:

Theorem 1.11. *We have the "rational functional calculus" formula*

$$\sigma(f(T))=f(\sigma(T))$$

valid for any rational function $f\in\mathbb{C}(X)$ having poles outside $\sigma(T)$.

Proof. We pick a scalar $\lambda\in\mathbb{C}$, we write $f=P/Q$, and we set:

$$F=P-\lambda Q$$

By using now Theorem 1.9, for this polynomial, we obtain:

$$\begin{aligned}\lambda\in\sigma(f(T))&\iff F(T)\notin B(H)^{-1}\\&\iff 0\in\sigma(F(T))\\&\iff 0\in F(\sigma(T))\\&\iff \exists\mu\in\sigma(T),F(\mu)=0\\&\iff \lambda\in f(\sigma(T))\end{aligned}$$

Thus, we are led to the formula in the statement. □

As an application of the above methods, we can investigate certain special classes of operators, such as the self-adjoint ones, and the unitary ones. Let us start with:

Proposition 1.12. *The following happen:*

(1) *We have $\sigma(T^*)=\overline{\sigma(T)}$, for any $T\in B(H)$.*
(2) *If $T=T^*$ then $X=\sigma(T)$ satisfies $X=\overline{X}$.*
(3) *If $U^*=U^{-1}$ then $X=\sigma(U)$ satisfies $X^{-1}=\overline{X}$.*

PROOF. We have several assertions here, the idea being as follows:

(1) The spectrum of the adjoint operator T^* can be computed as follows:

$$\begin{aligned}\sigma(T^*) &= \left\{\lambda \in \mathbb{C}\Big|T^* - \lambda \notin B(H)^{-1}\right\} \\ &= \left\{\lambda \in \mathbb{C}\Big|T - \bar{\lambda} \notin B(H)^{-1}\right\} \\ &= \overline{\sigma(T)}\end{aligned}$$

(2) This is clear indeed from (1).

(3) For a unitary operator, $U^* = U^{-1}$, Theorem 1.11 and (1) give:

$$\sigma(U)^{-1} = \sigma(U^{-1}) = \sigma(U^*) = \overline{\sigma(U)}$$

Thus, we are led to the conclusion in the statement. □

In analogy with what happens for the usual matrices, we would like to improve now (2,3) above, with results stating that the spectrum $X = \sigma(T)$ satisfies $X \subset \mathbb{R}$ for self-adjoints, and $X \subset \mathbb{T}$ for unitaries. This will be tricky. Let us start with:

THEOREM 1.13. *The spectrum of a unitary operator*

$$U^* = U^{-1}$$

is on the unit circle, $\sigma(U) \subset \mathbb{T}$.

PROOF. Assuming $U^* = U^{-1}$, we have the following norm computation:

$$||U|| = \sqrt{||UU^*||} = \sqrt{1} = 1$$

Now if we denote by D the unit disk, we obtain from this:

$$\sigma(U) \subset D$$

On the other hand, once again by using $U^* = U^{-1}$, we have as well:

$$||U^{-1}|| = ||U^*|| = ||U|| = 1$$

Thus, as before with D being the unit disk in the complex plane, we have:

$$\sigma(U^{-1}) \subset D$$

Now by using Theorem 1.11, we obtain $\sigma(U) \subset D \cap D^{-1} = \mathbb{T}$, as desired. □

We have as well a similar result for the self-adjoints, as follows:

THEOREM 1.14. *The spectrum of a self-adjoint operator*

$$T = T^*$$

consists of real numbers, $\sigma(T) \subset \mathbb{R}$.

PROOF. The idea is that we can deduce the result from Theorem 1.13, by using the following remarkable rational function, depending on a parameter $r \in \mathbb{R}$:

$$f(z) = \frac{z + ir}{z - ir}$$

Indeed, for $r >> 0$ the operator $f(T)$ is well-defined, and we have:

$$\left(\frac{T + ir}{T - ir}\right)^* = \frac{T - ir}{T + ir} = \left(\frac{T + ir}{T - ir}\right)^{-1}$$

Thus $f(T)$ is unitary, and by using Theorem 1.13 we obtain:

$$\begin{aligned}
\sigma(T) &\subset f^{-1}(f(\sigma(T))) \\
&= f^{-1}(\sigma(f(T))) \\
&\subset f^{-1}(\mathbb{T}) \\
&= \mathbb{R}
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

One key thing that we know about matrices, which is clear for the diagonalizable matrices, and then in general follows by density, is the following formula:

$$\sigma(e^A) = e^{\sigma(A)}$$

We would like to have such formulae for the general operators $T \in B(H)$, but this is something quite technical. Consider the rational calculus morphism from Definition 1.10, which is as follows, with the exponent standing for "having poles outside $\sigma(T)$":

$$\mathbb{C}(X)^T \to B(H) \quad , \quad f \to f(T)$$

As mentioned before, the rational functions are holomorphic outside their poles, and this raises the question of extending this morphism, as follows:

$$Hol(\sigma(T)) \to B(H) \quad , \quad f \to f(T)$$

But for this, we can use the Cauchy formula. Indeed, given a function $f \in \mathbb{C}(X)^T$, the operator $f(T) \in B(H)$ from Definition 1.10 can be recaptured as follows:

$$f(T) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - T}\, dz$$

Now given an arbitrary function $f \in Hol(\sigma(T))$, we can define $f(T) \in B(H)$ by the exactly same formula, and we obtain in this way the desired correspondence:

$$Hol(\sigma(T)) \to B(H) \quad , \quad f \to f(T)$$

This was for the plan. In practice now, all this needs a bit of care, with many verifications needed, and with the technical remark that a winding number must be added to the above Cauchy formulae, for things to be correct. The result is as follows:

THEOREM 1.15. *Given $T\in B(H)$, we have a morphism of algebras as follows, where $Hol(\sigma(T))$ is the algebra of functions which are holomorphic around $\sigma(T)$,*

$$Hol(\sigma(T))\to B(H)\quad,\quad f\to f(T)$$

which extends the previous rational functional calculus $f\to f(T)$. We have:

$$\sigma(f(T))=f(\sigma(T))$$

Moreover, if $\sigma(T)$ is contained in an open set U and $f_n,f:U\to\mathbb C$ are holomorphic functions such that $f_n\to f$ uniformly on compact subsets of U then $f_n(T)\to f(T)$.

PROOF. This follows indeed by reasoning along the above lines, by making a heavy use of the Cauchy formula, and for full details here, we refer to any specialized operator theory book. In what follows, we will not really need this result. □

In order to formulate now our next result, we will need the following notion:

DEFINITION 1.16. *Given an operator $T\in B(H)$, its spectral radius*

$$\rho(T)\in\big[0,||T||\big]$$

is the radius of the smallest disk centered at 0 containing $\sigma(T)$.

Now with this notion in hand, we have the following key result, improving our key theoretical result so far about spectra, namely $\sigma(T)\neq\emptyset$, from Theorem 1.6:

THEOREM 1.17. *The spectral radius of an operator $T\in B(H)$ is given by*

$$\rho(T)=\lim_{n\to\infty}||T^n||^{1/n}$$

and in this formula, we can replace the limit by an inf.

PROOF. We have several things to be proved, the idea being as follows:

(1) Our first claim is that the numbers $u_n=||T^n||^{1/n}$ satisfy:

$$(n+m)u_{n+m}\leq nu_n+mu_m$$

Indeed, we have the following estimate, using the Young inequality $ab\leq a^p/p+b^q/q$, with exponents $p=(n+m)/n$ and $q=(n+m)/m$:

$$\begin{aligned}u_{n+m}&=||T^{n+m}||^{1/(n+m)}\\&\leq||T^n||^{1/(n+m)}||T^m||^{1/(n+m)}\\&\leq||T^n||^{1/n}\cdot\frac{n}{n+m}+||T^m||^{1/m}\cdot\frac{m}{n+m}\\&=\frac{nu_n+mu_m}{n+m}\end{aligned}$$

(2) Our second claim is that the second assertion holds, namely:

$$\lim_{n\to\infty}||T^n||^{1/n}=\inf_n||T^n||^{1/n}$$

For this purpose, we just need the inequality found in (1). Indeed, fix $m \geq 1$, let $n \geq 1$, and write $n = lm + r$ with $0 \leq r \leq m - 1$. By using twice $u_{ab} \leq u_b$, we get:

$$\begin{aligned} u_n &\leq \frac{1}{n}(lmu_{lm} + ru_r) \\ &\leq \frac{1}{n}(lmu_m + ru_1) \\ &\leq u_m + \frac{r}{n} u_1 \end{aligned}$$

It follows that we have $\limsup_n u_n \leq u_m$, which proves our claim.

(3) Summarizing, we are left with proving the main formula, which is as follows, and with the remark that we already know that the sequence on the right converges:

$$\rho(T) = \lim_{n \to \infty} ||T^n||^{1/n}$$

In one sense, we can use the polynomial calculus formula $\sigma(T^n) = \sigma(T)^n$. Indeed, this gives the following estimate, valid for any n, as desired:

$$\begin{aligned} \rho(T) &= \sup_{\lambda \in \sigma(T)} |\lambda| \\ &= \sup_{\rho \in \sigma(T)^n} |\rho|^{1/n} \\ &= \sup_{\rho \in \sigma(T^n)} |\rho|^{1/n} \\ &= \rho(T^n)^{1/n} \\ &\leq ||T^n||^{1/n} \end{aligned}$$

(4) For the reverse inequality, we fix a number $\rho > \rho(T)$, and we want to prove that we have $\rho \geq \lim_{n \to \infty} ||T^n||^{1/n}$. By using the Cauchy formula, we have:

$$\begin{aligned} \frac{1}{2\pi i} \int_{|z|=\rho} \frac{z^n}{z - T} \, dz &= \frac{1}{2\pi i} \int_{|z|=\rho} \sum_{k=0}^{\infty} z^{n-k-1} T^k \, dz \\ &= \sum_{k=0}^{\infty} \frac{1}{2\pi i} \left(\int_{|z|=\rho} z^{n-k-1} dz \right) T^k \\ &= \sum_{k=0}^{\infty} \delta_{n,k+1} T^k \\ &= T^{n-1} \end{aligned}$$

By applying the norm we obtain from this formula:

$$||T^{n-1}|| \leq \frac{1}{2\pi} \int_{|z|=\rho} \left|\left| \frac{z^n}{z - T} \right|\right| \, dz \leq \rho^n \cdot \sup_{|z|=\rho} \left|\left| \frac{1}{z - T} \right|\right|$$

Since the sup does not depend on n, by taking n-th roots, we obtain in the limit:

$$\rho \geq \lim_{n\to\infty} ||T^n||^{1/n}$$

Now recall that ρ was by definition an arbitrary number satisfying $\rho > \rho(T)$. Thus, we have obtained the following estimate, valid for any $T \in B(H)$:

$$\rho(T) \geq \lim_{n\to\infty} ||T^n||^{1/n}$$

Thus, we are led to the conclusion in the statement. □

In the case of the normal elements, we have the following finer result:

THEOREM 1.18. *The spectral radius of a normal element,*

$$TT^* = T^*T$$

is equal to its norm.

PROOF. We can proceed in two steps, as follows:

Step 1. In the case $T = T^*$ we have $||T^n|| = ||T||^n$ for any exponent of the form $n = 2^k$, by using the formula $||TT^*|| = ||T||^2$, and by taking n-th roots we get:

$$\rho(T) \geq ||T||$$

Thus, we are done with the self-adjoint case, with the result $\rho(T) = ||T||$.

Step 2. In the general normal case $TT^* = T^*T$ we have $T^n(T^n)^* = (TT^*)^n$, and by using this, along with the result from Step 1, applied to TT^*, we obtain:

$$\begin{aligned}
\rho(T) &= \lim_{n\to\infty} ||T^n||^{1/n}\\
&= \sqrt{\lim_{n\to\infty} ||T^n(T^n)^*||^{1/n}}\\
&= \sqrt{\lim_{n\to\infty} ||(TT^*)^n||^{1/n}}\\
&= \sqrt{\rho(TT^*)}\\
&= \sqrt{||T||^2}\\
&= ||T||
\end{aligned}$$

Thus, we are led to the conclusion in the statement. □

1c. Normal operators

By using Theorem 1.18 we can say a number of non-trivial things about the normal operators, commonly known as "spectral theorem for normal operators". As a first result here, we can improve the polynomial functional calculus formula, as follows:

Theorem 1.19. *Given $T\in B(H)$ normal, we have a morphism of algebras*

$$\mathbb{C}[X]\to B(H)\quad,\quad P\to P(T)$$

having the properties $||P(T)||=||P_{|\sigma(T)}||$, and $\sigma(P(T))=P(\sigma(T))$.

Proof. This is an improvement of Theorem 1.8 in the normal case, with the extra assertion being the norm estimate. But the element $P(T)$ being normal, we can apply to it the spectral radius formula for normal elements, and we obtain:

$$\begin{aligned}||P(T)|| &= \rho(P(T))\\ &= \sup_{\lambda\in\sigma(P(T))}|\lambda|\\ &= \sup_{\lambda\in P(\sigma(T))}|\lambda|\\ &= ||P_{|\sigma(T)}||\end{aligned}$$

Thus, we are led to the conclusions in the statement. □

We can improve as well the rational calculus formula, and the holomorphic calculus formula, in the same way. Importantly now, at a more advanced level, we have:

Theorem 1.20. *Given $T\in B(H)$ normal, we have a morphism of algebras*

$$C(\sigma(T))\to B(H)\quad,\quad f\to f(T)$$

which is isometric, $||f(T)||=||f||$, and has the property $\sigma(f(T))=f(\sigma(T))$.

Proof. The idea here is to "complete" the morphism in Theorem 1.19, namely:

$$\mathbb{C}[X]\to B(H)\quad,\quad P\to P(T)$$

Indeed, we know from Theorem 1.19 that this morphism is continuous, and is in fact isometric, when regarding the polynomials $P\in\mathbb{C}[X]$ as functions on $\sigma(T)$:

$$||P(T)||=||P_{|\sigma(T)}||$$

Thus, by Stone-Weierstrass, we have a unique isometric extension, as follows:

$$C(\sigma(T))\to B(H)\quad,\quad f\to f(T)$$

It remains to prove $\sigma(f(T))=f(\sigma(T))$, and we can do this by double inclusion:

"$\subset$" Given a continuous function $f\in C(\sigma(T))$, we must prove that we have:

$$\lambda\notin f(\sigma(T))\implies\lambda\notin\sigma(f(T))$$

For this purpose, consider the following function, which is well-defined:

$$\frac{1}{f-\lambda}\in C(\sigma(T))$$

We can therefore apply this function to T, and we obtain:

$$\left(\frac{1}{f-\lambda}\right)T=\frac{1}{f(T)-\lambda}$$

In particular $f(T)-\lambda$ is invertible, so $\lambda\notin\sigma(f(T))$, as desired.

"$\supset$" Given a continuous function $f\in C(\sigma(T))$, we must prove that we have:

$$\lambda\in f(\sigma(T))\implies\lambda\in\sigma(f(T))$$

But this is the same as proving that we have:

$$\mu\in\sigma(T)\implies f(\mu)\in\sigma(f(T))$$

For this purpose, we approximate our function by polynomials, $P_n\to f$, and we examine the following convergence, which follows from $P_n\to f$:

$$P_n(T)-P_n(\mu)\to f(T)-f(\mu)$$

We know from polynomial functional calculus that we have:

$$P_n(\mu)\in P_n(\sigma(T))=\sigma(P_n(T))$$

Thus, the operators $P_n(T)-P_n(\mu)$ are not invertible. On the other hand, we know that the set formed by the invertible operators is open, so its complement is closed. Thus the limit $f(T)-f(\mu)$ is not invertible either, and so $f(\mu)\in\sigma(f(T))$, as desired. □

As an important comment, Theorem 1.20 is not exactly in final form, because it misses an important point, namely that our correspondence maps:

$$\bar{z}\to T^*$$

However, this is something non-trivial, and we will be back to this later. Observe however that Theorem 1.20 is fully powerful for the self-adjoint operators, $T=T^*$, where the spectrum is real, so where $z=\bar{z}$ on the spectrum. We will be back to this.

As a second result now, along the same lines, we can further extend Theorem 1.20 into a measurable functional calculus theorem, as follows:

Theorem 1.21. *Given $T\in B(H)$ normal, we have a morphism of algebras as follows, with L^∞ standing for abstract measurable functions, or Borel functions,*

$$L^\infty(\sigma(T))\to B(H)\quad,\quad f\to f(T)$$

which is isometric, $||f(T)||=||f||$, and has the property $\sigma(f(T))=f(\sigma(T))$.

Proof. As before, the idea will be that of "completing" what we have. To be more precise, we can use the Riesz theorem and a polarization trick, as follows:

(1) Given a vector $x\in H$, consider the following functional:

$$C(\sigma(T))\to\mathbb{C}\quad,\quad g\to<g(T)x,x>$$

By the Riesz theorem, this functional must be the integration with respect to a certain measure μ on the space $\sigma(T)$. Thus, we have a formula as follows:

$$< g(T)x, x >= \int_{\sigma(T)} g(z) d\mu(z)$$

Now given an arbitrary Borel function $f \in L^\infty(\sigma(T))$, as in the statement, we can define a number $< f(T)x, x > \in \mathbb{C}$, by using exactly the same formula, namely:

$$< f(T)x, x >= \int_{\sigma(T)} f(z) d\mu(z)$$

Thus, we have managed to define numbers $< f(T)x, x > \in \mathbb{C}$, for all vectors $x \in H$, and in addition we can recover these numbers as follows, with $g_n \in C(\sigma(T))$:

$$< f(T)x, x >= \lim_{g_n \to f} < g_n(T)x, x >$$

(2) In order to define now numbers $< f(T)x, y > \in \mathbb{C}$, for all vectors $x, y \in H$, we can use a polarization trick. Indeed, for any operator $S \in B(H)$ we have:

$$< S(x+y), x+y >=< Sx, x > + < Sy, y > + < Sx, y > + < Sy, x >$$

By replacing $y \to iy$, we have as well the following formula:

$$< S(x+iy), x+iy >=< Sx, x > + < Sy, y > -i < Sx, y > +i < Sy, x >$$

By multiplying this formula by i, and summing with the first one, we obtain:

$$\begin{aligned} < S(x+y), x+y > +i < S(x+iy), x+iy > &= (1+i)[< Sx, x > + < Sy, y >] \\ &+ 2 < Sx, y > \end{aligned}$$

(3) But with this, we can now finish. Indeed, by combining (1,2), given a Borel function $f \in L^\infty(\sigma(T))$, we can define numbers $< f(T)x, y > \in \mathbb{C}$ for any $x, y \in H$, and it is routine to check, by using approximation by continuous functions $g_n \to f$ as in (1), that we obtain in this way an operator $f(T) \in B(H)$, having all the desired properties. □

As a comment here, the above result and its proof provide us with more than a Borel functional calculus, because what we got is a certain measure on the spectrum $\sigma(T)$, along with a functional calculus for the L^∞ functions with respect to this measure. We will be back to this later, and for the moment we will only need Theorem 1.21 as formulated, with $L^\infty(\sigma(T))$ standing, a bit abusively, for the Borel functions on $\sigma(T)$.

We can now diagonalize the normal operators. We will do this in 3 steps, first for the self-adjoint operators, then for the families of commuting self-adjoint operators, and finally for the general normal operators, by using a trick of the following type:

$$T = Re(T) + iIm(T)$$

The diagonalization in infinite dimensions is more tricky than in finite dimensions, and instead of writing a formula of type $T = UDU^*$, with $U, D \in B(H)$ being respectively unitary and diagonal, we will express our operator as $T = U^*MU$, with $U : H \to K$ being a certain unitary, and $M \in B(K)$ being a certain diagonal operator. We first have:

THEOREM 1.22. *Any self-adjoint operator $T \in B(H)$ can be diagonalized,*

$$T = U^* M_f U$$

with $U : H \to L^2(X)$ being a unitary operator from H to a certain L^2 space associated to T, with $f : X \to \mathbb{R}$ being a certain function, once again associated to T, and with

$$M_f(g) = fg$$

being the usual multiplication operator by f, on the Hilbert space $L^2(X)$.

PROOF. The construction of U, f can be done in several steps, as follows:

(1) We first prove the result in the special case where our operator T has a cyclic vector $x \in H$, with this meaning that the following holds:

$$\overline{span\left(T^k x \Big| n \in \mathbb{N}\right)} = H$$

For this purpose, let us go back to the proof of Theorem 1.21. We will use the following formula from there, with μ being the measure on $X = \sigma(T)$ associated to x:

$$< g(T)x, x >= \int_{\sigma(T)} g(z) d\mu(z)$$

Our claim is that we can define a unitary $U : H \to L^2(X)$, first on the dense part spanned by the vectors $T^k x$, by the following formula, and then by continuity:

$$U[g(T)x] = g$$

Indeed, the following computation shows that U is well-defined, and isometric:

$$\begin{aligned}
||g(T)x||^2 &= < g(T)x, g(T)x > \\
&= < g(T)^* g(T)x, x > \\
&= < |g|^2(T)x, x > \\
&= \int_{\sigma(T)} |g(z)|^2 d\mu(z) \\
&= ||g||_2^2
\end{aligned}$$

We can then extend U by continuity into a unitary $U : H \to L^2(X)$, as claimed. Now observe that we have the following formula:

$$\begin{aligned}
UTU^* g &= U[Tg(T)x] \\
&= U[(zg)(T)x] \\
&= zg
\end{aligned}$$

Thus our result is proved in the present case, with U as above, and with $f(z) = z$.

