THE HYPEROCTAHEDRAL QUANTUM GROUP

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Abstract. We consider the hypercube in $\mathbb{R}^n$, and show that its quantum symmetry group is a q-deformation of O_n at $q = -1$. Then we consider the graph formed by n segments, and show that its quantum symmetry group is free in some natural sense. This latter quantum group, denoted H_n^+, enlarges Wang's series S_n^+, O_n^+, U_n^+.

Introduction

The idea of noncommuting coordinates goes back to Heisenberg. Several theories emerged from Heisenberg's work, most complete being Connes' noncommutative geometry, where the base space is a Riemannian manifold. See [19].

A natural question is about studying algebras of free coordinates on algebraic groups. Given a group $G \subset U_n$, the matrix coordinates $u_{ij} \in C(G)$ commute with each other, and satisfy certain relations R. One can define then the universal algebra generated by abstract variables u_{ij}, subject to the relations R. The spectrum of this algebra is an abstract object, called noncommutative version of G. The noncommutative version is not unique, because it depends on R. We have the following examples:

(1) Free quantum semigroups. The algebra generated by variables u_{ij} with relations making $u = (u_{ij})$ a unitary matrix was considered by Brown [14]. This algebra has a comultiplication and a counit, but no antipode. In other words, the corresponding noncommutative version U_n^{nc} is a quantum semigroup.

(2) Quantum groups. A remarkable discovery is that for a compact Lie group G, once commutativity is included into the relations R, one can introduce a complex parameter q, as to get deformed relations R^q. The corresponding noncommutative versions G^q are called quantum groups. See Drinfeld [21].

(3) Compact quantum groups. The quantum group G^q is semisimple for q not a root of unity, and compact under the slightly more restrictive assumption $q \in \mathbb{R}$. Woronowicz gave in [35] a simple list of axioms for compact quantum groups, which allows construction of many other noncommutative versions.

(4) Free quantum groups. The Brown algebras don't fit into Woronowicz's axioms, but a slight modification leads to free quantum groups. The quantum groups O_n^+, U_n^+ appeared in Wang's thesis [32]. Then Connes suggested use of symmetric groups, and the quantum group S_n^+ was constructed in [33].

The quantum groups S_n^+, O_n^+, U_n^+ have been intensively studied in the last years.

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The purpose of this paper is to introduce and study a new free quantum group. This is H_n^+, the free analogue of the hyperoctahedral group H_n.

The various algebraic results in [3], [4], [11] and integration techniques in [7], [8], [9] apply to H_n^+, and our results can be summarized in the following table:

	S_n^+	H_n^+	O_n^+	U_n^+
Jones diagrams	TL	colored TL	double TL	free TL
Speicher partitions	NC	colored NC	double NC	free NC
Voiculescu law	free Poisson	free Bessel	semicircular	circular
Di Francesco matrix	full meander	colored meander	meander	free meander

Here the S_n^+, O_n^+, U_n^+ columns are from [7], [8], and the H_n^+ column is new. The precise meaning of this table, and the connection with [20], [23], [28], [30] and with the paper of Bisch and Jones [13] will be explained in the body of the paper. Before proceeding, we would like to make a few comments:

(1) The quantum group H_n^+ is by definition a subgroup of S_{2n}^+, and emerges from the recent litterature on quantum permutation groups, via a combination of the approach in [4], [5], [6] with the one in [7], [8], [9].
(2) The origins of H_n^+ go back to results in [11], with the construction of free wreath products, and with the exceptional isomorphism $H_2^+ = O_2^{-1}$. Besides the above table, we have the quite surprising result that O_n^{-1} is a noncommutative analogue of H_n, for any n.
(3) The fact that the Fuss-Catalan algebra can be realized with compact quantum groups is known since [3]. However, there are many choices for such a quantum group. The results in this paper show that H_n^+ is the good choice.
(4) The computations for H_n^+ belong to a certain order 1 problematics, for the Fuss-Catalan algebra. For general free wreath products the situation is discussed in [4], with a conjectural order 0 statement. The plan here would be to prove and improve this statement, then to compare it with results of Śniady in [27].
(5) The quantum groups considered in this paper satisfy $S_n^+ \subset G \subset U_n^+$. This condition is very close to freeness, and we think that quantum groups satisfying it should deserve a systematic investigation. Some general results in this sense, at level of integration formulae, are worked out here.

