Weingarten calculus explained

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ABSTRACT. This is an introduction to Weingarten calculus and its applications. We first discuss the various groups of matrices, discrete or continuous, with general theory, examples, and the Weingarten integration formula for them. Armed with this new calculus, we then discuss some basic applications, notably to questions regarding random matrices, and other basic objects from probability. In view of some further, sharper applications, we then fine-tune our calculus, with a number of fine estimates for the Weingarten functions, using more advanced mathematics, that we explain. Finally, we discuss a number of generalizations, quite often in relation with theoretical physics.

Preface

This is an introduction to Weingarten calculus and its applications. We first discuss the various groups of matrices, discrete or continuous, with general theory, examples, and the Weingarten integration formula for them. Armed with this new calculus, we then discuss some basic applications, notably to questions regarding random matrices, and other basic objects from probability. In view of some further, sharper applications, we then fine-tune our calculus, with a number of fine estimates for the Weingarten functions, using more advanced mathematics, that we explain. Finally, we discuss a number of generalizations, quite often in relation with theoretical physics.

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And answers are a prison for oneself
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Congratulations for having read this book, and no exercises for this final chapter.

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