(2) We discuss now the general case. Our first claim is that H has a decomposition as follows, with each H_i being invariant under T, and admitting a cyclic vector x_i:

$$H = \bigoplus_i H_i$$

Indeed, this is something elementary, the construction being by recurrence in finite dimensions, in the obvious way, and by using the Zorn lemma in general. Now with this decomposition in hand, we can make a direct sum of the diagonalizations obtained in (1), for each of the restrictions $T_{|H_i}$, and we obtain the formula in the statement. □

We have the following technical generalization of the above result:

THEOREM 1.23. *Any family of commuting self-adjoint operators $T_i \in B(H)$ can be jointly diagonalized,*

$$T_i = U^* M_{f_i} U$$

with $U : H \to L^2(X)$ being a unitary operator from H to a certain L^2 space associated to $\{T_i\}$, with $f_i : X \to \mathbb{R}$ being certain functions, once again associated to T_i, and with

$$M_{f_i}(g) = f_i g$$

being the usual multiplication operator by f_i, on the Hilbert space $L^2(X)$.

PROOF. This is similar to the proof of Theorem 1.22, by suitably modifying the measurable calculus formula, and μ itself, as to have this working for all operators T_i. □

We can now discuss the case of the arbitrary normal operators, as follows:

THEOREM 1.24. *Any normal operator $T \in B(H)$ can be diagonalized,*

$$T = U^* M_f U$$

with $U : H \to L^2(X)$ being a unitary operator from H to a certain L^2 space associated to T, with $f : X \to \mathbb{C}$ being a certain function, once again associated to T, and with

$$M_f(g) = fg$$

being the usual multiplication operator by f, on the Hilbert space $L^2(X)$.

PROOF. This is our main diagonalization theorem, the idea being as follows:

(1) Consider the decomposition of T into its real and imaginary parts, namely:

$$T = \frac{T + T^*}{2} + i \cdot \frac{T - T^*}{2i}$$

We know that the real and imaginary parts are self-adjoint operators. Now since T was assumed to be normal, $TT^* = T^*T$, these real and imaginary parts commute:

$$\left[\frac{T + T^*}{2}, \frac{T - T^*}{2i} \right] = 0$$

Thus Theorem 1.23 applies to these real and imaginary parts, and gives the result. □

This was for our series of diagonalization theorems. There is of course one more result here, regarding the families of commuting normal operators, as follows:

THEOREM 1.25. *Any family of commuting normal operators $T_i \in B(H)$ can be jointly diagonalized,*

$$T_i = U^* M_{f_i} U$$

with $U : H \to L^2(X)$ being a unitary operator from H to a certain L^2 space associated to $\{T_i\}$, with $f_i : X \to \mathbb{C}$ being certain functions, once again associated to T_i, and with

$$M_{f_i}(g) = f_i g$$

being the usual multiplication operator by f_i, on the Hilbert space $L^2(X)$.

PROOF. This is similar to the proof of Theorem 1.23 and Theorem 1.24, by combining the arguments there. To be more precise, this follows as Theorem 1.23, by using the decomposition trick from the proof of Theorem 1.24. □

With the above diagonalization results in hand, we can now "fix" the continuous and measurable functional calculus theorems, with a key complement, as follows:

THEOREM 1.26. *Given a normal operator $T \in B(H)$, the following hold, for both the functional calculus and the measurable calculus morphisms:*

1. *These morphisms are $*$-morphisms.*
2. *The function $\bar{z}$ gets mapped to T^*.*
3. *The functions $Re(z), Im(z)$ get mapped to $Re(T), Im(T)$.*
4. *The function $|z|^2$ gets mapped to $TT^* = T^*T$.*
5. *If f is real, then $f(T)$ is self-adjoint.*

PROOF. These assertions are more or less equivalent, with (1) being the main one, which obviously implies everything else. But this assertion (1) follows from the diagonalization result for normal operators, from Theorem 1.24. □

1d. Normed algebras

Good news, we can now talk about operator algebras. Let us start with the following broad definition, obtained by imposing the "minimal" set of reasonable axioms:

DEFINITION 1.27. *An operator algebra is an algebra of bounded operators $A \subset B(H)$ which contains the unit, is closed under taking adjoints,*

$$T \in A \implies T^* \in A$$

and is closed as well under the norm.

As a first result now regarding the operator algebras, in relation with the normal operators, where most of the non-trivial results that we have so far are, we have:

THEOREM 1.28. *The operator algebra $< T >\subset B(H)$ generated by a normal operator $T\in B(H)$ appears as an algebra of continuous functions,*

$$< T >= C(\sigma(T))$$

where $\sigma(T)\subset\mathbb{C}$ denotes as usual the spectrum of T.

PROOF. We know that we have a continuous morphism of $*$-algebras, as follows:

$$C(\sigma(T))\to B(H)\quad,\quad f\to f(T)$$

Moreover, by the general properties of the continuous calculus, also established in the above, this morphism is injective, and its image is the norm closed algebra $< T >$ generated by T,T^*. Thus, we obtain the isomorphism in the statement. □

The above result is very nice, and it is possible to further build on it, as follows:

THEOREM 1.29. *The operator algebra $< T_i >\subset B(H)$ generated by a family of normal operators $T_i\in B(H)$ appears as an algebra of continuous functions,*

$$< T >= C(X)$$

where $X\subset\mathbb{C}$ is a certain compact space associated to the family $\{T_i\}$. Equivalently, any commutative operator algebra $A\subset B(H)$ is of the form $A=C(X)$.

PROOF. We have two assertions here, the idea being as follows:

(1) Regarding the first assertion, this follows exactly as in the proof of Theorem 1.28, by using this time the spectral theorem for families of normal operators.

(2) As for the second assertion, this is clear from the first one, because any commutative algebra $A\subset B(H)$ is generated by its elements $T\in A$, which are all normal. □

All this is good to know, but Theorem 1.28 and Theorem 1.29 remain something quite heavy, based on the spectral theorem. We would like to present now an alternative proof for these results, which is rather elementary, and has the advantage of reconstructing the compact space X directly from the knowledge of the algebra A. We will need:

THEOREM 1.30. *Given an operator $T\in A\subset B(H)$, define its spectrum as:*

$$\sigma(T)=\left\{\lambda\in\mathbb{C}\Big|T-\lambda\notin A^{-1}\right\}$$

The following spectral theory results hold, exactly as in the $A=B(H)$ case:

1. *We have $\sigma(ST)\cup\{0\}=\sigma(TS)\cup\{0\}$.*
2. *We have polynomial, rational and holomorphic calculus.*
3. *As a consequence, the spectra are compact and non-empty.*
4. *The spectra of unitaries ($U^*=U^{-1}$) and self-adjoints ($T=T^*$) are on $\mathbb{T},\mathbb{R}$.*
5. *The spectral radius of normal elements ($TT^*=T^*T$) is given by $\rho(T)=||T||$.*

In addition, assuming $T\in A\subset B$, the spectra of T with respect to A and to B coincide.

PROOF. This is something that we know from before, in the case $A = B(H)$. In general the proof is similar, the idea being as follows:

(1) Regarding the assertions (1-5), which are of course formulated a bit informally, the proofs here are perfectly similar to those for the full operator algebra $A = B(H)$. All this is standard material, and in fact, things before were written in such a way as for their extension now, to the general operator algebra setting, to be obvious.

(2) Regarding the last assertion, the inclusion $\sigma_B(T) \subset \sigma_A(T)$ is clear. For the converse, assume $T - \lambda \in B^{-1}$, and consider the following self-adjoint element:

$$S = (T - \lambda)^*(T - \lambda)$$

The difference between the two spectra of $S \in A \subset B$ is then given by:

$$\sigma_A(S) - \sigma_B(S) = \left\{\mu \in \mathbb{C} - \sigma_B(S)\Big|(S - \mu)^{-1} \in B - A\right\}$$

Thus this difference in an open subset of $\mathbb{C}$. On the other hand S being self-adjoint, its two spectra are both real, and so is their difference. Thus the two spectra of S are equal, and in particular S is invertible in A, and so $T - \lambda \in A^{-1}$, as desired.

(3) As an observation, the last assertion applied with $B = B(H)$ shows that the spectrum $\sigma(T)$ as constructed in the statement coincides with the spectrum $\sigma(T)$ as constructed and studied before, so the fact that (1-5) hold indeed is no surprise.

(4) Finally, I can hear you screaming that I should have concieved this book differently, matter of not proving the same things twice. Good point, with my distinguished colleague Bourbaki saying the same, and in answer, wait for chapter 3 below, where we will prove exactly the same things a third time. We can discuss pedagogy at that time. □

We can now get back to the commutative algebras, and we have the following result, due to Gelfand, which provides an alternative to Theorem 1.28 and Theorem 1.29:

THEOREM 1.31. *Any commutative operator algebra $A \subset B(H)$ is of the form*

$$A = C(X)$$

with the "spectrum" X of such an algebra being the space of characters $\chi : A \to \mathbb{C}$, with topology making continuous the evaluation maps $ev_T : \chi \to \chi(T)$.

PROOF. Given a commutative operator algebra A, we can define X as in the statement. Then X is compact, and $T \to ev_T$ is a morphism of algebras, as follows:

$$ev : A \to C(X)$$

(1) We first prove that ev is involutive. We use the following formula, which is similar to the $z = Re(z) + iIm(z)$ formula for the usual complex numbers:

$$T = \frac{T + T^*}{2} + i \cdot \frac{T - T^*}{2i}$$

Thus it is enough to prove the equality $ev_{T^*} = ev_T^*$ for self-adjoint elements T. But this is the same as proving that $T = T^*$ implies that ev_T is a real function, which is in turn true, because $ev_T(\chi) = \chi(T)$ is an element of $\sigma(T)$, contained in $\mathbb{R}$.

(2) Since A is commutative, each element is normal, so ev is isometric:

$$||ev_T|| = \rho(T) = ||T||$$

(3) It remains to prove that ev is surjective. But this follows from the Stone-Weierstrass theorem, because $ev(A)$ is a closed subalgebra of $C(X)$, which separates the points. □

The above theorem of Gelfand is something very beautiful, and far-reaching. It is possible to further build on it, indefinitely high. We will be back to this, later.

1e. Exercises

Exercises:

Exercise 1.32.

Exercise 1.33.

Exercise 1.34.

Exercise 1.35.

Exercise 1.36.

Exercise 1.37.

Exercise 1.38.

Exercise 1.39.

Bonus exercise.

CHAPTER 2

Operator algebras

2a. Operator algebras

Instead of further building on the above results, which are already quite non-trivial, let us return to our modest status of apprentice operator algebraists, and declare ourselves unsatisfied with the formalism from chapter 1, on the following intuitive grounds:

Thought 2.1. *Our assumption that $A \subset B(H)$ is norm closed is not satisfying, because we would like A to be stable under polar decomposition, under taking spectral projections, and more generally, under measurable functional calculus.*

So, let us get now into this, topologies on $B(H)$, and fine-tunings of our operator algebra formalism, based on them. The result that we will need is as follows:

Proposition 2.2. *For a subalgebra $A \subset B(H)$, the following are equivalent:*

1. *A is closed under the weak operator topology, making each of the linear maps $T \to < Tx, y >$ continuous.*
2. *A is closed under the strong operator topology, making each of the linear maps $T \to Tx$ continuous.*

In the case where these conditions are satisfied, A is closed under the norm topology.

Proof. There are several statements here, the proof being as follows:

(1) It is clear that the norm topology is stronger than the strong operator topology, which is in turn stronger than the weak operator topology. At the level of the subsets $S \subset B(H)$ which are closed things get reversed, in the sense that weakly closed implies strongly closed, which in turn implies norm closed. Thus, we are left with proving that for any algebra $A \subset B(H)$, strongly closed implies weakly closed.

(2) Consider the Hilbert space obtained by summing n times H with itself:

$$K = H \oplus \ldots \oplus H$$

The operators over K can be regarded as being square matrices with entries in $B(H)$, and in particular, we have a representation $\pi : B(H) \to B(K)$, as follows:

$$\pi(T) = \begin{pmatrix} T & & \\ & \ddots & \\ & & T \end{pmatrix}$$

Assume now that we are given an operator $T \in \bar{A}$, with the bar denoting the weak closure. We have then, by using the Hahn-Banach theorem, for any $x \in K$:

$$\begin{aligned} T \in \bar{A} &\implies \pi(T) \in \overline{\pi(A)} \\ &\implies \pi(T)x \in \overline{\pi(A)x} \\ &\implies \pi(T)x \in \overline{\pi(A)x}^{||\cdot||} \end{aligned}$$

Now observe that the last formula tells us that for any $x = (x_1, \ldots, x_n)$, and any $\varepsilon > 0$, we can find $S \in A$ such that the following holds, for any i:

$$||Sx_i - Tx_i|| < \varepsilon$$

Thus T belongs to the strong operator closure of A, as desired. □

Observe that in the above the terminology is a bit confusing, because the norm topology is stronger than the strong operator topology. As a solution, we agree to call the norm topology "strong", and the weak and strong operator topologies "weak", whenever these two topologies coincide. With this convention made, the algebras $A \subset B(H)$ in Proposition 2.2 are those which are weakly closed. Thus, we can now formulate:

Definition 2.3. *A von Neumann algebra is an operator algebra*

$$A \subset B(H)$$

which is closed under the weak topology.

These algebras will be our main objects of study, in what follows. As basic examples, we have the algebra $B(H)$ itself, then the singly generated algebras, $A =< T >$ with $T \in B(H)$, and then the multiply generated algebras, $A =< T_i >$ with $T_i \in B(H)$. But for the moment, let us keep things simple, and build directly on Definition 2.3, by using basic functional analysis methods. We will need the following key result:

Theorem 2.4. *For an operator algebra $A \subset B(H)$, we have*

$$A'' = \bar{A}$$

with A'' being the bicommutant inside $B(H)$, and $\bar{A}$ being the weak closure.

Proof. We can prove this by double inclusion, as follows:

"$\supset$" Since any operator commutes with the operators that it commutes with, we have a trivial inclusion $S \subset S''$, valid for any set $S \subset B(H)$. In particular, we have:

$$A \subset A''$$

Our claim now is that the algebra A'' is closed, with respect to the strong operator topology. Indeed, assuming that we have $T_i \to T$ in this topology, we have:

$$\begin{aligned} T_i \in A'' &\implies ST_i = T_iS,\ \forall S \in A' \\ &\implies ST = TS,\ \forall S \in A' \\ &\implies T \in A \end{aligned}$$

Thus our claim is proved, and together with Proposition 2.2, which allows us to pass from the strong to the weak operator topology, this gives $\bar{A} \subset A''$, as desired.

"$\subset$" Here we must prove that we have the following implication, valid for any $T \in B(H)$, with the bar denoting as usual the weak operator closure:

$$T \in A'' \implies T \in \bar{A}$$

For this purpose, we use the same amplification trick as in the proof of Proposition 2.2. Consider the Hilbert space obtained by summing n times H with itself:

$$K = H \oplus \ldots \oplus H$$

The operators over K can be regarded as being square matrices with entries in $B(H)$, and in particular, we have a representation $\pi : B(H) \to B(K)$, as follows:

$$\pi(T) = \begin{pmatrix} T && \\ & \ddots & \\ && T \end{pmatrix}$$

The idea will be that of doing the computations in this representation. First, in this representation, the image of our algebra $A \subset B(H)$ is given by:

$$\pi(A) = \left\{ \begin{pmatrix} T && \\ & \ddots & \\ && T \end{pmatrix} \Big| T \in A \right\}$$

We can compute the commutant of this image, exactly as in the usual scalar matrix case, and we obtain the following formula:

$$\pi(A)' = \left\{ \begin{pmatrix} S_{11} & \ldots & S_{1n} \\ \vdots && \vdots \\ S_{n1} & \ldots & S_{nn} \end{pmatrix} \Big| S_{ij} \in A' \right\}$$

We conclude from this that, given an operator $T \in A''$ as above, we have:

$$\begin{pmatrix} T && \\ & \ddots & \\ && T \end{pmatrix} \in \pi(A)''$$

In other words, the conclusion of all this is that we have:

$$T \in A'' \implies \pi(T) \in \pi(A)''$$

Now given a vector $x \in K$, consider the orthogonal projection $P \in B(K)$ on the norm closure of the vector space $\pi(A)x \subset K$. Since the subspace $\pi(A)x \subset K$ is invariant under the action of $\pi(A)$, so is its norm closure inside K, and we obtain from this:

$$P \in \pi(A)'$$

By combining this with what we found above, we conclude that we have:

$$T \in A'' \implies \pi(T)P = P\pi(T)$$

Since this holds for any $x \in K$, we conclude that any operator $T \in A''$ belongs to the strong operator closure of A. By using now Proposition 2.2, which allows us to pass from the strong to the weak operator closure, we conclude that we have:

$$A'' \subset \bar{A}$$

Thus, we have the desired reverse inclusion, and this finishes the proof. □

Now by getting back to the von Neumann algebras, from Definition 2.3, we have the following result, which is a reformulation of Theorem 2.4, by using this notion:

THEOREM 2.5. *For an operator algebra $A \subset B(H)$, the following are equivalent:*

(1) *A is weakly closed, so it is a von Neumann algebra.*
(2) *A equals its algebraic bicommutant A'', taken inside $B(H)$.*

PROOF. This follows from the formula $A'' = \bar{A}$ from Theorem 2.4, along with the trivial fact that the commutants are automatically weakly closed. □

The above statement, called bicommutant theorem, and due to von Neumann [89], is quite interesting, philosophically speaking. Among others, it shows that the von Neumann algebras are exactly the commutants of the self-adjoint sets of operators:

PROPOSITION 2.6. *Given a subset $S \subset B(H)$ which is closed under $*$, the commutant*

$$A = S'$$

is a von Neumann algebra. Any von Neumann algebra appears in this way.

PROOF. We have two assertions here, the idea being as follows:

(1) Given $S \subset B(H)$ satisfying $S = S^*$, the commutant $A = S'$ satisfies $A = A^*$, and is also weakly closed. Thus, A is a von Neumann algebra. Note that this follows as well from the following "tricommutant formula", which follows from Theorem 2.5:

$$S''' = S'$$

(2) Given a von Neumann algebra $A \subset B(H)$, we can take $S = A'$. Then S is closed under the involution, and we have $S' = A$, as desired. □

Observe that Proposition 2.6 can be regarded as yet another alternative definition for the von Neumann algebras, and with this definition being probably the best one when talking about quantum mechanics, where the self-adjoint operators $T : H \to H$ can be though of as being "observables" of the system, and with the commutants $A = S'$ of the sets of such observables $S = \{T_i\}$ being the algebras $A \subset B(H)$ that we are interested in. And with all this actually needing some discussion about self-adjointness, and about boundedness too, but let us not get into this here, and stay mathematical, as before.

As another interesting consequence of Theorem 2.5, we have:

PROPOSITION 2.7. *Given a von Neumann algebra $A \subset B(H)$, its center*

$$Z(A) = A \cap A'$$

regarded as an algebra $Z(A) \subset B(H)$, is a von Neumann algebra too.

PROOF. This follows from the fact that the commutants are weakly closed, that we know from the above, which shows that $A' \subset B(H)$ is a von Neumann algebra. Thus, the intersection $Z(A) = A \cap A'$ must be a von Neumann algebra too, as claimed. □

2b. Basic results

In order to develop some general theory, let us start by investigating the finite dimensional case. Here the ambient algebra is $B(H) = M_N(\mathbb{C})$, any linear subspace $A \subset B(H)$ is automatically closed, for all 3 topologies in Proposition 2.2, and we have:

THEOREM 2.8. *The $*$-algebras $A \subset M_N(\mathbb{C})$ are exactly the algebras of the form*

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

depending on parameters $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \mathbb{N}$ satisfying

$$n_1 + \ldots + n_k = N$$

embedded into $M_N(\mathbb{C})$ via the obvious block embedding, twisted by a unitary $U \in U_N$.