Summarizing, the quantum group H_n^+ appears to be a central object of the theory, of the same type as Wang's fundamental examples S_n^+, O_n^+, U_n^+.

Finally, let us mention that the semigroup of measures for H_n^+, that we call "free Bessel", seems to be new. Here are the even moments of these measures:

$$\int x^{2k}\, d\mu_t(x) = \sum_{b=1}^{k} \frac{1}{b} \binom{k-1}{b-1} \binom{2k}{b-1} t^b$$

As for the odd moments, these are zero. The relation with Bessel functions is that in the classical case, namely for the group H_n, the measures that we get are supported on $\mathbb{Z}$, with density given by Bessel functions of the first kind.

The paper is organized in four parts, as follows:

(1) In 1-2 we discuss the integration problem, with a presentation of some previously known results, and with an explicit computation for H_n.
(2) In 3-4 we show that O_n^{-1} is the quantum symmetry group of the hypercube in $\mathbb{R}^n$. Thus O_n^{-1} can be regarded as a noncommutative version of H_n.
(3) In 5-10 we construct the free version H_n^+, as quantum symmetry group of the graph formed by n segments. We perform a detailed combinatorial study of H_n^+, the main result being the asymptotic freeness of diagonal coefficients.
(4) In 11-12 we extend a part of the results to a certain class of quantum groups, that we call free. We give some other examples of free quantum groups, by using a free product construction, and we discuss the classification problem.

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1. The integration problem

In this section we present the basic idea of Weingarten's approach to the computation of integrals over compact groups, and its extension to quantum groups.

Consider a compact group $G \subset O_n$. This can be either a Lie group, like O_n itself, or just a finite group, like S_n, or one of its subgroups.

Definition 1.1. *For a compact group $G \subset O_n$, the formula*

$$g = \begin{pmatrix} u_{11}(g) & & u_{1n}(g) \\ & \ddots & \\ u_{n1}(g) & & u_{nn}(g) \end{pmatrix}$$

defines n^2 functions $u_{ij} : G \to \mathbb{R}$, that we call matrix coordinates of G.

These functions are coordinates in the usual sense, when G viewed as a real algebraic variety. We regard them as complex functions, $u_{ij} : G \to \mathbb{C}$.

Consider now the Haar measure of G. The integration problem is to compute the integrals of various products of matrix coordinates:

$$I(i,j) = \int_G u_{i_1 j_1}(g) \ldots u_{i_k j_k}(g)\, dg$$

It is convenient at this point to switch to operator algebraic language. We regard integration as being an abstract functional on the algebra $C(G)$:

$$\int : C(G) \to \mathbb{C}$$

The integration problem is to compute this functional. Indeed, by norm continuity the functional is uniquely determined on the dense subalgebra generated by entries of u, and by linearity computation is reduced to that of the above integrals.

Definition 1.2. *The integration problem is the computation of*

$$I(i,j)=\int u_{i_1j_1}\dots u_{i_kj_k}$$

in terms of multi-indices $i=(i_1,\dots,i_k)$ and $j=(j_1,\dots,j_k)$.

The very first example of such an integral, or rather of a sum of such integrals, is the following well-known quantity:

$$\int (u_{11}+u_{22}+\dots+u_{nn})^k=\#\left\{1\in u^{\otimes k}\right\}$$

Here the number on the right is the multiplicity of 1 into the k-th tensor power of u, whose computation is a fundamental problem in representation theory. The number on the left is the integral of the k-th power of the character of u.

This relation with representation theory provides actually a solution to the integration problem. The algorithm is as follows:

(1) Decompose the representation $u^{\otimes k}$, as sum of irreducibles.
(2) Find a basis D_k for the linear space F_k of fixed vectors of $u^{\otimes k}$.
(3) Compute the Gram matrix G_k of this basis D_k.
(4) Compute the Weingarten matrix, $W_k=G_k^{-1}$.
(5) Express the orthogonal projection P_k onto F_k in terms of W_k.
(6) Use the fact that the matrix coefficients of P_k are the integrals $I(i,j)$.