PROOF. We have two assertions to be proved, the idea being as follows:

(1) Given numbers $n_1, \ldots, n_k \in \mathbb{N}$ satisfying $n_1 + \ldots + n_k = N$, we have indeed an obvious embedding of $*$-algebras, via matrix blocks, as follows:

$$M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C}) \subset M_N(\mathbb{C})$$

In addition, we can twist this embedding by a unitary $U \in U_N$, as follows:

$$M \to UMU^*$$

(2) In the other sense now, consider a $*$-algebra $A \subset M_N(\mathbb{C})$. It is elementary to prove that the center $Z(A) = A \cap A'$, as an algebra, is of the following form:

$$Z(A) \simeq \mathbb{C}^k$$

Consider now the standard basis $e_1,\ldots,e_k\in\mathbb C^k$, and let $p_1,\ldots,p_k\in Z(A)$ be the images of these vectors via the above identification. In other words, these elements $p_1,\ldots,p_k\in A$ are central minimal projections, summing up to 1:

$$p_1+\ldots+p_k=1$$

The idea is then that this partition of the unity will eventually lead to the block decomposition of A, as in the statement. We prove this in 4 steps, as follows:

Step 1. We first construct the matrix blocks, our claim here being that each of the following linear subspaces of A are non-unital $*$-subalgebras of A:

$$A_i=p_iAp_i$$

But this is clear, with the fact that each A_i is closed under the various non-unital $*$-subalgebra operations coming from the projection equations $p_i^2=p_i^*=p_i$.

Step 2. We prove now that the above algebras $A_i\subset A$ are in a direct sum position, in the sense that we have a non-unital $*$-algebra sum decomposition, as follows:

$$A=A_1\oplus\ldots\oplus A_k$$

As with any direct sum question, we have two things to be proved here. First, by using the formula $p_1+\ldots+p_k=1$ and the projection equations $p_i^2=p_i^*=p_i$, we conclude that we have the needed generation property, namely:

$$A_1+\ldots+A_k=A$$

As for the fact that the sum is indeed direct, this follows as well from the formula $p_1+\ldots+p_k=1$, and from the projection equations $p_i^2=p_i^*=p_i$.

Step 3. Our claim now, which will finish the proof, is that each of the $*$-subalgebras $A_i=p_iAp_i$ constructed above is a full matrix algebra. To be more precise here, with $n_i=rank(p_i)$, our claim is that we have isomorphisms, as follows:

$$A_i\simeq M_{n_i}(\mathbb C)$$

In order to prove this claim, recall that the projections $p_i\in A$ were chosen central and minimal. Thus, the center of each of the algebras A_i reduces to the scalars:

$$Z(A_i)=\mathbb C$$

But this shows, either via a direct computation, or via the bicommutant theorem, that the each of the algebras A_i is a full matrix algebra, as claimed.

Step 4. We can now obtain the result, by putting together what we have. Indeed, by using the results from Step 2 and Step 3, we obtain an isomorphism as follows:

$$A\simeq M_{n_1}(\mathbb C)\oplus\ldots\oplus M_{n_k}(\mathbb C)$$

Moreover, a more careful look at the isomorphisms established in Step 3 shows that at the global level, that of the algebra A itself, the above isomorphism simply comes by twisting the following standard multimatrix embedding, discussed in the beginning of the proof, (1) above, by a certain unitary matrix $U\in U_N$:

$$M_{n_1}(\mathbb{C})\oplus\ldots\oplus M_{n_k}(\mathbb{C})\subset M_N(\mathbb{C})$$

Now by putting everything together, we obtain the result. □

In relation with the bicommutant theorem, we have the following result, which fully clarifies the situation, with a very explicit proof, in finite dimensions:

Proposition 2.9. *Consider a $*$-algebra $A\subset M_N(\mathbb{C})$, written as above:*

$$A=M_{n_1}(\mathbb{C})\oplus\ldots\oplus M_{n_k}(\mathbb{C})$$

The commutant of this algebra is then, with respect with the block decomposition used,

$$A'=\mathbb{C}\oplus\ldots\oplus\mathbb{C}$$

and by taking one more time the commutant we obtain A itself, $A=A''$.

Proof. Let us decompose indeed our algebra A as in Theorem 2.8:

$$A=M_{n_1}(\mathbb{C})\oplus\ldots\oplus M_{n_k}(\mathbb{C})$$

The center of each matrix algebra being reduced to the scalars, the commutant of this algebra is then as follows, with each copy of $\mathbb{C}$ corresponding to a matrix block:

$$A'=\mathbb{C}\oplus\ldots\oplus\mathbb{C}$$

By taking once again the commutant we obtain A itself, and we are done. □

As another interesting application of Theorem 2.8, clarifying this time the relation with operator theory, in finite dimensions, we have the following result:

Theorem 2.10. *Given an operator $T\in B(H)$ in finite dimensions, $H=\mathbb{C}^N$, the von Neumann algebra $A=<T>$ that it generates inside $B(H)=M_N(\mathbb{C})$ is*

$$A=M_{n_1}(\mathbb{C})\oplus\ldots\oplus M_{n_k}(\mathbb{C})$$

with the sizes of the blocks $n_1,\ldots,n_k\in\mathbb{N}$ coming from the spectral theory of the associated matrix $M\in M_N(\mathbb{C})$. In the normal case $TT^*=T^*T$, this decomposition comes from

$$T=UDU^*$$

with $D\in M_N(\mathbb{C})$ diagonal, and with $U\in U_N$ unitary.

Proof. This is something which is routine, by using the linear algebra and spectral theory developed in chapter 1, for the matrices $M\in M_N(\mathbb{C})$. To be more precise:

(1) The fact that $A=<T>$ decomposes into a direct sum of matrix algebras is something that we already know, coming from Theorem 2.8.

(2) By using standard linear algebra, we can compute the block sizes $n_1,\ldots,n_k\in\mathbb{N}$, from the knowledge of the spectral theory of the associated matrix $M\in M_N(\mathbb{C})$.

(3) In the normal case, $TT^* = T^*T$, we can simply invoke the spectral theorem, and by suitably changing the basis, we are led to the conclusion in the statement. □

Let us get now to infinite dimensions, with Theorem 2.10 as our main source of inspiration. The same argument applies, provided that we are in the normal case, and we have the following result, summarizing our basic knowledge here:

THEOREM 2.11. *Given a bounded operator $T \in B(H)$ which is normal, $TT^* = T^*T$, the von Neumann algebra $A =< T >$ that it generates inside $B(H)$ is*

$$< T >= L^\infty(\sigma(T))$$

with $\sigma(T) \subset \mathbb{C}$ being as usual its spectrum.

PROOF. The measurable functional calculus theorem for the normal operators tells us that we have a weakly continuous morphism of $*$-algebras, as follows:

$$L^\infty(\sigma(T)) \to B(H) \quad , \quad f \to f(T)$$

Moreover, by the general properties of the measurable calculus, also established in chapter 1, this morphism is injective, and its image is the weakly closed algebra $< T >$ generated by T, T^*. Thus, we obtain the isomorphism in the statement. □

More generally now, along the same lines, we have the following result:

THEOREM 2.12. *Given operators $T_i \in B(H)$ which are normal, and which commute, the von Neumann algebra $A =< T_i >$ that these operators generates inside $B(H)$ is*

$$< T_i >= L^\infty(X)$$

with X being a certain measured space, associated to the family $\{T_i\}$.

PROOF. This is once again routine, by using the spectral theory for the families of commuting normal operators $T_i \in B(H)$ developed in chapter 1. □

As a fundamental consequence now of the above results, we have:

THEOREM 2.13. *The commutative von Neumann algebras are the algebras*

$$A = L^\infty(X)$$

with X being a measured space.

PROOF. We have two assertions to be proved, the idea being as follows:

(1) In one sense, we must prove that given a measured space X, we can realize the $A = L^\infty(X)$ as a von Neumann algebra, on a certain Hilbert space H. But this is something that we know since chapter 1, the representation being as follows:

$$L^\infty(X) \subset B(L^2(X)) \quad , \quad f \to (g \to fg)$$

(2) In the other sense, given a commutative von Neumann algebra $A \subset B(H)$, we must construct a certain measured space X, and an identification $A = L^\infty(X)$. But this follows from Theorem 2.12, because we can write our algebra as follows:

$$A = < T_i >$$

To be more precise, A being commutative, any element $T \in A$ is normal, so we can pick a basis $\{T_i\} \subset A$, and then we have $A = < T_i >$ as above, with $T_i \in B(H)$ being commuting normal operators. Thus Theorem 2.12 applies, and gives the result.

(3) Alternatively, and more explicitly, we can deduce this from Theorem 2.11, applied with $T = T^*$. Indeed, by using $T = Re(T) + iIm(T)$, we conclude that any von Neumann algebra $A \subset B(H)$ is generated by its self-adjoint elements $T \in A$. Moreover, by using measurable functional calculus, we conclude that A is linearly generated by its projections. But then, assuming $A = \overline{span}\{p_i\}$, with p_i being projections, we can set:

$$T = \sum_{i=0}^{\infty} \frac{p_i}{3^i}$$

Then $T = T^*$, and by functional calculus we have $p_0 \in < T >$, then $p_1 \in < T >$, and so on. Thus $A = < T >$, and $A = L^\infty(X)$ comes now via Theorem 2.11, as claimed. □

The above result is the foundation for all the advanced von Neumann algebra theory, that we will discuss in the remainder of this book, and there are many things that can be said about it. To start with, in relation with the general theory of the normed closed algebras, that we developed in the beginning of this chapter, we have:

Warning 2.14. *Although the von Neumann algebras are norm closed, the theory of norm closed algebras does not always apply well to them. For instance for* $A = L^\infty(X)$ *Gelfand gives* $A = C(\widehat{X})$, *with* $\widehat{X}$ *being a certain technical compactification of* X.

In short, this would be my advice, do not mess up the two theories that we will be developing in this book, try finding different rooms for them, in your brain. At least at this stage of things, because later, do not worry, we will be playing with both.

Now forgetting about Gelfand, and taking Theorem 2.13 as such, tentative foundation for the theory that we want to develop, as a first consequence of this, we have:

Theorem 2.15. *Given a von Neumann algebra* $A \subset B(H)$, *we have*

$$Z(A) = L^\infty(X)$$

with X *being a certain measured space.*

Proof. We know from Proposition 2.7 that the center $Z(A) \subset B(H)$ is a von Neumann algebra. Thus Theorem 2.13 applies, and gives the result. □

It is possible to further build on this, with a powerful decomposition result as follows, over the measured space X constructed in Theorem 2.15:

$$A=\int_X A_x\,dx$$

But more on this later, after developing the appropriate tools for this program, which is something non-trivial. Among others, before getting into such things, we will have to study the von Neumann algebras A having trivial center, $Z(A)=\mathbb{C}$, called factors, which include the fibers A_x in the above decomposition result. More on this later.

2c. Random matrices

Our main results so far on the von Neumann algebras concern the finite dimensional case, where the algebra is of the form $A=\oplus_i M_{n_i}(\mathbb{C})$, and the commutative case, where the algebra is of the form $A=L^\infty(X)$. In order to advance, we must solve:

QUESTION 2.16. *What are the next simplest von Neumann algebras, generalizing at the same time the finite dimensional ones, $A=\oplus_i M_{n_i}(\mathbb{C})$, and the commutative ones, $A=L^\infty(X)$, that we can use as input for our study?*

In this formulation, our question is a no-brainer, the answer to it being that of looking at the direct integrals of matrix algebras, over an arbitrary measured space X:

$$A=\int_X M_{n_x}(\mathbb{C})dx$$

However, when thinking a bit, all this looks quite tricky, with most likely lots of technical functional analysis and measure theory involved. So, we will leave the investigation of such algebras, which are indeed quite basic, and called of type I, for later.

Nevermind. Let us replace Question 2.16 with something more modest, as follows:

QUESTION 2.17 (update). *What are the next simplest von Neumann algebras, generalizing at the same time the usual matrix algebras, $A=M_N(\mathbb{C})$, and the commutative ones, $A=L^\infty(X)$, that we can use as input for our study?*

But here, what we have is again a no-brainer, because in relation to what has been said above, we just have to restrict the attention to the "isotypic" case, where all fibers are isomorphic. And in this case our algebra is a random matrix algebra:

$$A=\int_X M_N(\mathbb{C})dx$$

Which looks quite nice, and so good news, we have our algebras. In practice now, although there is some functional analysis to be done with these algebras, the main questions regard the individual operators $T\in A$, called random matrices. Thus, we are basically back to good old operator theory. Let us begin our discussion with:

Definition 2.18. *A random matrix algebra is a von Neumann algebra of the following type, with X being a probability space, and with $N \in \mathbb{N}$ being an integer:*

$$A = M_N(L^\infty(X))$$

In other words, A appears as a tensor product, as follows,

$$A = M_N(\mathbb{C}) \otimes L^\infty(X)$$

of a matrix algebra and a commutative von Neumann algebra.

As a first observation, our algebra can be written as well as follows, with this latter convention being quite standard in the probability literature:

$$A = L^\infty(X, M_N(\mathbb{C}))$$

In connection with the tensor product notation, which is often the most useful one for computations, we have as well the following possible writing, also used in probability:

$$A = L^\infty(X) \otimes M_N(\mathbb{C})$$

Importantly now, each random matrix algebra A is naturally endowed with a canonical von Neumann algebra trace $tr : A \to \mathbb{C}$, which appears as follows:

Proposition 2.19. *Given a random matrix algebra $A = M_N(L^\infty(X))$, consider the linear form $tr : A \to \mathbb{C}$ given by:*

$$tr(T) = \frac{1}{N}\sum_{i=1}^N \int_X T_{ii}^x dx$$

In tensor product notation, $A = M_N(\mathbb{C}) \otimes L^\infty(X)$, we have then the formula

$$tr = \frac{1}{N} Tr \otimes \int_X$$

and this functional $tr : A \to \mathbb{C}$ is a faithful positive unital trace.

Proof. The first assertion, regarding the tensor product writing of tr, is clear from definitions. As for the second assertion, regarding the various properties of tr, this follows from this, because these properties are stable under taking tensor products. □

As before, there is a discussion here in connection with the other possible writings of A. With the probabilistic notation $A = L^\infty(X, M_N(\mathbb{C}))$, the trace appears as:

$$tr(T) = \int_X \frac{1}{N} Tr(T^x)\, dx$$

Also, with the probabilistic tensor notation $A = L^\infty(X) \otimes M_N(\mathbb{C})$, the trace appears exactly as in the second part of Proposition 2.19, with the order inverted:

$$tr = \int_X \otimes \frac{1}{N} Tr$$

To summarize, the random matrix algebras appear to be very basic objects, and the only difficulty, in the beginning, lies in getting familiar with the 4 possible notations for them. Or perhaps 5 possible notations, because we have $A = \int_X M_N(\mathbb{C})dx$ as well.

Getting to work now, as already said, the main questions about random matrix algebras regard the individual operators $T \in A$, called random matrices. To be more precise, we are interested in computing the laws of such matrices, constructed according to:

THEOREM 2.20. *Given an operator algebra $A \subset B(H)$ with a faithful trace $tr : A \to \mathbb{C}$, any normal element $T \in A$ has a law, namely a probability measure μ satisfying*

$$tr(T^k) = \int_{\mathbb{C}} z^k d\mu(z)$$

with the powers being with respect to colored exponents $k = \circ\bullet\bullet\circ\ldots$, defined via

$$a^\emptyset = 1 \quad , \quad a^\circ = a \quad , \quad a^\bullet = a^*$$

and multiplicativity. This law is unique, and is supported by the spectrum $\sigma(T) \subset \mathbb{C}$. In the non-normal case, $TT^* \neq T^*T$, such a law does not exist.

PROOF. We have two assertions here, the idea being as follows:

(1) In the normal case, $TT^* = T^*T$, we know from chapter 1, based on the continuous functional calculus theorem, that we have:

$$< T >= C(\sigma(T))$$

Thus the functional $f(T) \to tr(f(T))$ can be regarded as an integration functional on the algebra $C(\sigma(T))$, and by the Riesz theorem, this latter functional must come from a probability measure μ on the spectrum $\sigma(T)$, in the sense that we must have:

$$tr(f(T)) = \int_{\sigma(T)} f(z)d\mu(z)$$

We are therefore led to the conclusions in the statement, with the uniqueness assertion coming from the fact that the operators T^k, taken as usual with respect to colored integer exponents, $k = \circ\bullet\bullet\circ\ldots$, generate the whole operator algebra $C(\sigma(T))$.

(2) In the non-normal case now, $TT^* \neq T^*T$, we must show that such a law does not exist. For this purpose, we can use a positivity trick, as follows:

$$\begin{aligned}
TT^* - T^*T \neq 0 \quad &\implies \quad (TT^* - T^*T)^2 > 0\\
&\implies \quad TT^*TT^* - TT^*T^*T - T^*TTT^* + T^*TT^*T > 0\\
&\implies \quad tr(TT^*TT^* - TT^*T^*T - T^*TTT^* + T^*TT^*T) > 0\\
&\implies \quad tr(TT^*TT^* + T^*TT^*T) > tr(TT^*T^*T + T^*TTT^*)\\
&\implies \quad tr(TT^*TT^*) > tr(TTT^*T^*)
\end{aligned}$$

Now assuming that T has a law $\mu \in \mathcal{P}(\mathbb{C})$, in the sense that the moment formula in the statement holds, the above two different numbers would have to both appear by integrating $|z|^2$ with respect to this law μ, which is contradictory, as desired. □

Back now to the random matrices, as a basic example, assume $X = \{.\}$, so that we are dealing with a usual scalar matrix, $T \in M_N(\mathbb{C})$. By changing the basis of $\mathbb{C}^N$, which won't affect our trace computations, we can assume that T is diagonal:

$$T \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}$$

But for such a diagonal matrix, we have the following formula:

$$tr(T^k) = \frac{1}{N}(\lambda_1^k + \ldots + \lambda_N^k)$$

Thus, the law of T is the average of the Dirac masses at the eigenvalues:

$$\mu = \frac{1}{N}\left(\delta_{\lambda_1} + \ldots + \delta_{\lambda_N}\right)$$

As a second example now, assume $N = 1$, and so $T \in L^\infty(X)$. In this case we obtain the usual law of T, because the equation to be satisfied by μ is:

$$\int_X \varphi(T) = \int_{\mathbb{C}} \varphi(x) d\mu(x)$$

At a more advanced level, the main problem regarding the random matrices is that of computing the law of various classes of such matrices, coming in series:

Question 2.21. *What is the law of random matrices coming in series*

$$T_N \in M_N(L^\infty(X))$$

in the $N >> 0$ regime?

The general strategy here, coming from physicists, is that of computing first the asymptotic law μ^0, in the $N \to \infty$ limit, and then looking for the higher order terms as well, as to finally reach to a series in N^{-1} giving the law of T_N, as follows:

$$\mu_N = \mu^0 + N^{-1}\mu^1 + N^{-2}\mu^2 + \ldots$$

As a basic example here, of particular interest are the random matrices having i.i.d. complex normal entries, under the constraint $T = T^*$. Here the asymptotic law μ^0 is the Wigner semicircle law on $[-2, 2]$. We will discuss this later in this book.

2d. Quantum spaces

Let us end this preliminary chapter on operator algebras with some philosophy, a bit a la Heisenberg. In relation with general "quantum space" goals, Theorem 2.13 is something very interesting, philosophically speaking, suggesting us to formulate:

Definition 2.22. *Given a von Neumann algebra $A\subset B(H)$, we write*

$$A=L^\infty(X)$$

and call X a quantum measured space.

As an example here, for the simplest noncommutative von Neumann algebra that we know, namely the usual matrix algebra $A=M_N(\mathbb C)$, the formula that we want to write is as follows, with M_N being a certain mysterious quantum space:

$$M_N(\mathbb C)=L^\infty(M_N)$$

So, what can we say about this space M_N? As a first observation, this is a finite space, with its cardinality being defined and computed as follows:

$$|M_N|=\dim_{\mathbb C}M_N(\mathbb C)=N^2$$

Now since this is the same as the cardinality of the set $\{1,\ldots,N^2\}$, we are led to the conclusion that we should have a twisting result as follows, with the twisting operation $X\to X^\sigma$ being something that destroys the points, but keeps the cardinality:

$$M_N=\{1,\ldots,N^2\}^\sigma$$

From an analytic viewpoint now, we would like to understand what is the integration over M_N, giving rise to the corresponding L^∞ functions. And here, we can set:

$$\int_{M_N}A=tr(A)$$

To be more precise, on the left we have the integral of an arbitrary function on M_N, which according to our conventions, should be a usual matrix:

$$A\in L^\infty(M_N)=M_N(\mathbb C)$$

As for the quantity on the right, the outcome of the computation, this can only be the trace of A. In addition, it is better to choose this trace to be normalized, by $tr(1)=1$, and this in order for our measure on M_N to have mass 1, as it is ideal:

$$tr(A)=\frac{1}{N}Tr(A)$$

We can say even more about this. Indeed, since the traces of positive matrices are positive, we are led to the following formula, to be taken with the above conventions, which shows that the measure on M_N that we constructed is a probability measure:

$$A > 0 \implies \int_{M_N} A > 0$$

Before going further, let us record what we found, for future reference:

THEOREM 2.23. *The quantum measured space M_N formally given by*

$$M_N(\mathbb{C}) = L^\infty(M_N)$$

has cardinality N^2, appears as a twist, in a purely algebraic sense,

$$M_N = \{1, \ldots, N^2\}^\sigma$$

and is a probability space, its uniform integration being given by

$$\int_{M_N} A = tr(A)$$

where at right we have the normalized trace of matrices, $tr = Tr/N$.

PROOF. This is something half-informal, mostly for fun, which basically follows from the above discussion, the details and missing details being as follows:

(1) In what regards the formula $|M_N| = N^2$, coming by computing the complex vector space dimension, as explained above, this is obviously something rock-solid.

(2) Regarding twisting, we would like to have a formula as follows, with the operation $A \to A^\sigma$ being something that destroys the commutativity of the multiplication:

$$L^\infty(M_N) = L^\infty(1, \ldots, N^2)^\sigma$$

In more familiar terms, with usual complex matrices on the left, and with a better-looking product of sets being used on the right, this formula reads:

$$M_N(\mathbb{C}) = L^\infty\Big(\{1, \ldots, N\} \times \{1, \ldots, N\}\Big)^\sigma$$

In order to establish this formula, consider the algebra on the right. As a complex vector space, this algebra has the standard basis $\{f_{ij}\}$ formed by the Dirac masses at the points (i, j), and the multiplicative structure of this algebra is given by:

$$f_{ij} f_{kl} = \delta_{ij,kl}$$

Now let us twist this multiplication, according to the formula $e_{ij} e_{kl} = \delta_{jk} e_{il}$. We obtain in this way the usual combination formulae for the standard matrix units $e_{ij} : e_j \to e_i$ of the algebra $M_N(\mathbb{C})$, and so we have our twisting result, as claimed.

(3) In what regards the integration formula in the statement, with the conclusion that the underlying measure on M_N is a probability one, this is something that we fully explained before, and as for the result (1) above, it is something rock-solid.

(4) As a last technical comment, observe that the twisting operation performed in (2) destroys both the involution, and the trace of the algebra. This is something quite interesting, which cannot be fixed, and we will back to it, later on. □

In order to advance now, based on the above result, the key point there is the construction and interpretation of the trace $tr : M_N(\mathbb{C}) \to \mathbb{C}$, as an integration functional. But this leads us into the following natural, and quite puzzling question:

QUESTION 2.24. *In the general context of Definition 2.22, where we formally wrote $A = L^\infty(X)$, what is the underlying integration functional $tr : A \to \mathbb{C}$?*

This is a quite subtle question, and there are several possible answers here. For instance, we would like the integration functional to have the following property:

$$tr(ab) = tr(ba)$$

And the problem is that certain von Neumann algebras do not possess such traces. This is actually something quite advanced, that we do not know yet, but by anticipating a bit, we are in trouble, and we must modify Definition 2.22, as follows:

DEFINITION 2.25 (update). *Given a von Neumann algebra $A \subset B(H)$, coming with a faithful positive unital trace $tr : A \to \mathbb{C}$, we write*

$$A = L^\infty(X)$$

and call X a quantum probability space. We also write the trace as $tr = \int_X$, and call it integration with respect to the uniform measure on X.

At the level of examples, passed the classical probability spaces X, we know from Theorem 2.23 that the quantum space M_N is a finite quantum probability space. But this raises the question of understanding what the finite quantum probability spaces are, in general. For this purpose, we need to examine the finite dimensional von Neumann algebras. And the result here, extending Theorem 2.8, is as follows:

THEOREM 2.26. *The finite dimensional von Neumann algebras $A \subset B(H)$ over an arbitrary Hilbert space H are exactly the direct sums of matrix algebras,*

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

embedded into $B(H)$ by using a partition of unity of $B(H)$ with rank 1 projections

$$1 = P_1 + \ldots + P_k$$

with the "factors" $M_{n_i}(\mathbb{C})$ being each embedded into the algebra $P_iB(H)P_i$.

PROOF. This is standard, as in the case $A \subset M_N(\mathbb{C})$. Consider the center of A, which is a finite dimensional commutative von Neumann algebra, of the following form:

$$Z(A) = \mathbb{C}^k$$

Now let P_i be the Dirac mass at $i \in \{1, \ldots, k\}$. Then $P_i \in B(H)$ is an orthogonal projection, and these projections form a partition of unity, as follows:

$$1 = P_1 + \ldots + P_k$$

With $A_i = P_i A P_i$, we have then a non-unital $*$-algebra decomposition, as follows:

$$A = A_1 \oplus \ldots \oplus A_k$$

On the other hand, it follows from the minimality of each of the projections $P_i \in Z(A)$ that we have unital $*$-algebra isomorphisms $A_i \simeq M_{n_i}(\mathbb{C})$, and this gives the result. $\square$

We can now deduce what the finite quantum measured spaces are, in the sense of the old Definition 2.22. Indeed, we must solve here the following equation:

$$L^\infty(X) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

Now since the direct unions of sets correspond to direct sums at the level of the associated algebras of functions, in the classical case, we can take the following formula as a definition for a direct union of sets, in the general, noncommutative case:

$$L^\infty(X_1 \sqcup \ldots \sqcup X_k) = L^\infty(X_1) \oplus \ldots \oplus L^\infty(X_k)$$

With this, and by remembering the definition of M_N, we are led to the conclusion that the solution to our quantum measured space equation above is as follows:

$$X = M_{n_1} \sqcup \ldots \sqcup M_{n_k}$$

For fully solving our problem, in the spirit of the new Definition 2.25, we still have to discuss the traces on $L^\infty(X)$. We are led in this way to the following statement:

Theorem 2.27. *The finite quantum measured spaces are the spaces*

$$X = M_{n_1} \sqcup \ldots \sqcup M_{n_k}$$

according to the following formula, for the associated algebras of functions:

$$L^\infty(X) = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

The cardinality $|X|$ of such a space is the following number,

$$N = n_1^2 + \ldots + n_k^2$$

and the possible traces are as follows, with $\lambda_i > 0$ summing up to 1:

$$tr = \lambda_1 tr_1 \oplus \ldots \oplus \lambda_k tr_k$$

Among these traces, we have the canonical trace, appearing as

$$tr : L^\infty(X) \subset \mathcal{L}(L^\infty(X)) \to \mathbb{C}$$

via the left regular representation, having weights $\lambda_i = n_i^2/N$.

PROOF. We have many assertions here, basically coming from the above discussion, with only the last one needing some explanations. Consider the left regular representation of our algebra $A = L^\infty(X)$, which is given by the following formula:

$$\pi : A \subset \mathcal{L}(A) \quad , \quad \pi(a) : b \to ab$$

We know that the algebra $\mathcal{L}(A)$ of linear operators $T : A \to A$ is isomorphic to a matrix algebra, and more specifically to $M_N(\mathbb{C})$, with $N = |X|$ being as before:

$$\mathcal{L}(A) \simeq M_N(\mathbb{C})$$

Thus, this algebra has a trace $tr : \mathcal{L}(A) \to \mathbb{C}$, and by composing this trace with the representation π, we obtain a certain trace $tr : A \to \mathbb{C}$, that we can call "canonical":

$$tr : A \subset \mathcal{L}(A) \to \mathbb{C}$$

We can compute the weights of this trace by using a multimatrix basis of A, formed by matrix units e^i_{ab}, with $i \in \{1, \ldots, k\}$ and with $a, b \in \{1, \ldots, n_i\}$, and we obtain:

$$\lambda_i = \frac{n_i^2}{N}$$

Thus, we are led to the conclusion in the statement. □

We will be back to quantum spaces on several occasions, in what follows. In fact, the present book is as much on operator algebras as it is on quantum spaces, and this because these two points of view are both useful, and complementary to each other.

2e. Exercises

Exercises:

EXERCISE 2.28.

EXERCISE 2.29.

EXERCISE 2.30.

EXERCISE 2.31.

EXERCISE 2.32.

EXERCISE 2.33.

EXERCISE 2.34.

EXERCISE 2.35.

Bonus exercise.

CHAPTER 3

Functional analysis

3a. Kaplansky density

Time now for some more advanced von Neumann algebra theory, and hang on, all this will be quite technical, using as main ingredient some fine functional analysis.

As a first objective, we would like to have a better understanding of the precise difference between the norm closed $*$-algebras, or C^*-algebras, $A \subset B(H)$, that we discussed in some detail at the end of chapter 1, and the weakly closed such algebras, which are the von Neumann algebras from chapter 2, from a functional analytic viewpoint.

Let us begin our study with some generalities. We first have:

PROPOSITION 3.1. *The weak operator topology on $B(H)$ is the topology having the following equivalent properties:*

1. *It makes $T \to < Tx, y >$ continuous, for any $x, y \in H$.*
2. *It makes $T_n \to T$ when $< T_n x, y > \to < Tx, y >$, for any $x, y \in H$.*
3. *Has as subbase the sets $U_T(x, y, \varepsilon) = \{S : | < (S - T)x, y > | < \varepsilon\}$.*
4. *Has as base $U_T(x_1, \ldots, x_n, y_1, \ldots, y_n, \varepsilon) = \{S : | < (S - T)x_i, y_i > | < \varepsilon, \forall i\}$.*

PROOF. The equivalences (1) $\iff$ (2) $\iff$ (3) $\iff$ (4) all follow from definitions, with of course (1,2) referring to the coarsest topology making that things happen. □

Similarly, in what regards the strong operator topology, we have:

PROPOSITION 3.2. *The strong operator topology on $B(H)$ is the topology having the following equivalent properties:*

1. *It makes $T \to Tx$ continuous, for any $x \in H$.*
2. *It makes $T_n \to T$ when $T_n x \to Tx$, for any $x \in H$.*
3. *Has as subbase the sets $V_T(x, \varepsilon) = \{S : ||(S - T)x|| < \varepsilon\}$.*
4. *Has as base the sets $V_T(x_1, \ldots, x_n, \varepsilon) = \{S : ||(S - T)x_i|| < \varepsilon, \forall i\}$.*

PROOF. Again, the equivalences (1) $\iff$ (2) $\iff$ (3) $\iff$ (4) are all clear, and with (1,2) referring to the coarsest topology making that things happen. □

We know from chapter 2 that an operator algebra $A \subset B(H)$ is weakly closed if and only if it is strongly closed. Here is a useful generalization of this fact:

THEOREM 3.3. *Given a convex set of bounded operators*

$$C \subset B(H)$$

its weak operator closure and strong operator closure coincide.

PROOF. Since the weak operator topology on $B(H)$ is weaker by definition than the strong operator topology on $B(H)$, we have, for any subset $C \subset B(H)$:

$$\overline{C}^{strong} \subset \overline{C}^{weak}$$

Now by assuming that $C \subset B(H)$ is convex, we must prove that:

$$T \in \overline{C}^{weak} \implies T \in \overline{C}^{strong}$$

In order to do so, let us pick vectors $x_1, \ldots, x_n \in H$ and $\varepsilon > 0$. We let $K = H^{\oplus n}$, and we consider the standard embedding $i : B(H) \subset B(K)$, given by:

$$iT(y_1, \ldots, y_n) = (Ty_1, \ldots, Ty_n)$$

We have then the following implications, which are all trivial:

$$T \in \overline{C}^{weak} \implies iT \in \overline{iC}^{weak} \implies iT(x) \in \overline{iC(x)}^{weak}$$

Now since the set $C \subset B(H)$ was assumed to be convex, the set $iC(x) \subset K$ is convex too, and by the Hahn-Banach theorem, for compact sets, it follows that we have:

$$iT(x) \in \overline{iC(x)}^{||.||}$$

Thus, there exists an operator $S \in C$ such that we have, for any i:

$$||Sx_i - Tx_i|| < \varepsilon$$

But this shows that we have $S \in V_T(x_1, \ldots, x_n, \varepsilon)$, and since $x_1, \ldots, x_n \in H$ and $\varepsilon > 0$ were arbitrary, by Proposition 3.2 it follows that we have $T \in \overline{C}^{strong}$, as desired. □

We will need as well the following standard result:

PROPOSITION 3.4. *Given a vector space $E \subset B(H)$, and a linear form $f : E \to \mathbb{C}$, the following conditions are equivalent:*

(1) *f is weakly continuous.*
(2) *f is strongly continuous.*
(3) *$f(T) = \sum_{i=1}^n < Tx_i, y_i >$, for certain vectors $x_i, y_i \in H$.*

PROOF. This is something standard, using the same tools at those already used in chapter 5, namely basic functional analysis, and amplification tricks:

(1) $\implies$ (2) Since the weak operator topology on $B(H)$ is weaker than the strong operator topology on $B(H)$, weakly continuous implies strongly continuous. To be more precise, assume $T_n \to T$ strongly. Then $T_n \to T$ weakly, and since f was assumed to be weakly continuous, we have $f(T_n) \to f(T)$. Thus f is strongly continuous, as desired.

(2) $\implies$ (3) Assume indeed that our linear form $f : E \to \mathbb{C}$ is strongly continuous. In particular f is strongly continuous at 0, and Proposition 3.2 provides us with vectors $x_1, \ldots, x_n \in H$ and a number $\varepsilon > 0$ such that, with the notations there:

$$f(V_0(x_1, \ldots, x_n, \varepsilon)) \subset D_0(1)$$

That is, we can find vectors $x_1, \ldots, x_n \in H$ and a number $\varepsilon > 0$ such that:

$$||Tx_i|| < \varepsilon, \forall i \implies |f(T)| < 1$$

But this shows that we have the following estimate:

$$\sum_{i=1}^n ||Tx_i||^2 < \varepsilon^2 \implies |f(T)| < 1$$

By linearity, it follows from this that we have the following estimate:

$$|f(T)| < \frac{1}{\varepsilon}\sqrt{\sum_{i=1}^n ||Tx_i||^2}$$

Consider now the direct sum $H^{\oplus n}$, and inside it, the following vector:

$$x = (x_1, \ldots, x_n) \in H^{\oplus n}$$

Consider also the following linear space, written in tensor product notation:

$$K = \overline{(E \otimes 1)x} \subset H^{\oplus n}$$

We can define a linear form $f' : K \to \mathbb{C}$ by the following formula, and continuity:

$$f'(Tx_1, \ldots, Tx_n) = f(T)$$

We conclude that there exists a vector $y \in K$ such that the following happens:

$$f'\big((T \otimes 1)y\big) = < (T \otimes 1)x, y >$$

But in terms of the original linear form $f : E \to \mathbb{C}$, this means that we have:

$$f(T) = \sum_{i=1}^n < Tx_i, y_i >$$

(3) $\implies$ (1) This is clear, because we have, with respect to the weak topology:

$$\begin{aligned} T_n \to T \quad &\implies \quad < T_n x_i, y_i > \to < Tx_i, y_i >, \forall i \\ &\implies \quad \sum_{i=1}^n < T_n x_i, y_i > \to \sum_{i=1}^n < Tx_i, y_i > \\ &\implies \quad f(T_n) \to f(T) \end{aligned}$$

Thus, our linear form f is weakly continuous, as desired. □

Here is one more well-known result, that we will need as well:

THEOREM 3.5. *The unit ball of $B(H)$ is weakly compact.*

PROOF. If we denote by $B_1 \subset B(H)$ the unit ball, and by $D_1 \subset \mathbb{C}$ the unit disk, we have a morphism as follows, which is continuous with respect to the weak topology on B_1, and with respect to the product topology on the set on the right:

$$B_1 \subset \prod_{||x||,||y||\leq 1} D_1 \quad , \quad T \to (< Tx, y >)_{x,y}$$

Since the set on the right is compact, by Tychonoff, it is enough to show that the image of B_1 is closed. So, let $(c_{xy}) \in \bar{B}_1$. We can then find $T_i \in B_1$ such that:

$$< T_i x, y > \to c_{xy} \quad , \quad \forall x, y$$

But this shows that the following map is a bounded sesquilinear form:

$$H \times H \to \mathbb{C} \quad , \quad (x, y) \to c_{xy}$$

Thus, we can find an operator $T \in B(H)$, and so $T \in B_1$, such that $< Tx, y >= c_{xy}$ for any $x, y \in H$, and this shows that we have $(c_{xy}) \in B_1$, as desired. □

Getting back to operator algebras, we have the following result, due to Kaplansky, which is something very useful, and of independent interest as well:

THEOREM 3.6. *Given an operator algebra $A \subset B(H)$, the following happen:*

(1) *The unit ball of A is strongly dense in the unit ball of A''.*
(2) *The same happens for the self-adjoint parts of the above unit balls.*

PROOF. This is something quite tricky, the idea being as follows:

(1) Consider the self-adjoint part $A_{sa} \subset A$. By taking real parts of operators, and using the fact that $T \to T^*$ is weakly continuous, we have then:

$$\overline{A_{sa}}^w \subset \left(\overline{A}^w\right)_{sa}$$

Now since the set A_{sa} is convex, and by Theorem 3.3 all weak operator topologies coincide on the convex sets, we conclude that we have in fact equality:

$$\overline{A_{sa}}^w = \left(\overline{A}^w\right)_{sa}$$

(2) With this result in hand, let us prove now the second assertion of the theorem. For this purpose, consider an element $T \in \overline{A}^w$, satisfying $T = T^*$ and $||T|| \leq 1$. Consider as well the following function, going from the interval $[-1, 1]$ to itself:

$$f(t) = \frac{2t}{1 + t^2}$$

By functional calculus we can find an element $S \in \left(\overline{A}^w\right)_{sa}$ such that:

$$f(S) = T$$

In other words, we can find an element $S\in\left(\overline{A}^w\right)_{sa}$ such that:

$$T=\frac{2S}{1+S^2}$$

Now given arbitrary vectors $x_1,\ldots,x_n\in H$ and an arbitrary number $\varepsilon>0$, let us pick an element $R\in A_{sa}$, subject to the following two inequalities:

$$||RTx_i-STx_i||\leq\varepsilon\quad,\quad\left|\left|\frac{R}{1+S^2}x_i-\frac{S}{1+S^2}x_i\right|\right|\leq\varepsilon$$

Finally, consider the following element, which has norm ≤ 1:

$$L=\frac{2R}{1+R^2}$$

We have then the following computation, using the above formulae:

$$\begin{aligned}
L-T &= \frac{2R}{1+R^2}-\frac{2S}{1+S^2}\\
&= 2\left(\frac{1}{1+R^2}\big(R(1+S^2)-(1+S^2)R\big)\frac{1}{1+S^2}\right)\\
&= 2\left(\frac{1}{1+R^2}(R-S)\frac{1}{1+S^2}+\frac{R}{1+R^2}(S-R)\frac{S}{1+S^2}\right)\\
&= \frac{2}{1+R^2}(R-S)\frac{1}{1+S^2}+\frac{L}{2}(S-R)T
\end{aligned}$$

Thus, we have the following estimate, for any $i\in\{1,\ldots,n\}$:

$$||(L-T)x_i||\leq\varepsilon$$

But this gives the second assertion of the theorem, as desired.

(3) Let us prove now the first assertion of the theorem. Given an arbitrary element $T\in\overline{A}^w$, satisfying $||T||\leq 1$, let us look at the following element:

$$T'=\begin{pmatrix}0&T\\T^*&0\end{pmatrix}\in M_2(\overline{A}^w)$$

This element is then self-adjoint, and we can use what we proved in the above, and we are led in this way to the first assertion in the statement, as desired. □

We can go back now to our original question, from the beginning of the present chapter, namely that of abstractly characterizing the von Neumann algebras, and we have:

THEOREM 3.7. *A norm closed operator $*$-algebra*

$$A\subset B(H)$$

is a von Neumann algebra precisely when its unit ball is weakly compact.

Proof. This is something which is now clear, coming from the Kaplansky density results established in Theorem 3.6. To be more precise:

(1) In one sense, assuming that $A\subset B(H)$ is a von Neumann algebra, this algebra is weakly closed. But since the unit ball of $B(H)$ is weakly compact, we are led to the conclusion that the unit ball of A is weakly compact too.

(2) Conversely, assume that an operator algebra $A\subset B(H)$ is such that its unit ball is weakly compact. In particular, the unit ball of A is weakly closed. Now if T satisfying $||T||\leq 1$ belongs to the weak closure of A, by Kaplansky density we conclude that we have $T\in A$. Thus our algebra A must be a von Neumann algebra, as claimed. □

3b. Projections, order

In order to further investigate the von Neumann algebras, the key idea, coming from the analysis of the finite dimensional algebras from chapter 2, will be that of looking at the projections. Let us start with some generalities. In analogy with what happens in finite dimensions, we have the following notions, over an arbitrary Hilbert space H:

Definition 3.8. *Associated to any two projections $P,Q\in B(H)$ are:*

(1) *The projection $P\wedge Q$, projecting on the common range.*
(2) *The projection $P\vee Q$, projecting on the span of the ranges.*

Abstractly speaking, these two operations can be thought of as being inf and sup type operations, and all the known algebraic formulae for inf and sup hold in this setting. For the moment we will not need all this, and we will be back to it later. Let us record however the following basic formula, which is something very useful:

Proposition 3.9. *We have the following formula,*

$$P+Q=P\wedge Q+P\vee Q$$

valid for any two projections $P,Q\in B(H)$.