The fact that this works follows from 5-6. We use here an elementary result from linear algebra: the projection onto a space having an orthogonal basis is given by a trivial formula, and in case the basis is not orthogonal, the Gram matrix (or rather its inverse) comes into the picture, and leads to a more complicated formula. The last assertion is also a standard fact, coming from Peter-Weyl theory.

The rest of this section contains a detailed discussion of the problem in 1-4. For the moment, let us summarize the above discussion, in the form of a vague statement:

Definition 1.3. *The Weingarten problem is to find a formula of type*

$$\int u_{i_1j_1}\dots u_{i_kj_k}=\sum_{pq}\delta_{pi}\delta_{qj}W_k(p,q)$$

where the sum is over diagrams, the δ-symbols encode the coupling of diagrams with multi-indices, and W_k is the inverse of the coupling matrix of diagrams.

The origins of this approach go back to Weingarten's paper [34]. The formula is worked out in detail in [15], [17], [18], for several classes of Lie groups.

As already mentioned, the general formula is quite theoretical, because of serious complexity problems in step 4. In fact, the Weingarten matrix cannot be computed explicitly in the general case, and only partial results are available.

Fortunately, we have here a hierarchy of reasonable questions:

(1) Order zero: computation of the law of $u_{11}+\dots+u_{nn}$.
(2) First order: computation of the law of $u_{11}+\dots+u_{ss}$, with $1\leq s\leq n$.

(3) Second order: integration of carefully chosen quantities, coming from advanced representation theory, or from theoretical physics.

The list is quite heuristic, and is based on recent work on the subject, rather than on some precise mathematical statement. The numbers $0, 1, 2$ should be thought of as being exponents of n^{-1} in some related asymptotic expansion. See e.g. [16].

In this paper we restrict attention to first order problems. It is convenient to consider arbitrary sums of diagonal entries of u, and to denote them as follows.

Definition 1.4. *To any union of intervals $I \subset [0,1]$ we associate the element*

$$u_I = \sum_{s \in nI} u_{ss}$$

of the algebra $C(G)$, where u is the matrix of coefficients of G.

The example to be kept in mind is with the interval $I = [0,t]$. We get here the element $u_I = u_{11} + \ldots + u_{ss}$ with $s = [tn]$, corresponding to the first order problem.

We will need a few notions from classical and free probability. It is convenient to use here the abstract framework of C^*-algebras endowed with positive traces:

$$\int : A \to \mathbb{C}$$

An element of A will be called random variable. The law of a self-adjoint random variable a is the real probability measure μ given by the following formula:

$$\int a^k = \int_{\mathbb{R}} x^k \, d\mu(x)$$

Recall that two elements $a \in A$ and $b \in B$ are independent in $A \otimes B$, and that there is an abstract notion of independence axiomatizing this situation. See [31].

The total length of a union of intervals I is denoted $l(I)$.

Theorem 1.5. *Consider the groups O_n, S_n.*

(1) *For O_n, the variable u_I is asymptotically Gaussian of parameter $l(I)$.*
(2) *For S_n, the variable u_I is asymptotically Poisson of parameter $l(I)$.*
(3) *The variables u_I, u_J with $I \cap J = \emptyset$ are asymptotically independent.*

This statement is pointed out in [8], in a slightly different form.

We present now a quantum analogue of this result. We use the following key definition, emerging from Wang's work [32], [33], which was in turn based on Woronowicz's axiomatization of compact quantum groups [35], [36].

Definition 1.6. *Let $u \in M_n(A)$ be a square matrix over a C^*-algebra.*

(1) *u is called orthogonal if all elements u_{ij} are self-adjoint, and u is unitary. In other words, we must have $u = \bar{u}$ and $u^t = u^{-1}$.*
(2) *u is called magic unitary if all elements u_{ij} are projections, and on each row and each column of u these projections are orthogonal, and sum up to 1.*

The matrix coordinates of O_n form an orthogonal matrix. The matrix coordinates of S_n are the characteristic functions $u_{ij} = \chi\{\sigma \in S_n \mid \sigma(j) = i\}$, which form a magic unitary. These remarks and the Gelfand theorem lead to the following formulae:

$$\begin{aligned} C(O_n) &= C^*_{com}\left(u_{ij} \mid u = n \times n \text{ orthogonal}\right) \\ C(S_n) &= C^*_{com}\left(u_{ij} \mid u = n \times n \text{ magic unitary}\right) \end{aligned}$$