Proof. This is clear from definitions, because when computing $P+Q$ we obtain the projection $P\vee Q$ on the span on the ranges, modulo the fact that the vectors in the common range are obtained twice, which amounts in saying that we must add $P\wedge Q$. □

With the above notions in hand, we have the following result:

Theorem 3.10. *Consider two projections $P,Q\in B(H)$.*

(1) *In finite dimensions, over $H=\mathbb{C}^N$, we have, in norm:*

$$(PQ)^n\to P\wedge Q$$

(2) *In infinite dimensions, we have the following convergence, for any $x\in H$,*

$$(PQ)^n x\to (P\wedge Q)x$$

but the operators $(PQ)^n$ do not necessarily converge in norm.

Proof. We have several assertions here, the proof being as follows:

(1) Assume that we are in the case $P,Q\in M_N(\mathbb{C})$. By substracting $P\wedge Q$ from both P,Q, we can assume $P\wedge Q=0$, and we must prove that we have:

$$P\wedge Q=0\implies (PQ)^n\to 0$$

Our claim is that we have $||PQ||<1$. Indeed, we know that we have:

$$||PQ||\leq ||P||\cdot||Q||=1$$

Assuming now by contradiction that we have $||PQ||=1$, since we are in finite dimensions, we must have, for a certain norm one vector, $||x||=1$:

$$||PQx||=1$$

Thus, we must have equalities in the following estimate:

$$||PQx||\leq||Qx||\leq||x||$$

But the second equality tells us that we must have $x\in Im(Q)$, and with this in hand, the first equality tells us that we must have $x\in Im(P)$. But this contradicts $P\wedge Q=0$, so we have proved our claim, and the convergence $(PQ)^n\to 0$ follows.

(2) In infinite dimensions now, as before by substracting $P\wedge Q$ from both P,Q, we can assume $P\wedge Q=0$, and we must prove that we have, for any $x\in H$:

$$P\wedge Q=0\implies (PQ)^nx\to 0$$

For this purpose, we use a trick. Consider the following operator:

$$R=PQP$$

This operator is positive, because we have $R=(PQ)(PQ)^*$, and we have:

$$||R||\leq||P||\cdot||Q||\cdot||P||=1$$

Our claim, which will finish the proof, is that for any $x\in H$ we have:

$$R^nx\to 0$$

In order to prove this claim, let us diagonalize R, by using the spectral theorem for self-adjoint operators, from chapter 1. If all the eigenvalues are <1 then we are done. If not, this means that we can find a nonzero vector $x\in H$ such that:

$$||Rx||=||x||$$

But this condition means that we must have equalities in the following estimate:

$$||PQPx||\leq||QPx||\leq||Px||\leq||x||$$

The point now is that this is impossible, due to our assumption $P\wedge Q=0$. Indeed, the last equality tells us that we must have $x\in Im(P)$, and with this in hand, the middle equality tells us that we must have $x\in Im(Q)$. But this contradicts $P\wedge Q=0$, so we have proved our claim, and the convergence $(PQ)^nx\to 0$ follows.

(3) Finally, for a counterexample to $(PQ)^n \to 0$, in infinite dimensions, we can take $H = l^2(\mathbb{N})$, and then find projections P, Q such that $(PQ)^n e_k \to 0$ for any k, but with the convergence arbitrarily slowing down with $k \to \infty$. Thus, $(PQ)^n \not\to 0$. $\square$

As a consequence, in connection with the von Neumann algebras, we have:

THEOREM 3.11. *Given two projections $P, Q \in B(H)$, the projections*

$$P \wedge Q \quad , \quad P \vee Q$$

both belong to the von Neumann algebra generated by P, Q.

PROOF. This comes from the above. Indeed, in what regards $P \wedge Q$, this is something that follows from Theorem 3.10. As for $P \vee Q$, here the result follows from the result for $P \wedge Q$, and from the formula $P + Q = P \wedge Q + P \vee Q$, from Proposition 3.9. $\square$

The idea now will be that of studying the von Neumann algebras $A \subset B(H)$ by using their projections, $p \in A$. Let us start with the following result:

THEOREM 3.12. *Any von Neumann algebra is generated by its projections.*

PROOF. This is something that we know from chapter 2, coming from the measurable functional calculus, which can cut any normal operator into projections. $\square$

There are many other things that can be said about projections, in the general setting. In what follows we will just discuss the most important and useful such results. A first such result, providing us with some geometric intuition on projections, is as follows:

THEOREM 3.13. *Given a von Neumann algebra $A \subset B(H)$, and a projection $p \in A$, we have the following equalities, between von Neumann algebras on pH:*

(1) $pAp = (A'p)'$.
(2) $(pAp)' = A'p$.

PROOF. This is not exactly obvious, but can be proved as follows:

(1) As a first observation, the von Neumann algebras pAp and $A'p$ commute on pH. Thus, we must prove that we have the following implication:

$$x \in (A'p)' \implies x \in pAp$$

For this purpose, consider the element $y = xp$. Then for any $z \in A'$ we have:

$$\begin{aligned} zy &= zxp \\ &= zpxp \\ &= xpzp \\ &= xpz \\ &= yz \end{aligned}$$

But this shows that we have $y \in A$, and so we obtain, as desired:

$$x = pyp \in pAp$$

(2) As before, one of the inclusions being clear, we must prove that we have:

$$x \in (pAp)' \implies x \in A'p$$

By using the standard fact that any bounded operator appears as a linear combination of 4 unitaries, that we know from the end of chapter 1, it is enough to prove this for a unitary element, $x = u$. So, assume that we have a unitary as follows:

$$u \in (pAp)'$$

In order to prove our claim, consider the following vector space:

$$K = \overline{ApH}$$

This space being invariant under both the algebras A, A', we conclude that the projection $q = Proj(K)$ onto it belongs to the center of our von Neumann algebra:

$$q \in Z(A)$$

Our claim now, which will quickly lead to the result that we want to prove, is that we can extend the above unitary $u \in (pAp)'$ to the space $K = \overline{ApH}$ via the following formula, valid for any elements $x_i \in A$, and any vectors $\xi_i \in pH$:

$$v\left(\sum_i x_i\xi_i\right) = \sum_i x_i u\xi_i$$

In order to prove this latter claim, we can use the following computation:

$$\begin{aligned}
\left\|v\left(\sum_i x_i\xi_i\right)\right\|^2 &= \sum_{ij} < x_iu\xi_i, x_ju\xi_j > \\
&= \sum_{ij} < x_j^*x_iu\xi_i, u\xi_j > \\
&= \sum_{ij} < px_j^*x_ipu\xi_i, u\xi_j > \\
&= \sum_{ij} < upx_j^*x_ip\xi_i, u\xi_j > \\
&= \sum_{ij} < px_j^*x_ip\xi_i, \xi_j > \\
&= \sum_{ij} < x_j^*x_i\xi_i, \xi_j > \\
&= \sum_{ij} < x_i\xi_i, x_j\xi_j > \\
&= \left\|\sum_i x_i\xi_i\right\|^2
\end{aligned}$$

Thus v is well-defined by the above formula, and is an isometry of K. Now observe that this element v commutes with the algebra A on the space ApH, and so on K. Thus $vq \in A'$, and so $u = vqp$, which proves that we have $u \in A'p$, as desired. ☐

As a second result now, once again in the general setting, we have:

PROPOSITION 3.14. *Given a von Neumann algebra $A \subset B(H)$, the formula*

$$p \simeq q \iff \exists u, \begin{cases} uu^* = p \\ u^*u = q \end{cases}$$

defines an equivalence relation for the projections $p \in A$.

PROOF. This is something elementary, which follows from definitions, with the transitivity coming by composing the corresponding partial isometries. ☐

As a third result, once again in the general setting, which once again provides us with some intuition, but this time of somewhat abstract type, we have:

THEOREM 3.15. *Given a von Neumann algebra $A\subset B(H)$, we have a partial order on the projections $p\in A$, constructed as follows, with u being a partial isometry,*

$$p\preceq q\iff\exists u,\ \begin{cases}uu^*=p\\ u^*u\leq q\end{cases}$$

which is related to the equivalence relation $\simeq$ constructed above by:

$$p\simeq q\iff p\preceq q,\ q\preceq p$$

Thus, $\preceq$ is a partial order on the equivalence classes of projections $p\in A$.

PROOF. We have several assertions here, the idea being as follows:

(1) The fact that we have indeed a partial order is clear, with the transitivity coming, as before, by composing the corresponding partial isometries.

(2) Regarding now the relation with $\simeq$, via the equivalence in the statement, the implication $\implies$ is clear. Thus, we are left with proving $\impliedby$, which reads:

$$p\preceq q,\ q\preceq p\implies p\simeq q$$

Our assumption is that we have partial isometries u,v such that:

$$uu^*=p\quad,\quad u^*u\leq q$$

$$v^*v\leq p\quad,\quad vv^*=q$$

We can construct then two sequences of decreasing projections, as follows:

$$p_0=p\quad,\quad p_{n+1}=v^*q_nv$$

$$q_0=q\quad,\quad q_{n+1}=u^*p_nu$$

Consider now the limits of these two sequences of projections, namely:

$$p_\infty=\bigwedge_i p_i\quad,\quad q_\infty=\bigwedge_i q_i$$

In terms of all these projections that we constructed, we have the following decomposition formulae for the original projections p,q:

$$p=(p-p_1)+(p_1-p_2)+\ldots+p_\infty$$

$$q=(q-q_1)+(q_1-q_2)+\ldots+q_\infty$$

Now observe that the summands are equivalent, with this being clear from the definition of p_n,q_n at the finite indices $n<\infty$, and with $p_\infty\simeq q_\infty$ coming from:

$$v^*q_\infty v=p_\infty\quad,\quad q_\infty vv^*q_\infty=q_\infty$$

Thus we obtain that we have $p\simeq q$, as desired, by summing.

(3) Finally, the fact that the order relation $\preceq$ factorizes indeed to the equivalence classes under $\simeq$ follows from the equivalence established in (2). □

Summarizing, in view of Theorem 3.12, and of Theorem 3.15, we can formulate:

Conclusion 3.16. *We can think of a von Neumann algebra $A \subset B(H)$ as being a kind of object belonging to "mathematical logic", consisting of equivalence classes of projections $p \in A$, ordered via the relation $\preceq$, and producing A itself via transport by partial isometries, and then linear combinations, and weak limits.*

Which is something quite remarkable, who on Earth could have guessed, say when we were back in chapter 2, struggling with the basics of the von Neumann algebra theory, or even at the beginning of the present chapter 3, again struggling with some sort of basics, of the more advanced theory, that we will end up with something that luminous.

Well, that person on Earth who found this was von Neumann himself, back in the 1930s. And his Conclusion 3.16, called "von Neumann vision" of the operator algebras, has been extremely useful ever since, and is still largely used nowadays.

Very nice all this, first class mathematics, but in what concerns us, however, we will rather stick to our $A = L^\infty(X)$ viewpoint, with X being a quantum measured space, and the most often being a "quantum manifold". This is more of a "continuous" philosophy, and in order to keep it intact, and powerful, we will have to take sometimes distances with the von Neumann philosophy, especially in what concerns the terminology.

In short, we will be definitely users of the von Neumann projection technology, which is extremely powerful, and is quite often the only available tool, but keeping in mind however that we are dealing with continuous objects X, and choosing the terminology and notations accordingly, inspired from continuous geometry.

3c. States, isomorphism

Getting back now to general questions concerning the von Neumann algebras, one question that we met on several occasions, and that we would like to clarify now, is the relation between abstract isomorphism and spatial isomorphism.

To be more precise, we would like to understand when two von Neumann algebras $A \subset B(H)$ and $B \subset B(K)$ are isomorphic, in an algebraic and topological sense, but without reference to the ambient Hilbert spaces H, K. With the idea in mind that, once this understood, we will be able to talk about the von Neumann algebras A as being abstract objects, a bit as were the C^*-algebras, discussed in chapter 1.

In order to discuss this, let us start with some technical preliminaries. Here is a definition that I have been postponing for long, but which is now unavoidable:

Definition 3.17. *We call ultraweak and ultrastrong topologies on $B(H)$ the topologies defined exactly as the weak and strong operator topologies, but by using infinite families of vectors $(x_i)_{i\in\mathbb{N}} \subset H$ instead of finite families $(x_i)_{i=1,\ldots,N} \subset H$.*

And up to you to tell me if you love such things or not, and I will be here listening, like Sigmund Freud. Anyway. With this convention, we have the following result:

PROPOSITION 3.18. *Given a vector space $E \subset B(H)$, and a linear form $f : E \to \mathbb{C}$, the following conditions are equivalent:*

(1) *f is ultraweakly continuous.*
(2) *f is ultrastrongly continuous.*
(3) *$f(T) = \sum_{i=1}^\infty < Tx_i, y_i >$, for certain vectors $x_i, y_i \in H$.*

PROOF. This is similar to the proof of Proposition 3.4, as follows:

(1) $\implies$ (2) Since the ultraweak operator topology is weaker than the ultrastrong operator topology, ultraweakly continuous implies ultrastrongly continuous.

(2) $\implies$ (3) Assume that $f : E \to \mathbb{C}$ is ultrastrongly continuous. By continuity we can find vectors $x_i \in H$ and a number $\varepsilon > 0$ such that:

$$\sum_{i=1}^\infty ||Tx_i||^2 < \varepsilon^2 \implies |f(T)| < 1$$

It follows from this that we have the following estimate:

$$|f(T)| < \frac{1}{\varepsilon}\sqrt{\sum_{i=1}^\infty ||Tx_i||^2}$$

Consider now the direct sum $H^{\oplus\infty}$, and inside it, the following vector:

$$x = (x_i) \in H^{\oplus\infty}$$

Consider also the following linear space, written in tensor product notation:

$$K = \overline{(E \otimes 1)x} \subset H^{\oplus\infty}$$

We can define a linear form $f' : K \to \mathbb{C}$ by the following formula, and continuity:

$$f'((Tx_i)_i) = f(T)$$

We conclude that there exists a vector $y \in K$ such that:

$$f'\big((T \otimes 1)y\big) =< (T \otimes 1)x, y >$$

But in terms of the original linear form $f : E \to \mathbb{C}$, this means that we have:

$$f(T) = \sum_{i=1}^\infty < Tx_i, y_i >$$

(3) $\implies$ (1) This is indeed clear from definitions. □

As a consequence of the above result, we have:

THEOREM 3.19. *Given a von Neumann algebra $A \subset B(H)$, and a positive linear form $f : A \to \mathbb{C}$, the following are equivalent:*

(1) *f is normal, in the sense that $f(\sup_i x_i) = \sup_i f(x_i)$, for any increasing sequence of positive elements $x_i \in A$.*
(2) *f is completely additive, in the sense that $f(\bigvee_i p_i) = \sum_i f(p_i)$, for any family of pairwise orthogonal projections $p_i \in A$.*
(3) *f is ultraweakly continuous, or equivalently, f is a vector state, $f =< Tx, x >$, when suitably extending it to the space $H \otimes l^2(\mathbb{N})$.*

PROOF. This is something very standard, as follows:

(1) $\implies$ (2) Given a family of pairwise orthogonal projections $\{p_i\}$, we can consider the following increasing sequence of positive elements:

$$x_n = \sum_{i=1}^n p_i$$

By using now the formula in (1) for these elements we obtain, as desired:

$$\begin{aligned} f\left(\bigvee_i p_i\right) &= f\left(\sup_n x_n\right) \\ &= \sup_n f(x_n) \\ &= \sup_n \sum_{i=1}^n f(p_i) \\ &= \sum_i f(p_i) \end{aligned}$$

(2) $\implies$ (3) This is something more technical, that we will prove in several steps. Let us fix a projection $q \in A$, and consider a vector $\xi \in Im(q)$ such that:

$$< q\xi, \xi >> 1$$

Our claim is that there exists a projection $p \leq q$ such that, for any $x \in A$:

$$f(pxp) \leq < pxp\xi, \xi >$$

Indeed, in order to prove this, let us pick, by using the Zorn lemma, a maximal family of pairwise orthogonal projections $\{p_i\} \subset A$ such that, for any i, we have:

$$f(p_i) \geq < p_i\xi, \xi >$$

By using our complete additivity assumption, we have then:

$$\begin{aligned} f\left(\bigvee_i p_i\right) &= \sum_i f(p_i) \\ &\geq \sum_i < p_i\xi, \xi > \\ &= \left\langle \left(\bigvee_i p_i\right)\xi, \xi \right\rangle \end{aligned}$$

Now consider the following projection, which is nonzero:

$$p = q - \bigvee_i p_i$$

By maximality of the family $\{p_i\}$, for any nonzero projection $r \leq p$, we have:

$$f(r) << r\xi, \xi >$$

We therefore obtain the following estimate, valid for any $x \in A_+$, as desired:

$$f(pxp) \leq < pxp\xi, \xi >$$

Now by Cauchy-Schwarz we obtain that for any $x \in A$, $||x|| \leq 1$, we have:

$$\begin{aligned} |f(xp)|^2 &\leq f(px^*xp)f(1) \\ &\leq < px^*xp\xi, \xi > \\ &= ||xp\xi||^2 \end{aligned}$$

Thus the following linear form is strongly continuous on the unit ball of A:

$$x \to f(px)$$

In order to finish now, once again by using the Zorn lemma, let us pick a maximal family of pairwise orthogonal projections $\{p_i\} \subset A$ such that $x \to f(p_i x)$ is strongly continuous on the unit ball of A, for any i. By maximality we have then:

$$\sum_i f(p_i) = f\left(\sum_i p_i\right) = f(1) = 1$$

Now given $\varepsilon > 0$, let us choose a finite subset of our index set, $F \subset I$, such that for all the finite subsets $F \subset J \subset I$, we have an inequality as follows:

$$1 - f\left(\sum_{j \in J} p_j\right) \leq \varepsilon$$

By Cauchy-Schwarz we have then, for any $x\in A$, $||x||=1$, the following estimate:

$$\begin{aligned}\left|f\left(x\left(1-\sum_{j\in J}p_j\right)\right)\right|^2 &\leq f\left(1-\sum_{j\in J}p_j\right)f(xx^*)\\ &\leq \varepsilon\end{aligned}$$

We conclude from this that we have the following estimate:

$$\left|\left|f-f\left(\cdot\left(1-\sum_{j\in J}p_j\right)\right)\right|\right|\leq\sqrt{\varepsilon}$$

Thus we obtain $f\in A_*$, as desired.

(3) $\implies$ (1) This is something trivial, coming from definitions. □

We can now go back to our original question, and we have:

THEOREM 3.20. *Given two von Neumann algebras $A\subset B(H)$ and $B\subset B(K)$, acting on possibly different Hilbert spaces H,K, any algebraic isomorphism*

$$\Phi:A\simeq B$$

is spatial up to amplification, in the sense that we have a formula as follows,

$$\Phi(T)\otimes 1=U(T\otimes 1)U^*$$

for a certain Hilbert space L, and a certain unitary $U:H\otimes L\to K\otimes L$.

PROOF. This is something standard, coming from Theorem 3.19, as follows:

(1) As a first observation, assuming that a positive unital linear form $f:A\to\mathbb C$ is a vector state, given by a certain vector $x\in H$, then by Theorem 3.19 the linear form $f\Phi^{-1}$ is also a vector state, say given by a vector $y\in K$.

(2) We conclude from this that we have a unitary as follows, intertwining the corresponding actions of the von Neumann algebras A and B:

$$U_x:\overline{Ax}\to\overline{By}$$

Now by making the above vector $x\in H$ vary, and performing a direct sum, we obtain with $L=l^2(\mathbb N)$ an isometry as in the statement, namely:

$$U:H\otimes L\to K\otimes L$$

Our construction shows that U intertwines indeed the actions of the von Neumann algebras A and B, and what is left to do is to study the unitarity of U.

(3) We will prove now that, up to a suitable replacement, the above operator U can be taken to be unitary, still intertwining the actions of the von Neumann algebras A and

B. For this purpose, consider the action of von Neumann algebra A on the direct sum Hilbert space $(H \otimes L) \oplus (K \otimes L)$ given by the following matrices:

$$x' = \begin{pmatrix} x \otimes 1 & 0 \\ 0 & \Phi(x) \otimes 1 \end{pmatrix}$$

Since U intertwines the actions of the von Neumann algebras A and B, in terms of 2×2 matrices, we are led to the following conclusion:

$$\begin{pmatrix} 0 & 0 \\ U & 0 \end{pmatrix} \in A'$$

Thus, the following happens inside the von Neumann algebra A':

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \preceq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

On the other hand, the same reasoning applied to the isomorphism Φ^{-1} shows that we have as well, once again inside the von Neumann algebra A':

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \preceq \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

(4) We are now in position to finish. By combining the above two conclusions, we obtain an equivalence of projections inside A', as follows:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \simeq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Now pick a partial isometry implementing this equivalence. This partial isometry must be of the following form, with U' being now a unitary:

$$V = \begin{pmatrix} 0 & 0 \\ U' & 0 \end{pmatrix}$$

Thus, we have a unitary as follows, which intertwines the actions of A and B:

$$U' : H \otimes L \to K \otimes L$$

But this is the unitary we were looking for, and we are done. □

The above result is something quite fundamental, allowing us to talk about von Neumann algebras A as abstract objects, without reference to the exact Hilbert space H where the elements $a \in A$ live as operators $a \in B(H)$, and with this being of course possible modulo some functional analysis knowledge. We will heavily use this point of view in chapter 4 below, and then in chapters 5-8 too, when talking about II_1 factors.