We can proceed now with liberation. We call comultiplication, counit and antipode any morphisms of C^*-algebras of the following type:

$$\begin{aligned} \Delta &: A \to A \otimes A \\ \varepsilon &: A \to \mathbb{C} \\ S &: A \to A^{op} \end{aligned}$$

An orthogonal Hopf algebra is a C^*-algebra A given with an orthogonal matrix u, such that the following formulae define a comultiplication, a counit and an antipode:

$$\begin{aligned} \Delta(u_{ij}) &= \sum u_{ik} \otimes u_{kj} \\ \varepsilon(u_{ij}) &= \delta_{ij} \\ S(u_{ij}) &= u_{ji} \end{aligned}$$

The basic example is the algebra $C(G)$, where $G \subset O_n$ is a compact group. Here u is the matrix of coordinates of G, and the axioms are clear from definitions.

Theorem 1.7. *The universal algebras*

$$\begin{aligned} A_o(n) &= C^*\left(u_{ij} \mid u = n \times n \text{ orthogonal}\right) \\ A_s(n) &= C^*\left(u_{ij} \mid u = n \times n \text{ magic unitary}\right) \end{aligned}$$

are orthogonal Hopf algebras.

See Wang [32], [33]. As a conclusion, we have the following heuristic equalities, where plus superscripts denote free versions of the corresponding classical groups:

$$\begin{aligned} A_o(n) &= C(O_n^+) \\ A_s(n) &= C(S_n^+) \end{aligned}$$

We discuss now the liberation aspect of the operation $G \to G^+$. There are several interpretations here. The idea that we follow, pointed out in [8], is that with $n \to \infty$ the integral geometry of G is governed by independence, while that of G^+, by freeness.

Recall that two elements $a \in A$ and $b \in B$ are free in $A * B$, and that there is an abstract notion of freeness axiomatizing this situation. See [30], [31].

Theorem 1.8. *Consider the algebras $A_o(n), A_s(n)$.*

(1) *For $A_o(n)$, the variable u_I is asymptotically semicircular of parameter $l(I)$.*
(2) *For $A_s(n)$, the variable u_I is asymptotically free Poisson of parameter $l(I)$.*
(3) *The variables u_I, u_J with $I \cap J = \emptyset$ are asymptotically free.*

This statement appears in [8], in a slightly different form. The assertions follow from results in [7], [8]. This will be explained in detail later on.

2. The hyperoctahedral group

In this section we review a few well-known facts regarding the symmetry group of the cube. The whole material is here in order to be carefully studied in next sections: several problems will appear from the simple facts explained below.

Definition 2.1. *H_n is the common symmetry group of the following spaces:*

1. *The cube $\mathrm{K}_n \subset \mathbb{R}^n$, regarded as a metric space (2^n points).*
2. *The graph K_n which looks like a cube (2^n vertices, $n2^{n-1}$ edges).*
3. *The space $\mathrm{I}_n \subset \mathbb{R}^n$ formed by the ± 1 points on each axis ($2n$ points).*
4. *The graph I_n formed by n segments ($2n$ vertices, n edges).*

The equivalences $1 \Leftrightarrow 2$ and $3 \Leftrightarrow 4$ are consequences of the following general principle: when drawing a geometric object O, the symmetry group of the graph O that we obtain is the same as the symmetry group of O.

As for the equivalence $1 \Leftrightarrow 3$, this follows from the fact that a rotation of $\mathbb{R}^n$ leaves invariant the cube if and only if it leaves invariant the set of middles of each face.

For each segment I^i consider the element $\tau_i \in H_n$ which returns I^i, and keeps the other segments fixed. The elements $\tau_1, \ldots, \tau_n$ have order two and commute with each other, so they generate the product of n copies of Z_2, denoted in this paper L_n:

$$L_n = Z_2 \times \ldots \times Z_2$$

Now the symmetric group S_n acts as well on X_n by permuting the segments, and it is routine to check that we have $H_n = L_n \rtimes S_n$. But this latter crossed product is by definition the wreath product $Z_2 \wr S_n$.