3d. Predual theory

We have seen so far, as a consequence of the Kaplansky density theorem, that an operator algebra $A \subset B(H)$ is a von Neumann algebra precisely when its unit ball is weakly compact. This is certainly useful, but there are many other possible characterizations of the von Neumann algebras, as operator algebras, which are useful as well.

To be more precise, going ahead now with more abstract functional analysis, that we will be using in what follows, on several occasions, let us formulate:

Definition 3.21. *Given a von Neumann algebra $A \subset B(H)$, we set*

$$A_* = \left\{ f : A \to \mathbb{C},\ \text{weakly continuous} \right\}$$

regarded as a linear subspace, $A_* \subset A^*$, of the usual dual, given by

$$A^* = \left\{ f : A \to \mathbb{C},\ \text{norm continuous} \right\}$$

and we call this space A_* predual of our von Neumann algebra A.

Our first goal will be that of proving that we have the following duality formula, between the linear space A_* constructed above, and the algebra A itself:

$$A = (A_*)^*$$

In order to do so, let us first discuss the case of the full operator algebra $A = B(H)$ itself. This is actually the key case, with the extension to the arbitrary von Neumann algebras $A \subset B(H)$ being something coming afterwards, quite straightforward.

We will need some standard operator theory, as a continuation of the material from chapter 1. First, we have the following result, regarding the trace class operators:

Theorem 3.22. *The space of trace class operators, which appears as an intermediate space between the finite rank operators and the compact operators,*

$$F(H) \subset B_1(H) \subset K(H)$$

is a two-sided $*$-ideal of $K(H)$. The following is a Banach space norm on $B_1(H)$,

$$||T||_1 = Tr|T|$$

satisfying $||T|| \leq ||T||_1$, and for $T \in B_1(H)$ and $S \in B(H)$ we have:

$$||ST||_1 \leq ||S|| \cdot ||T||_1$$

Also, the subspace $F(H)$ is dense inside $B_1(H)$, with respect to this norm.

PROOF. There are several assertions here, the idea being as follows:

(1) In order to prove that $B_1(H)$ is a linear space, and that $||T||_1 = Tr|T|$ is a norm on it, the only non-trivial point is that of proving the following inequality:

$$Tr|S+T| \leq Tr|S| + Tr|T|$$

For this purpose, consider the polar decompositions of these operators:

$$S = U|S| \quad , \quad T = V|T| \quad , \quad S+T = W|S+T|$$

Given an orthonormal basis $\{e_n\}$, we have the following formula:

$$\begin{aligned} Tr|S+T| &= \sum_n < |S+T|e_n, e_n > \\ &= \sum_n < W^*(S+T)e_n, e_n > \\ &= \sum_n < W^*U|S|e_n, e_n > + \sum_n < W^*V|T|e_n, e_n > \end{aligned}$$

The point now is that the first sum can be estimated as follows:

$$\begin{aligned} & \sum_n < W^*U|S|e_n, e_n > \\ = & \sum_n < \sqrt{|S|}e_n, \sqrt{|S|}U^*We_n > \\ \leq & \sum_n \left|\left|\sqrt{|S|}e_n\right|\right| \cdot \left|\left|\sqrt{|S|}U^*We_n\right|\right| \\ \leq & \sqrt{\sum_n \left|\left|\sqrt{|S|}e_n\right|\right|^2} \cdot \sqrt{\sum_n \left|\left|\sqrt{|S|}U^*We_n\right|\right|^2} \end{aligned}$$

In order to estimate the terms on the right, we can proceed as follows:

$$\begin{aligned} \sum_n \left|\left|\sqrt{|S|}U^*We_n\right|\right|^2 &= \sum_n < W^*U|S|U^*We_n, e_n > \\ &= Tr(W^*U|S|U^*W) \\ &\leq Tr(U|S|U^*) \\ &\leq Tr(|S|) \end{aligned}$$

The second sum in the above formula of $Tr|S+T|$ can be estimated in the same way, and in the end we obtain, as desired:

$$Tr|S+T| \leq Tr|S| + Tr|T|$$

(2) The estimate $||T|| \leq ||T||_1$ can be established as follows:

$$\begin{aligned} ||T|| &= \big|\big||T|\big|\big| \\ &= \sup_{||x||=1} < |T|x, x > \\ &\leq Tr|T| \end{aligned}$$

(3) The fact that $B_1(H)$ is indeed a Banach space follows by constructing a limit for any Cauchy sequence, by using the singular value decomposition.

(4) The fact that $B_1(H)$ is indeed closed under the involution follows from:

$$\begin{aligned} Tr(T^*) &= \sum_n < T^* e_n, e_n > \\ &= \sum_n < e_n, Te_N > \\ &= \overline{Tr(T)} \end{aligned}$$

(5) In order to prove now the ideal property of $B_1(H)$, we use the standard fact, that we know from chapter 1, that any bounded operator $T \in B(H)$ can be written as a linear combination of 4 unitary operators, as follows:

$$T = \lambda_1 U_1 + \lambda_2 U_2 + \lambda_3 U_3 + \lambda_4 U_4$$

Indeed, by taking the real and imaginary part we can first write T as a linear combination of 2 self-adjoint operators, and then by functional calculus each of these 2 self-adjoint operators can be written as a linear linear combination of 2 unitary operators.

(6) With this trick in hand, we can now prove the ideal property of $B_1(H)$. Indeed, it is enough to prove that we have:

$$T \in B_1(H), U \in U(H) \implies UT, TU \in B_1(H)$$

But this latter result follows by using the polar decomposition theorem.

(7) With a bit more care, we obtain from this the estimate $||ST||_1 \leq ||S|| \cdot ||T||_1$ from the statement. As for the last assertion, this is clear as well. □

Let us discuss as well another important class of operators, namely the Hilbert-Schmidt ones. These operators, that we will often need, are introduced as follows:

Definition 3.23. *An operator $T \in B(H)$ is said to be Hilbert-Schmidt if:*

$$Tr(T^*T) < \infty$$

The set of such operators is denoted $B_2(H)$.

As before with other sets of operators, in finite dimensions we obtain in this way all the operators. In general, we have the following result, regarding such operators:

THEOREM 3.24. *The space $B_2(H)$ of Hilbert-Schmidt operators, which appears as an intermediate space between the trace class operators and the compact operators,*

$$F(H) \subset B_1(H) \subset B_2(H) \subset K(H)$$

is a two-sided $*$-ideal of $K(H)$. This ideal has the property

$$S, T \in B_2(H) \implies ST \in B_1(H)$$

and conversely, each $T \in B_1(H)$ appears as product of two operators in $B_2(H)$. In terms of the singular values (λ_n), the Hilbert-Schmidt operators are characterized by:

$$\sum_n \lambda_n^2 < \infty$$

Also, the following formula, whose output is finite by Cauchy-Schwarz,

$$< S, T >= Tr(ST^*)$$

defines a scalar product of $B_2(H)$, making it a Hilbert space.

PROOF. All this is quite standard, from the results that we have already, and more specifically from the singular value decomposition theorem, and its applications. To be more precise, the proof of the various assertions goes as follows:

(1) First of all, the fact that the space of Hilbert-Schmidt operators $B_2(H)$ is stable under taking sums, and so is a vector space, follows from:

$$\begin{aligned}(S+T)^*(S+T) &\leq (S+T)^*(S+T) + (S-T)^*(S-T) \\ &= (S^*+T^*)(S+T) + (S^*-T^*)(S-T) \\ &= 2(S^*S + T^*T)\end{aligned}$$

Regarding now multiplicative properties, we can use here the following inequality:

$$(ST)^*(ST) = T^*S^*ST \leq ||S||^2 T^*T$$

Thus, the space $B_2(H)$ is a two-sided $*$-ideal of $K(H)$, as claimed.

(2) In order to prove now that the product of any two Hilbert-Schmidt operators is a trace class operator, we can use the following formula, which is elementary:

$$S^*T = \sum_{k=1}^{4} i^k (S - iT)^*(S - iT)$$

Conversely, given an arbitrary trace class operator $T \in B_1(H)$, we have:

$$T \in B_1(H) \implies |T| \in B_1(H) \implies \sqrt{|T|} \in B_2(H)$$

Thus, by using the polar decomposition $T = U|T|$, we obtain the following decomposition for T, with both components being Hilbert-Schmidt operators:

$$T = U|T| = U\sqrt{|T|} \cdot \sqrt{|T|}$$

(3) The condition for the singular values is clear.

(4) The fact that we have a scalar product is clear as well.

(5) The proof of the completness property is routine as well. □

We have as well the following key result, regarding the Hilbert-Schmidt operators:

THEOREM 3.25. *We have the following formula,*

$$Tr(ST) = Tr(TS)$$

valied for any Hilbert-Schmidt operators $S, T \in B_2(H)$.

PROOF. We can prove this in two steps, as follows:

(1) Assume first that $|S|$ is trace class. Consider the polar decomposition $S = U|S|$, and choose an orthonormal basis $\{x_i\}$ for the image of U, suitably extended to an orthonormal basis of H. We have then the following computation, as desired:

$$\begin{aligned} Tr(ST) &= \sum_i < U|S|Tx_i, x_i > \\ &= \sum_i < |S|TUU^*x_i, U^*x_i > \\ &= Tr(|S|TU) \\ &= Tr(TU|S|) \\ &= Tr(TS) \end{aligned}$$

(2) Assume now that we are in the general case, where S is only assumed to be Hilbert-Schmidt. For any finite rank operator S' we have then:

$$\begin{aligned} |Tr(ST) - Tr(TS)| &= |Tr((S - S')T) - Tr(T(S - S'))| \\ &\leq 2||S - S'||_2 \cdot ||T||_2 \end{aligned}$$

Thus by choosing S' with $||S - S'||_2 \to 0$, we obtain the result. □

Good news, with the above ingredients in hand, and getting back now to von Neumann algebras, and to our predual questions raised before, we first have the following result:

THEOREM 3.26. *The linear space $B(H)_* \subset B(H)^*$ consisting of the linear forms $f : B(H) \to \mathbb{C}$ which are weakly continuous is given by*

$$B(H)_* = \left\{ T \to Tr(ST) \Big| S \in B_1(H) \right\}$$

and we have the following duality formula

$$B(H) = (B(H)_*)^*$$

as a duality in the usual Banach space sense.

Proof. There are several things to be proved, the idea being as follows:

(1) First of all, any linear form of type $T \to Tr(ST)$, with S being trace class, is weakly continuous. Thus, if we denote by $B(H)_\circ$ the subspace of $B(H)$ in the statement, consisting of such linear forms, we have an inclusion as follows:

$$B(H)_\circ \subset B(H)_*$$

(2) In order to prove now the reverse inclusion, consider an arbitrary weakly continuous linear form $f \in B(H)_*$. We can then find vectors (x_i) and (y_i) such that:

$$f(T) = \sum_i < Tx_i, y_i >$$

Let us consider now the following operators, going by definition from the Hilbert space $l^2(\mathbb{N})$ to our Hilbert space H, and which are both Hilbert-Schmidt:

$$Q : e_i \to x_i \quad , \quad R : e_i \to y_i$$

In terms of these operators, our linear form can be written as follows:

$$f(T) = Tr(R^*TQ)$$

On the other hand, by using the formula in Theorem 9.24 we obtain:

$$Tr(R^*TQ) = Tr(TQR^*)$$

Thus, with $S = QR^*$, which is trace class, we have the following formula:

$$f(T) = Tr(TS)$$

Thus, we have proved that we have an inclusion as follows:

$$B(H)_* \subset B(H)_\circ$$

(3) Summing up, from (1) and (2) we conclude that we have an equality as follows, which proves the first assertion in the statement:

$$B(H)_* = B(H)_\circ$$

(4) It remains to prove that $B(H)$ is indeed the dual of $B(H)_*$. For this purpose, we use the above identification, which ultimately identifies $B(H)_*$ with the space of trace class operators $B_1(H)$. So, assume that we have a linear form, as follows:

$$f : B_1(H) \to \mathbb{C}$$

It is then routine to show that f must come from evaluation on a certain operator $T \in B(H)$, and this leads to the conclusion that $B(H)$ is indeed the dual of $B(H)_*$. □

More generally now, for the arbitrary von Neumann algebras $A \subset B(H)$, we have:

THEOREM 3.27. *Given a von Neumann algebra $A \subset B(H)$, if we set*

$$A_* = \left\{ f : A \to \mathbb{C},\ \text{weakly continuous} \right\}$$

regarded as a linear subspace, $A_* \subset A^*$, of the usual dual, given by:

$$A^* = \left\{ f : A \to \mathbb{C},\ \text{norm continuous} \right\}$$

then we have the duality formula $A = (A_*)^*$, in the usual Banach space sense.

PROOF. This can be proved in several steps, as follows:

(1) First of all, we know from the above that the result holds for the von Neumann algebra $A = B(H)$ itself, in the sense that we have:

$$B(H) = (B(H)_*)^*$$

(2) The point now is that for any von Neumann subalgebra $A \subset B(H)$, or more generally for any weakly closed linear subspace $A \subset B(H)$, we have an equality as follows, coming as a consequence of the Hahn-Banach theorem:

$$A = A^{\perp\perp}$$

(3) Thus, modulo some standard algebra, and some standard identifications for quotient spaces and their duals, we are led to the conclusion in the statement. □

In fact, we have the following result, due to Sakai:

THEOREM 3.28. *The von Neumann algebras are exactly the C^*-algebras which have a predual, in the above sense.*

PROOF. This is a variation of the above, which caps the above series of results, and closes any further discussions, and for details here, we refer to Sakai's book [81]. □

3e. Exercises

Exercises:

EXERCISE 3.29.

EXERCISE 3.30.

EXERCISE 3.31.

EXERCISE 3.32.

EXERCISE 3.33.

EXERCISE 3.34.

EXERCISE 3.35.

EXERCISE 3.36.

Bonus exercise.

CHAPTER 4

Projections, factors

4a. Finite factors

In this chapter we go for the real thing, namely the study of the II_1 factors, following Murray and von Neumann [69], [70], [71], [89], [90], which is the basis for everything more advanced, in relation with the operator algebras.

We will only present here the very basic theory of the II_1 factors, and we will come back to them, on a regular basis, later. In fact, as we will soon discover, these II_1 factors are the "building blocks" of the whole von Neumann algebra theory.

Let us first talk about general factors. There are several possible ways of introducing them, and dividing them into several classes, for further study. In what concerns us, we will use a rather intuitive approach. The general idea, which is quite natural, is that among the von Neumann algebras $A \subset B(H)$, of particular interest are the "free" ones, having trivial center, $Z(A) = \mathbb{C}$. These algebras are called factors:

DEFINITION 4.1. *A factor is a von Neumann algebra $A \subset B(H)$ whose center*

$$Z(A) = A \cap A'$$

which is a commutative von Neumann algebra, reduces to the scalars, $Z(A) = \mathbb{C}$.

This notion is in fact something that we already met in the above, in the context of various comments or exercises, and time now to clarify all this. The idea is that there are two main motivations for the study of factors, with each of them being more than enough, as to serve as a strong motivation. First, at the intuitive level, we have:

PRINCIPLE 4.2 (Freeness). *The following happen:*

1. *The condition $Z(A) = \mathbb{C}$ defining the factors is, obviously, opposite to the condition $Z(A) = A$ defining the commutative von Neumann algebras.*
2. *Therefore, the factors are the von Neumann algebras which are "free", meaning as far as possible from the commutative ones.*
3. *Equivalently, with $A = L^\infty(X)$, the quantum spaces X coming from factors are those which are "free", meaning as far as possible from the classical spaces.*

So, this was for our first principle, which is something reasonable, intuitive, and self-explanatory, and which can surely serve as a strong motivation for the study of factors.

In fact, all that has being said above comes straight from the structure theorem for the commutative von Neumann algebras, $A = L^\infty(X)$, with X being a measured space, that we know since chapter 2, and the above principle is just a corollary of that theorem.

At a more advanced level, another motivation for the study of factors, which among others justifies the name "factors" for them, comes from the reduction theory of von Neumann [91], which is something non-trivial, that can be summarized as follows:

Principle 4.3 (Reduction theory). *Given a von Neumann algebra $A \subset B(H)$, if we write its center $Z(A) \subset A$, which is a commutative von Neumann algebra, as*

$$Z(A) = L^\infty(X)$$

with X being a measured space, then the whole algebra decomposes as

$$A = \int_X A_x \, dx$$

with the fibers A_x being factors, that is, satisfying $Z(A_x) = \mathbb{C}$.

As a first comment, we have already seen an instance of such decomposition results in chapter 2, when talking about finite dimensional algebras. Indeed, such algebras decompose, in agreement with the above, as direct sums of matrix algebras, as follows:

$$A = \bigoplus_x M_{n_x}(\mathbb{C})$$

In general, however, things are more complicated than this, and technically speaking, and as opposed to Principle 4.2, which was more of a triviality, Principle 4.3 is a tough theorem, due to von Neumann [91]. More on this later, in Part II below.

This was for the story, and let us close this philosophical discussion with:

Conclusion 4.4. *Regardless of the approach and technical level, be that beginner or advanced, the von Neumann factors are the algebras that matter.*

Getting to work now, there are many things that can be said about factors. In order to get started, as a direct continuation of the work from chapter 3, for the general von Neumann algebras, let us first study their projections. We will see that many interesting things happen here, with everything coming from the following technical result:

Proposition 4.5. *Given two projections $p, q \neq 0$ in a factor A, we have*

$$puq \neq 0$$

for a certain unitary $u \in A$.

PROOF. Assume by contradiction $puq = 0$, for any unitary $u \in A$. This gives:

$$u^*puq = 0$$

By using this for all the unitaries $u \in A$, we obtain the following formula:

$$\left(\bigvee_{u \in U_A} u^*pu\right) q = 0$$

On the other hand, from $p \neq 0$ we obtain, by factoriality of A:

$$\bigvee_{u \in U_A} u^*pu = 1$$

Thus, our previous formula is in contradiction with $q \neq 0$, as desired. □

Getteing back now to the order on projections from chapter 3, and to the whole von Neumann projection philosophy, in the case of factors things simplify, as follows:

THEOREM 4.6. *Given two projections $p, q \in A$ in a factor, we have*

$$p \preceq q \quad \text{or} \quad q \preceq p$$

and so $\preceq$ is a total order on the equivalence classes of projections $p \in A$.

PROOF. This basically follows from Proposition 4.5, and from the Zorn lemma, by using some standard functional analysis arguments. To be more precise:

(1) Consider indeed the following set of partial isometries:

$$S = \left\{ u \middle| uu^* \leq p, u^*u \leq q \right\}$$

We can then order this set S by saying that we have $u \leq v$ when $u^*u \leq v^*v$, and when $u = v$ holds on the initial domain u^*uH of u. With this convention made, the Zorn lemma applies, and provides us with a maximal element $u \in S$.

(2) In the case where this maximal element $u \in S$ satisfies $uu^* = p$ or $u^*u = q$, we are led to one of the conditions $p \preceq q$ or $q \preceq p$ in the statement, and we are done.

(3) So, assume that we are in the case left, $uu^* \neq p$ and $u^*u \neq q$. By Proposition 4.5 we obtain a unitary $v \neq 0$ satisfying the following conditions:

$$vv^* \leq p - uu^*$$

$$v^*v \leq q - u^*u$$

But these conditions show that the element $u + v \in S$ is strictly bigger than $u \in S$, which is a contradiction, and we are done. □

Moving ahead now, as explained time and again throughout this book, for a variety of reasons, which can be elementary or advanced, and also mathematical or physical, we are mainly interested in the case where our algebras have traces:

$$tr : A \to \mathbb{C}$$

And in relation with the factors, by leaving aside the rather trivial case of the matrix algebras $A = M_N(\mathbb{C})$, we are led in this way to the following key notion:

Definition 4.7. *A* II_1 *factor is a von Neumann algebra* $A \subset B(H)$ *which:*

(1) *Is infinite dimensional,* $\dim A = \infty$.
(2) *Has trivial center,* $Z(A) = \mathbb{C}$.
(3) *Has a trace* $tr : A \to \mathbb{C}$.

Here the order of the axioms is a bit random, with any of the possible $3! = 6$ choices making sense, and corresponding to a slightly different vision on what the II_1 factors truly are. With the above order, with (1) we are making it clear, right from the beginning, that we are not here for revolutionizing linear algebra. Then with (2) we adhere to Definition 4.1, and to what was said next about it, on freeness and reduction. And finally with (3) we adhere to the above principle, that von Neumann algebras must have traces.

More technically now, and leaving aside anything subjective, the above definition is motivated by the heavy classification work of Murray, von Neumann and Connes [17], [18], [69], [70], [71], [89], [90], [91], whose conclusion is more or less that everything in von Neumann algebras reduces, via some quite complicated procedures, we should mention that, to the study of the II_1 factors. With the mantra here being as follows:

Fact 4.8. *The* II_1 *factors are the building blocks of the whole von Neumann algebra theory.*

To be more precise, this statement, that we will get to understand later, is something widely agreed upon, at least among operator algebra experts who are familiar with von Neumann algebras, and with this agreement being something great. What remains controversial, however, is how to start playing with these Lego bricks that we have:

(1) A first option is that of adding the matrix algebras $M_N(\mathbb{C})$, not to be forgotten, and then stacking together such Lego bricks. According to the von Neumann reduction theory, this leads to the von Neumann algebras having traces, $tr : A \to \mathbb{C}$.

(2) A second option, perhaps even more playful, is that of taking crossed products of such Lego bricks by their automorphisms scaling the trace, or performing more general constructions inspired by advanced ergodic theory. This leads to general factors.

(3) And the third option is that of being a bad kid, or perhaps some kind of nerd, engineer in the becoming, and picking such a Lego brick, or a handful of them, and breaking them, see what's inside. Good option too, and more on this later.

Getting to work now, in practice, and forgetting about reduction theory, which raises the possibility of decomposing any tracial von Neumann algebra into factors, in order to obtain explicit examples of $\mathrm{II}_!$ factors, it is not even clear that such beasts exist. Fortunately the group von Neumann algebras are there, and we have the following result, which provides us with some examples of II_1 factors, to start with:

THEOREM 4.9. *The center of a group von Neumann algebra $L(\Gamma)$ is*

$$Z(L(\Gamma))=\left\{\sum_g\lambda_g g\Big|\lambda_{gh}=\lambda_{hg}\right\}''$$

and if $\Gamma\neq\{1\}$ has infinite conjugacy classes, in the sense that

$$\left|\{ghg^{-1}|g\in G\}\right|=\infty\quad,\quad\forall h\neq 1$$

with this being called ICC property, the algebra $L(\Gamma)$ is a II_1 factor.

PROOF. There are two assertions here, the idea being as follows:

(1) Consider a linear combination of group elements, which is in the weak closure of $\mathbb{C}[\Gamma]$, and so defines an element of the group von Neumann algebra $L(\Gamma)$:

$$a=\sum_g\lambda_g g$$

By linearity, this element $a\in L(\Gamma)$ belongs to the center of $L(\Gamma)$ precisely when it commutes with all the group elements $h\in\Gamma$, and this gives:

$$\begin{aligned}
a\in Z(A)&\iff ah=ha\\
&\iff \sum_g\lambda_g gh=\sum_g\lambda_g hg\\
&\iff \sum_k\lambda_{kh^{-1}}k=\sum_k\lambda_{h^{-1}k}k\\
&\iff \lambda_{kh^{-1}}=\lambda_{h^{-1}k}
\end{aligned}$$

Thus, we obtain the formula for $Z(L(\Gamma))$ in the statement.

(2) We have to examine the 3 conditions defining the II_1 factors. We already know from chapter 2 that the group algebra $L(G)$ has a trace, given by:

$$tr(g)=\delta_{g,1}$$

Regarding now the center, the condition $\lambda_{gh}=\lambda_{hg}$ that we found is equivalent to the fact that $g\to\lambda_g$ is constant on the conjugacy classes, and we obtain:

$$Z(L(\Gamma))=\mathbb{C}\iff\Gamma=\text{ICC}$$

Finally, assuming that this ICC condition is satisfied, with $\Gamma\neq\{1\}$, then our group Γ is infinite, and so the algebra $L(\Gamma)$ is infinite dimensional, as desired. □

In order to look now for more examples of II_1 factors, an idea would be that of attempting to decompose into factors the group von Neumann algebras $L(\Gamma)$, but this is something difficult, and in fact we won't really exit the group world in this way. Difficult as well is to investigate the factoriality of the von Neumann algebras of discrete quantum groups $L(\Gamma)$, because the basic computations from the proof of Theorem 4.9 won't extend to this setting, where the group elements $g\in\Gamma$ become corepresentations $g\in M_N(L(\Gamma))$. Despite years of efforts, it is presently not known at all what the "quantum ICC" condition should mean, and the problem comes from this. But more on this later.

In short, we have to stop here the construction of examples, and Theorem 4.9 will be what we have, at least for the moment. With this being actually not a big issue, the group factors $L(\Gamma)$ being known to be quite close to the generic II_1 factors.

4b. Basic results

Getting away now from the above difficulties, let us go back to the abstract II_1 factors, as axiomatized in Definition 4.7. In order to investigate them, the idea will be that from chapter 3, namely looking at the projections, and their equivalence classes.

In the case of the II_1 factors, as a first interesting remark, the presence of the trace trivializes the proof of the main result that we have about projections, as follows:

Theorem 4.10. *Given two projections $p,q\in A$ in a II_1 factor we have, trivially*

$$p\preceq q\quad\text{or}\quad q\preceq p$$

and so $\preceq$ is a total order on the equivalence classes of projections $p\in A$.

Proof. This is something that we already know, from Theorem 4.6, and which actually holds for any factor, with the non-trivial part being the following implication:

$$p\preceq q,\ q\preceq p\implies p\simeq q$$

But this implication is clear in the present II_1 factor setting, by using the trace. □

The above theorem and its proof, which are remarkable, are the first in a series of mysteries, in what concerns the special case of the II_1 factors. More such mysteries to follow. In order to study now the trace of the II_1 factors, we will need:

Proposition 4.11. *Given a weakly closed left ideal $I \subset A$ in a von Neumann algebra, there exists a unique projection $p \in A$ such that:*

$$I = Ap$$

Moreover, if $I \subset A$ is assumed to be a two-sided ideal, then $p \in Z(A)$.

Proof. We have several things to be proved, the idea being as follows:

(1) Given an ideal $I \subset A$ as in the statement, consider the following intersection:

$$I \cap I^* \subset A$$

This is a weakly closed non-unital $*$-subalgebra of A, so if we denote by $p \in A$ its largest projection, or unit, then we have an inclusion $Ap \subset I$.

(2) Conversely now, let us pick $x \in I$. By polar decomposition we can write $x = u|x|$, and we have the following implications, which prove the reverse inclusion $I \subset Ap$:

$$\begin{aligned} x \in I &\implies |x| = u^*x \in I \\ &\implies |x| \in I \cap I^* \\ &\implies |x|p = |x| \\ &\implies x = u|x| = u|x|p \in Ap \end{aligned}$$

(3) The uniqueness assertion is clear from the comparison theorem for projections.

(4) Regarding now the last assertion, assume that $I \subset A$ is a two-sided weakly closed ideal. Then for any unitary $u \in A$ we have:

$$\begin{aligned} I = uIu^* &\implies uIu^* = Ap \\ &\implies I = Aupu^* \end{aligned}$$

Thus by uniqueness we obtain $upu^* = p$, and so $p \in Z(A)$, as desired. □

As a first main result now regarding the II_1 factors, following the paper of Murray and von Neumann [71], which by the way is a must-read, we have:

Theorem 4.12. *Given a II_1 factor A, any weakly continuous positive trace*

$$tr : A \to \mathbb{C}$$

is automatically faithful.

Proof. Consider the null space of the trace, which is by definition:

$$I = \left\{ x \in A \Big| tr(x^*x) = 0 \right\}$$

We have the following inequality, which shows that I is a left ideal:

$$x^*a^*ax \leq ||a||^2 x^*x$$

Now by using the trace condition $tr(ab) = tr(ba)$, we conclude that I is a two-sided ideal. Also, the Cauchy-Schwarz inequality gives:

$$tr(x^*x) = 0 \iff tr(xy) = 0, \forall y \in A$$

We conclude from this that I is an intersection of kernels of weakly closed functionals, which are weakly closed, and so it is weakly closed. Thus the last assertion in Proposition 4.11 applies, and produces a projection $p \in Z(A)$ such that:

$$I = Ap$$

Now since A was assumed to be a factor, we have $Z(A) = \mathbb{C}$. Thus $p = 0$, and so the null ideal of the trace is $I = \{0\}$, and so our trace tr is faithful, as desired. □

Our goal now will be that of proving that the trace on a II_1 factor is unique, and takes on projections any value in $[0, 1]$. Let us start with a technical result, as follows:

Proposition 4.13. *Given a II_1 factor A, the traces of the projections*

$$tr(p) \in [0, 1]$$

can take arbitrarily small values.

Proof. Consider the set formed by all values of the trace on the projections:

$$S = \left\{ tr(p) \Big| p^2 = p = p^* \in A \right\}$$

We want to prove that the following number equals 0:

$$c = \inf(S - \{0\})$$

In order to do so, assume by contradiction $c > 0$, pick $\varepsilon > 0$ small, and pick a projection $p \in A$ such that the following condition is satisfied:

$$tr(p) < c + \varepsilon$$

Since we are in a II_1 factor, this projection $p \in A$ cannot be minimal, and so we can find another projection $q \in A$ satisfying $q < p$. Now observe that we have:

$$\begin{aligned} tr(p - q) &= tr(p) - tr(q) \\ &\leq tr(p) - c \\ &\leq \varepsilon \end{aligned}$$

Thus with $\varepsilon < c$ we obtain a contradiction, and so $c = 0$, as desired. □

In order to prove our next main result, we will need as well:

Proposition 4.14. *Given a II_1 factor A on a Hilbert space H and a projection $p \in A$, the von Neumann algebra pAp is a II_1 factor on the Hilbert space pH.*

PROOF. We have to prove that the von Neumann algebra pAp has a trace, and is infinite dimensional, and these two properties can be proved as follows:

(1) In what regards the trace, we know that the trace $tr : A \to \mathbb{C}$ restricts to a trace $tr : pAp \to \mathbb{C}$, which must be nonzero, as desired.

(2) In what regards the infinite dimensionality, this follows from the fact that a minimal projection in pAp would be minimal in A, which is impossible. □

Still following the fundamental paper of Murray and von Neumann [71], we can now formulate a second main result regarding the II_1 factors, as follows:

THEOREM 4.15. *Given a* II_1 *factor* A*, the traces of projections*

$$tr(p) \in [0,1]$$

can take any values in $[0,1]$*.*

PROOF. Given a number $c \in [0,1]$, consider the following set:

$$S = \left\{p^2 = p = p^* \in A \Big| tr(p) \leq c\right\}$$

This set satisfies the assumptions of the Zorn lemma, and so by this lemma we can find a maximal element $p \in S$. Assume by contradiction that we have:

$$tr(p) < c$$

The point now is that by using Proposition 4.13 and Proposition 4.14, we can slightly enlarge the trace of p, and we obtain a contradiction, as desired. □

As a third and last main result regarding the II_1 factors, also from [71], we have:

THEOREM 4.16. *The trace of a* II_1 *factor*

$$tr : A \to \mathbb{C}$$

is unique.

PROOF. This can be proved in many ways, a standard one being that of proving that any two traces agree on the projections, as a consequence of the above results:

(1) Assume indeed that we have a second trace $tr' : A \to \mathbb{C}$. Since A is generated by its projections, it is enough to show that we have $tr = tr'$ on projections.

(2) As a first observation, since traces on matrix algebras are unique, we obtain that we have $tr = tr'$ on the projections $p \in A$ having rational trace, $tr(p) \in \mathbb{Q}$.

(3) So, let us pick $p \in A$ having non-rational trace, $tr(p) \notin \mathbb{Q}$, and prove that we have $tr(p) = tr'(p)$. The idea will be that of using the result for the projections having rational traces, applied to an infinite direct sum of projections, converging to p.

(4) To be more precise, assume that we have constructed our sequence $p_i \to p$ up to order $n \in \mathbb{N}$, and let us try to construct p_{n+1}. The idea is to use the following algebra:

$$A_n = (p - p_n)A(p - p_n)$$

(5) Indeed this algebra is a II_1 factor, and we can choose inside it a projection p_{n+1} satisfying $p_n \leq p_{n+1} \leq p$, such that $tr = tr'$ on it, and such that:

$$tr(p - p_{n+1}) \leq \frac{1}{2} \cdot tr(p - p_n)$$

(6) According to our choices for these projections p_n, we have:

$$p = \bigvee_{n=1}^{\infty} p_n$$

Thus when evaluating tr, tr' on p we obtain the same result, as desired. □

In what regards illustrations for all this, as examples of II_1 factors we have so far the group von Neumann algebras $L(\Gamma)$, with Γ being an ICC group. In certain cases, it is possible to say more about all the above, and in particular about the projections, for instance with quite explicit procedures for constructing projections $p \in L(\Gamma)$ having an arbitrary prescribed trace $x \in [0, 1]$. We will be back to this later, when discussing more in detail the group von Neumann algebras $L(\Gamma)$, and their generalizations.

Back to theory, we have seen that the II_1 factors are very interesting objects, naturally lying above the matrix algebras $M_N(\mathbb{C})$, which are type I factors. From this perspective, a II_1 factor $A \subset B(H)$ is not really in need of the ambient Hilbert space H, and the question of "representing" it appears. We will discuss this question, in two steps:

(1) A first question is that of understanding the possible embeddings $A \subset B(H)$, with H being a Hilbert space. The main result here will be the construction of a numeric invariant $\dim_A H$, called coupling constant.

(2) A second question is that of understanding the possible embeddings $A \subset B$, with B being another II_1 factor. By using the coupling constant for both A, B we will construct a numeric invariant $[B : A]$, called index.

We will discuss now (1), and leave (2) for later, towards the end of this chapter. In order to get started, let us formulate the following definition:

DEFINITION 4.17. *Given a von Neumann algebra A with a trace $tr : A \to \mathbb{C}$, the emdedding*

$$A \subset B(L^2(A))$$

obtained by GNS construction is called standard form of A.

Here we use the GNS construction, explained in chapter 1. As the name indicates, the standard representation is something "standard", to be compared with any other representation $A \subset B(H)$, in order to understand this latter representation.

As already seen in chapter 1, the GNS construction has a number of unique features, that can be exploited. In the present setting, the main result is as follows:

Theorem 4.18. *In the context of the standard representation we have*

$$A' = JAJ$$

with $J : L^2(A) \to L^2(A)$ being the antilinear map given by $T \to T^*$.

Proof. Observe first that any $T \in A$ can be regarded as a vector $T \in L^2(A)$, to which we can associate, in an antilinear way, the vector $T^* \in L^2(A)$. Thus we have indeed an antilinear map J as in the statement. In terms of the standard cyclic and separating vector Ω for the GNS representation, the formula of this formula J is:

$$J(x\Omega) = x^*\Omega$$

(1) Our first claim is that we have the following formula:

$$< J\xi, J\eta > = < \xi, \eta >$$

Indeed, with $\xi = x\Omega$ and $\eta = y\Omega$, we have the following computation:

$$\begin{aligned} < J\xi, J\eta > &= < yx^*\Omega, \Omega > \\ &= tr(yx^*) \\ &= < \xi, \eta > \end{aligned}$$

(2) Our second claim is that we have the following formula:

$$JxJ(y\Omega) = yx^*\Omega$$

Indeed, this follows from the following computation:

$$JxJ(y\Omega) = J(xy^*\Omega) = yx^*\Omega$$

(3) Our claim now is that we have an inclusion as follows:

$$JAJ \subset A'$$

Indeed, this follows from the formula obtained in (2).

(4) In order to prove the reverse inclusion, our claim is that for $x \in A'$ we have:

$$Jx\Omega = x^*\Omega$$

Indeed, this follows from the following computation, valid for any $y \in A$:

$$\begin{aligned} < Jx\Omega, y\Omega > &= < Jy\Omega, x\Omega > \\ &= < y^*\Omega, x\Omega > \\ &= < \Omega, xy\Omega > \\ &= < x^*\Omega, y\Omega > \end{aligned}$$

(5) Our claim now is that the following formula defines a trace on A':

$$Tr(x) = < x\Omega, \Omega >$$

Indeed, for any two elements $x, y \in A'$ we have:

$$\begin{aligned} < xy\Omega, \Omega > &= < y\Omega, x^*\Omega > \\ &= < y\Omega, Jx\Omega > \\ &= < x\Omega, Jy\Omega > \\ &= < x\Omega, y^*\Omega > \\ &= < yx\Omega, \Omega > \end{aligned}$$

(6) We can now finish the proof. Indeed, by using the trace constructed in (5), we can apply our results obtained so far to A', and we obtain $JA'J \subset A$, as desired. □

As a basic illustration for the above result, we have:

THEOREM 4.19. *The commutant of a von Neumann group algebra $L(\Gamma)$, which is obtained by definition by using the left regular representation, is the von Neumann group algebra $R(\Gamma)$, obtained by using the right regular representation.*

PROOF. We recall that the left and the right representations of a discrete group Γ are given by the following formulae, by using the standard identification $\Gamma \subset l^2(\Gamma)$:

$$\lambda_g : h \to gh \quad , \quad \rho_g : h \to hg^{-1}$$

We have $Jg = g^{-1}$ for any group element $g \in \Gamma$, and by using this, we obtain:

$$\begin{aligned} J\lambda_g Jh &= J\lambda_g h^{-1} \\ &= Jgh^{-1} \\ &= hg^{-1} \\ &= \rho_g h \end{aligned}$$

Thus, the left and right representations are related by the following formula:

$$J\lambda_g J = \rho_g$$

By using now Theorem 4.18 we can compute commutants, as follows:

$$L(\Gamma)' = JL(\Gamma)J = R(\Gamma)$$

Finally, we have $L(\Gamma) = R(\Gamma)'$ too, by taking the commutant. □

As another application of the standard representation, let us go back to the uniqueness of the trace, that we know from Theorem 4.16. There are several alternative proofs for this fact, which are all instructive. As a first such statement and proof, we have:

THEOREM 4.20. *Given a* II_1 *factor* A*, and an element* $a \in A$*, we have the following Dixmier averaging property:*

$$\overline{span\left\{uau^*\Big|u \in U_A\right\}}^w \cap \mathbb{C}1 \neq \emptyset$$

In particular, the II_1 *factor trace* $tr : A \to \mathbb{C}$ *is unique.*

PROOF. We use the basic theory of the regular representation $A \subset L^2(A)$, with respect to the given trace $tr : A \to \mathbb{C}$, explained above. The proof goes as follows:

(1) Given an element $a \in A$, consider the space in the statement, obtained as the weak closure of the space spanned by the spinned versions of a, namely:

$$K_a = \overline{span\left\{uau^*\Big|u \in U_A\right\}}^w$$

This linear space $K_a \subset A$ is by definition weakly closed, and it follows that the subset $K_a\Omega \subset L^2(A)$, where $\Omega \in L^2(A)$ is the canonical trace vector, is a weakly closed convex subset. In particular, we see that $K_a\Omega \subset L^2(A)$ is a norm closed convex subset.

(2) In view of this, we can consider the unique element $b \in K_a$ having the property that $b\Omega$ has a minimal norm. We have then the following formula, for any unitary $u \in U_A$, where $J : L^2(A) \to L^2(A)$ is the standard antilinear map, given by $T \to T^*$:

$$||uJuJb\Omega|| = ||b\Omega||$$

By uniqueness of b, it follows that for any unitary $u \in U_A$, we have:

$$uJuJb\Omega = b\Omega$$

But this shows that for any unitary $u \in U_A$, we have:

$$ubu^* = b$$

We conclude that we have $b \in \mathbb{C}1$, and this proves the first assertion.

(3) Regarding now the second assertion, consider an arbitrary trace $tr : A \to \mathbb{C}$. By using $tr(uau^*) = tr(a)$, we conclude that this trace is constant on the following set:

$$K_a = \overline{span\left\{uau^*\Big|u \in U_A\right\}}^w$$

Now by using the first assertion, we conclude that we have the following formula:

$$\overline{span\left\{uau^*\Big|u \in U_A\right\}}^w \cap \mathbb{C}1 = \left\{tr(a)1\right\}$$

Summarizing, we have obtained a purely algebraic formula for our trace $tr : A \to \mathbb{C}$, and it follows that this trace is indeed unique, as claimed. □

In relation with the above, let us mention that there is as well a third proof for the uniqueness of the trace, due to Yeadon, based on nothing or almost, meaning the definition of the II_1 factors, along with some abstract functional analysis. For more on all this, basic theory of the II_1 factors, we refer to the standard operator algebra books, with some good choices here being the books of Connes [**19**], Takesaki [**84**] and Blackadar [**13**].

4c. Type II factors

Let us go back now to the general theory of the II_1 factors, with the aim of talking about representations of such II_1 factors, inside the category of the II_1 factors, $A \subset B$. For this purpose we will need a key notion, called coupling constant.

In order to discuss the construction of the coupling constant, we will need some further results on the type II factors, complementing those that we already have. The point indeed is that the class of II factors, to be axiomatized later, and with this being not something urgent, comprises, besides the II_1 factors discussed above, the II_∞ factors as well:

Definition 4.21. *A II_∞ factor is a von Neumann algebra of the form*

$$B = A \otimes B(H)$$

with A being a II_1 factor, and with H being an infinite dimensional Hilbert space.

We should mention that there are several possible ways of defining the II_∞ factors, and the above definition is something rather intuitive, the point being that, once you learn the theory of the II_∞ factors, as we will do here, what you remember at the end of the day is what has been said above, $B = A \otimes B(H)$, with A being a II_1 factor.

Getting started now, as a useful characterization of such factors, we have:

Proposition 4.22. *For an infinite factor B, the following are equivalent:*

1. *There exists a projection $p \in B$ such that pBp is a II_1 factor.*
2. *B is a II_∞ factor.*

Proof. This is something elementary, as follows:

(1) $\implies$ (2) Assume indeed that $p \in B$ is a projection such that pBp is a II_1 factor. We choose a maximal family of pairwise orthogonal projections $\{p_i\} \subset B$ satisfying $p_i \simeq p$, for any i, and we consider the following projection, which satisfies $q \preceq p$:

$$q = 1 - \sum_i p_i$$

Since the indexing set for our set of projections $\{p_i\}$ must be infinite, we can use a strict embedding of this index set into itself, as to write a formula as follows:

$$\begin{aligned} 1 &= q+\sum_i p_i \\ &\preceq p_0+\sum_{i\neq 0} p_i \\ &\preceq 1 \end{aligned}$$

Thus we have $\sum_i p_i \simeq 1$, and we may further suppose that we have in fact:

$$\sum_i p_i = 1$$

Thus the family $\{p_i\}$ can be used in order to construct a copy $B(H)\subset B$, with $H=l^2(\mathbb{N})$, and we must have $B=A\otimes B(H)$, with A being a II_1 factor, as desired.

(2) $\implies$ (1) This is clear, because when assuming $B=A\otimes B(H)$, as in Definition 4.21, we can take our projection $p\in B$ to be of the form $p=1\otimes q$, with $q\in B(H)$ being a rank 1 projection, and we have then $pBp=A$, which is a II_1 factor, as desired. □

Getting back now to the original interpretation of the II_∞ factors, from Definition 4.21, the tensor product writing there $B=A\otimes B(H)$ suggests tensoring the trace of the II_1 factor A with the usual operator trace of $B(H)$. We are led in this way to:

Definition 4.23. *Given a II_∞ factor B, written as $B=A\otimes B(H)$, with A being a II_1 factor and with H being an infinite dimensional Hilbert space, we define a map*

$$tr: B_+\to[0,\infty]\quad,\quad tr((x_{ij}))=\sum_i tr(x_{ii})$$

where we have chosen a basis of H, as to have $H\simeq l^2(\mathbb{N})$, and so $B(H)\subset M_\infty(\mathbb{C})$.

As an important observation, to start with, unlike in the II_1 factor case, that of the factor A, or in the I_∞ factor case, that of the factor $B(H)$, it is not possible to suitably normalize the trace constructed above. This follows indeed from the results below.

On the positive side now, the above trace has many useful properties, as follows:

Proposition 4.24. *The II_∞ factor trace that we constructed above*

$$tr: B_+\to[0,\infty]$$

has the following properties:

1. *$tr(x+y)=tr(x)+tr(y)$, and $tr(\lambda x)=\lambda tr(x)$ for $\lambda\geq 0$.*
2. *If $x_i\nearrow x$ then $tr(x_i)\to tr(x)$.*
3. *$tr(xx^*)=tr(x^*x)$.*
4. *$tr(uxu^*)=tr(x)$ for any $u\in U_B$.*

Proof. All this is elementary, the idea being as follows:

(1) This is clear from definitions.

(2) This is again clear from definitions.

(3) This is something which is elementary as well.

(4) This comes from (3), via the formula $uxu^* = u\sqrt{x} \cdot \sqrt{x}u^*$. □

As a main result now regarding the II_∞ factor trace, we have:

Theorem 4.25. *The II_∞ factor trace $tr : B_+ \to [0, \infty]$ constructed above, when restricted to the projections*

$$tr : P(B) \to [0, \infty]$$

induces an isomorphism between the totally ordered set of equivalence classes of projections in B and the interval $[0, \infty]$.

Proof. We have several things to be checked here, as follows:

(1) Our first claim is that a projection $p \in B$ is finite precisely when $tr(p) < \infty$.

– Indeed, in one sense, assume that we have $tr(p) < \infty$. If our projection p was to be infinite, we would have a subprojection $q \leq p$ having the same trace as p, and so $r = p-q$ would be a projection of trace 0, which is impossible. Thus p is indeed finite.

– In the other sense now, assuming $tr(p) = \infty$, we have to prove that p is infinite. For this purpose, let us pick a projection $q \leq p$ having finite trace. Then $r = p - q$ satisfies $tr(r) = \infty$, and so we can iterate the procedure, and we end up with an infinite sequence of pairwise orthogonal projections, which are all smaller than p. But this shows that p dominates an infinite projection, and so that p itself is infinite, as desired.

(2) Our second claim is that if $p, q \in B$ are projections, with p finite, then:

$$p \preceq q \iff tr(p) = tr(q)$$

But this follows exactly as in the II_1 factor case, discussed above.

(3) Our third and final claim, which will finish the proof, is that any infinite projection is equivalent to the identity. For this purpose, assume that $p \in B$ is infinite. By definition, this means that we can find a unitary $u \in B$ such that:

$$uu^* = p \quad , \quad u^*u \leq p \quad , \quad uu^* \neq p$$

But these conditions show that $(u^n)^i u^n$ is a strictly decreasing sequence of equivalent projections, and by using this sequence we conclude that we have $1 \preceq p$, as desired. □

Moving ahead now, in order to further investigate the II_∞ factors, we will need:

THEOREM 4.26. *Given a* II_1 *factor* $A \subset B(H)$*, there exists an isometry*

$$u : H \to L^2(A) \otimes l^2(\mathbb{N})$$

such that $ux = (x \otimes 1)u$*, for any* $x \in A$*.*

PROOF. We use a standard idea, that we used many times before, namely an amplification trick. Given a II_1 factor $A \subset B(H)$, consider the following Hilbert space:

$$K = H \oplus L^2(A) \otimes l^2(\mathbb{N})$$

Consider, as operators over this space K, the following projections:

$$p = id \oplus 0 \quad , \quad q = 0 \oplus id$$

Both these projections p, q belong then to A', which is a type II_∞ factor. Now since $q \in A'$ is infinite, by Theorem 4.25 we can find a partial isometry $u \in A'$ such that:

$$u^*u = p \quad , \quad uu^* \leq q$$

Now let us represent this partial isometry $u \in B(K)$ as a 2×2 matrix, as follows:

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The above conditions $u^*u = p$ and $uu^* \leq q$ reformulate then as follows:

$$b^*b + d^*d = 0 \quad , \quad aa^* + bb^* = 0$$

We conclude that our partial isometry $u \in B(K)$ has the following special form:

$$u = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$$

But the operator $c : H \to l^2(A) \otimes l^2(\mathbb{N})$ that we found in this way must be an isometry, and from $u \in A'$ we obtain $ux = (x \otimes 1)u$, for any $x \in A$, as desired. □

As a basic consequence of the above result, which is something good to know, and that we will use many times in what follows, we have:

THEOREM 4.27. *The commutant of a* II_1 *factor is a* II_1 *factor, or a* II_∞ *factor.*

PROOF. This follows indeed from the explicit interpretation of the operator algebra embedding $A \subset B(H)$ of our II_1 factor A, found in Theorem 4.26. □

Summarizing, we have an extension of the general theory of the II_1 factors, developed before, to the general case of the type II factors, which comprises by definition the II_1 factors and the II_∞ factors. All this is of course technically very useful.

4d. Coupling constant

We are now in position of constructing the coupling constant. The idea here, following as usual the key paper of Murray and von Neumann [71], will be that given a representation of a II_1 factor $A\subset B(H)$, we can try to understand how far is this representation from the standard form, where $H=L^2(A)$, from "above" or from "below".

In order to discuss this, which is something quite technical, let us start with:

PROPOSITION 4.28. *Given a II_1 factor $A\subset B(H)$, with its embedding into $B(H)$ being represented as above, in terms of an isometry*

$$u:H\to L^2(A)\otimes l^2(\mathbb N)\quad,\quad ux=(x\otimes 1)u$$

the following quantity does not depend on the choice of this isometry u:

$$C=tr(uu^*)$$

Moreover, for the standard form, where $H=L^2(A)$, this constant takes the value 1.

PROOF. Assume indeed that we have an isometry u as in the statement, and that we have as well a second such isometry, of the same type, namely:

$$v:H\to L^2(A)\otimes l^2(\mathbb N)\quad,\quad vx=(x\otimes 1)v$$

We have then $uu^*=uv^*vu^*$, and by using this, we obtain:

$$\begin{aligned}C_u&=tr(uu^*)\\&=tr(uv^*vu^*)\\&=tr(vu^*uv^*)\\&=tr(vv^*)\\&=C_v\end{aligned}$$

Thus, we are led to the conclusion in the statement. As for the last assertion, regarding the standard form, this is clear from definitions, because here we can take $u=1$. □

As a conclusion to all this, given a II_1 factor $A\subset B(H)$, we know from Theorem 4.26 that H must appear as an "inflated" version of $L^2(A)$. The corresponding inflation constant is a certain number, that we can call coupling constant, as follows:

DEFINITION 4.29. *Given a representation of a II_1 factor $A\subset B(H)$, we can talk about the corresponding coupling constant, as being the number*

$$\dim_A H\in(0,\infty]$$

constructed as follows, with $u:H\to L^2(A)\otimes l^2(\mathbb N)$ isometry satisfying $ux=(x\otimes 1)u$:

$$\dim_A H=tr(uu^*)$$

For the standard form, where $H=L^2(A)$, this coupling constant takes the value 1.

This definition might seem a bit complicated, but things here are quite non-trivial, and there is no way of doing something substantially simpler. Alternatively, we can define the coupling constant via the following formula, after proving first that the number on the right is indeed independent of the choice on a nonzero vector $x \in H$:

$$\dim_A H = \frac{tr_A(P_{A'x})}{tr_{A'}(P_{Ax})}$$

This latter formula was in fact the original definition of the coupling constant, by Murray and von Neumann [71]. However, technically speaking, things are slightly easier when using the approach in Definition 4.29. We will be back to this key formula of Murray and von Neumann, with full explanations, in a moment.

Let us start our study of the coupling constant with some basic results, coming from definitions and from what we already have, as results, as follows:

PROPOSITION 4.30. *The coupling constant* $\dim_A H \in (0, \infty]$ *associated to a* II_1 *factor representation* $A \subset B(H)$ *has the following properties:*

(1) *For the standard form,* $H = L^2(A)$*, we have* $\dim_A H = 1$.
(2) *For the usual representation on* $H = L^2(A) \otimes l^2(\mathbb{N})$*, we have* $\dim_A H = \infty$.
(3) *We have* $\dim_A H < \infty$ *precisely when* A' *is a* II_1 *factor.*
(4) *We have additivity,* $\dim_A(\oplus_i H_i) = \sum_i \dim_A H_i$.
(5) *We have* $\dim_A(L^2(A)p) = tr(p)$*, for any projection* $p \in A$.
(6) *The coupling constant can take any value in* $(0, \infty]$.

PROOF. All these assertions are elementary, the idea being as follows:

(1) This is something that we already know, coming from definitions.

(2) This is something that comes from definitions too.

(3) This comes from the general properties of the II_∞ factors, and their traces.

(4) Again, this is clear from the definition of the coupling constant.

(5) This follows by using $u(x) = x \otimes \xi$, with $\xi \in l^2(\mathbb{N})$ being of norm 1.

(6) This follows by starting with (5), and then making direct sums, as in (4). □

At a more advanced level now, in relation with projections and compressions, and getting towards the above-mentioned Murray-von Neumann approach, we have:

PROPOSITION 4.31. *We have the compression formula*

$$\dim_{pAp}(pH) = \frac{\dim_A H}{tr_A(p)}$$

valid for any projection $p \in A$.

PROOF. We can prove this result in two steps, as follows:

(1) Assume that H is as follows, with $q \in A$ being a projection satisfying $q \leq p$:

$$H = L^2(A)q$$

We can use the following unitary, intertwining the left and right actions of pAp:

$$L^2(pAp) \to pL^2(A)p \quad , \quad pxp\Omega \to p(x\Omega)p$$

Indeed, we obtain that the following algebras are unitarily equivalent:

$$pAp \subset B(pL^2(A)q) \quad , \quad pAp \subset B(L^2(pAp)q)$$

Thus, by using the formula (5) in Proposition 4.30 we obtain, as desired:

$$\begin{aligned} \dim_{pAp}(pH) &= tr_{pAp}(q) \\ &= \frac{tr_A(q)}{tr_A(p)} \\ &= \frac{\dim_A H}{tr_A(p)} \end{aligned}$$

(2) In the general case now, where H is arbitrary, the result follows from what we proved above, and from the additivity property from Proposition 4.30 (4). □

With all these properties established, we can now recover, as a theorem, the original definition of the coupling constant, due to Murray and von Neumann, as follows:

THEOREM 4.32. *Given a* II_1 *factor* $A \subset B(H)$*, with the commutant* $A' \subset B(H)$ *assumed to be finite, the corresponding coupling constant is finite, given by*

$$\dim_A H = \frac{tr_A(P_{A'x})}{tr_{A'}(P_{Ax})}$$

with the number on the right being independent of the choice on a nonzero vector $x \in H$*. In the case where* A' *is infinite, the corresponding coupling constant is infinite.*

PROOF. There are several things to be proved here, the idea being as follows:

(1) We know from Proposition 4.30 (3) that we have $\dim_A H < \infty$ precisely when the commutant $A' \subset B(H)$ is finite. Thus, we may assume that we are in this case.

(2) Assuming so, we have the following formula, valid for any projection $p \in A'$, which follows from the basic properties of the coupling constant, established above:

$$\dim_{Ap}(pH) = tr_{A'}(p) \dim_A H$$

(3) Now with this formula in hand, the formula in the statement follows as well, once again by doing a number of standard amplification and compression manipulations. □

As an illustration for all this, given an inclusion of ICC groups $\Lambda\subset\Gamma$, whose group algebras are both II_1 factors, we have the following formula:

$$\dim_{L(\Lambda)}L^2(\Gamma)=[\Gamma:\Lambda]$$

There are many other examples of explicit computations of the coupling constant, all leading into interesting mathematics. We will be back to this.

As a last topic for this chapter, given a II_1 factor A, let us discuss now the representations of type $A\subset B$, with B being another II_1 factor. This is a quite natural notion, perhaps even more natural than the representations $A\subset B(H)$, because we have previously decided that the II_1 factors B, and not the full operator algebras $B(H)$, are the correct infinite dimensional generalization of the usual matrix algebras $M_N(\mathbb{C})$.

This was for the philosophy, and one can of course agree or not with this. Or at least agree or not at the present point of the presentation, because once we will get into the structure of the subfactors $A\subset B$, which is something amazing, there is no way back.

In practice now, given an inclusion of II_1 factors $A\subset B$, a first question is that of defining its index, measuring how big is B compared to A. The first thought here goes into defining the index of $A\subset B$ as being a purely algebraic quantity, as follows:

$$N=\dim_A B$$

However, this is non-trivial, due to the fact that we are in the "continuous dimension" setting, and so our algebraic intuition, where indices are always integers, will not help us much. We will be back to this question later, with a technical solution to it.

In order to solve our index problem, a much better approach is by using the ambient operator algebra $B(H)$, or rather the ambient Hilbert space H, as follows:

THEOREM 4.33. *Given an inclusion of* II_1 *factors* $A\subset B$*, the number*

$$N=\frac{\dim_A H}{\dim_B H}$$

is independent of the ambient Hilbert space H*, and is called index.*

PROOF. The fact that the index of the subfactor $A\subset B$, as defined by the above formula, is indeed independent of the ambient Hilbert space H, comes from the various basic properties of the coupling constant, established above. □

There are many examples of subfactors coming from groups, and every time we obtain the intuitive index. More suprisingly now, Jones proved in [51] that the index, when small, is in fact "quantized", subject to the following unexpected restriction:

$$N\in\left\{4\cos^2\left(\frac{\pi}{n}\right)\Big|n\geq3\right\}\cup[4,\infty]$$

This is in fact part of a series of non-trivial results about the subfactors, due to Jones, and also Ocneanu, Popa, Wassermann and others, and involving as well the Temperley-Lieb algebra, and many more. We will be back to this, later in this book.

4e. Exercises

Exercises:

Exercise 4.34.

Exercise 4.35.

Exercise 4.36.

Exercise 4.37.

Exercise 4.38.

Exercise 4.39.

Exercise 4.40.

Exercise 4.41.

Bonus exercise.

Part II

Reduction, factors

Sanitarium
Leave me be
Sanitarium
Just leave me alone

CHAPTER 5

5a.

5b.

5c.

5d.

5e. Exercises

Exercises:

EXERCISE 5.1.

EXERCISE 5.2.

EXERCISE 5.3.

EXERCISE 5.4.

EXERCISE 5.5.

EXERCISE 5.6.

EXERCISE 5.7.

EXERCISE 5.8.

Bonus exercise.

CHAPTER 6

6a.

6b.

6c.

6d.

6e. Exercises

Exercises:

Exercise 6.1.

Exercise 6.2.

Exercise 6.3.

Exercise 6.4.

Exercise 6.5.

Exercise 6.6.

Exercise 6.7.

Exercise 6.8.

Bonus exercise.

CHAPTER 7

7a.

7b.

7c.

7d.

7e. Exercises

Exercises:

Exercise 7.1.

Exercise 7.2.

Exercise 7.3.

Exercise 7.4.

Exercise 7.5.

Exercise 7.6.

Exercise 7.7.

Exercise 7.8.

Bonus exercise.

CHAPTER 8

8a.

8b.

8c.

8d.

8e. Exercises

Exercises:

EXERCISE 8.1.

EXERCISE 8.2.

EXERCISE 8.3.

EXERCISE 8.4.

EXERCISE 8.5.

EXERCISE 8.6.

EXERCISE 8.7.

EXERCISE 8.8.

Bonus exercise.

Part III

Modular theory

You better run all day and run all night
And keep your dirty feelings deep inside
And if you're taking your girlfriend out tonight
You better park the car well out of sight

CHAPTER 9

9a.

9b.

9c.

9d.

9e. Exercises

Exercises:

EXERCISE 9.1.

EXERCISE 9.2.

EXERCISE 9.3.

EXERCISE 9.4.

EXERCISE 9.5.

EXERCISE 9.6.

EXERCISE 9.7.

EXERCISE 9.8.

Bonus exercise.

CHAPTER 10

10a.

10b.

10c.

10d.

10e. Exercises

Exercises:

Exercise 10.1.

Exercise 10.2.

Exercise 10.3.

Exercise 10.4.

Exercise 10.5.

Exercise 10.6.

Exercise 10.7.

Exercise 10.8.

Bonus exercise.

CHAPTER 11

11a.

11b.

11c.

11d.

11e. Exercises

Exercises:

EXERCISE 11.1.

EXERCISE 11.2.

EXERCISE 11.3.

EXERCISE 11.4.

EXERCISE 11.5.

EXERCISE 11.6.

EXERCISE 11.7.

EXERCISE 11.8.

Bonus exercise.

CHAPTER 12

12a.

12b.

12c.

12d.

12e. Exercises

Exercises:

Exercise 12.1.

Exercise 12.2.

Exercise 12.3.

Exercise 12.4.

Exercise 12.5.

Exercise 12.6.

Exercise 12.7.

Exercise 12.8.

Bonus exercise.

Part IV

Hyperfiniteness

Here comes the sun
In the form of a girl
She's the finest sweetest
Thing in the world

CHAPTER 13

13a.

13b.

13c.

13d.

13e. Exercises

Exercises:

Exercise 13.1.

Exercise 13.2.

Exercise 13.3.

Exercise 13.4.

Exercise 13.5.

Exercise 13.6.

Exercise 13.7.

Exercise 13.8.

Bonus exercise.

CHAPTER 14

14a.

14b.

14c.

14d.

14e. Exercises

Exercises:

EXERCISE 14.1.

EXERCISE 14.2.

EXERCISE 14.3.

EXERCISE 14.4.

EXERCISE 14.5.

EXERCISE 14.6.

EXERCISE 14.7.

EXERCISE 14.8.

Bonus exercise.

CHAPTER 15

15a.

15b.

15c.

15d.

15e. Exercises

Exercises:

Exercise 15.1.

Exercise 15.2.

Exercise 15.3.

Exercise 15.4.

Exercise 15.5.

Exercise 15.6.

Exercise 15.7.

Exercise 15.8.

Bonus exercise.

CHAPTER 16

16a.

16b.

16c.

16d.

16e. Exercises

Congratulations for having read this book, and no exercises for this final chapter.

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