Proposition 2.2. *H_n has the following properties:*

1. *We have $S_n \subset H_n \subset O_n$.*
2. *We have $H_n = L_n \rtimes S_n$.*
3. *We have $H_n = Z_2 \wr S_n$.*

We will see later on that the free analogue H_n^+ is a free wreath product. In terms of the hierarchy of integration problems explained in previous section, the available results for usual and free wreath products are as follows:

1. Second order results for wreath products are obtained in [27].
2. Order zero results for free wreath products are obtained in [4].

For the purposes of this paper, what we actually need are the first order results, for both H_n and H_n^+. Most convenient here is to present complete proofs for everything. We begin with a discussion of the first order problem for H_n.

Theorem 2.3. *The asymptotic law of $u_{11} + \ldots + u_{ss}$ with $s = [tn]$ is given by*

$$\beta_t = e^{-t} \sum_{k=-\infty}^{\infty} \delta_k \sum_{p=0}^{\infty} \frac{(t/2)^{|k|+2p}}{(|k|+p)!p!}$$

where δ_k is the Dirac mass at $k \in \mathbb{Z}$.

Proof. We regard H_n as being the symmetry group of the graph $I_n = \{I^1, \ldots, I^n\}$ formed by n segments. The diagonal coefficients are given by:

$$u_{ii}(g) = \begin{cases} 0 & \text{if } g \text{ moves } I^i \\ +1 & \text{if } g \text{ fixes } I^i \\ -1 & \text{if } g \text{ returns } I^i \end{cases}$$

We denote by $\uparrow g, \downarrow g$ the number of segments among $\{I^1, \ldots, I^s\}$ which are fixed, respectively returned by an element $g \in H_n$. With this notation, we have:

$$u_{11} + \ldots + u_{ss} = \uparrow g - \downarrow g$$

We denote by P_n probabilities computed over the group H_n. The density of the law of $u_{11} + \ldots + u_{ss}$ at a point $k \geq 0$ is given by the following formula:

$$\begin{aligned} D(k) &= \mathrm{P}_n(\uparrow g - \downarrow g = k) \\ &= \sum_{p=0}^{\infty} \mathrm{P}_n(\uparrow g = k+p, \downarrow g = p) \end{aligned}$$

Assume first that we have $t = 1$. We use the fact that the probability of $\sigma \in S_n$ to have no fixed points is asymptotically $1/e$. Thus the probability of $\sigma \in S_n$ to have m fixed points is asymptotically $1/(em!)$. In terms of probabilities over H_n, we get:

$$\begin{aligned} \lim_{n\to\infty} D(k) &= \lim_{n\to\infty} \sum_{p=0}^{\infty} (1/2)^{k+2p} \binom{k+2p}{k+p} \mathrm{P}_n(\uparrow g + \downarrow g = k+2p) \\ &= \sum_{p=0}^{\infty} (1/2)^{k+2p} \binom{k+2p}{k+p} \frac{1}{e(k+2p)!} \\ &= \frac{1}{e} \sum_{p=0}^{\infty} \frac{(1/2)^{k+2p}}{(k+p)!p!} \end{aligned}$$

The general case $0 < t \leq 1$ follows by performing some modifications in the above computation. The asymptotic density is computed as follows:

$$\begin{aligned} \lim_{n\to\infty} D(k) &= \lim_{n\to\infty} \sum_{p=0}^{\infty} (1/2)^{k+2p} \binom{k+2p}{k+p} \mathrm{P}_n(\uparrow g + \downarrow g = k+2p) \\ &= \sum_{p=0}^{\infty} (1/2)^{k+2p} \binom{k+2p}{k+p} \frac{t^{k+2p}}{e^t(k+2p)!} \\ &= e^{-t} \sum_{p=0}^{\infty} \frac{(t/2)^{k+2p}}{(k+p)!p!} \end{aligned}$$

Together with $D(-k) = D(k)$, this gives the formula in the statement. $\square$

Theorem 2.4. *The measures β_t form a truncated one-parameter semigroup with respect to convolution.*

Proof. Consider the Bessel function of the first kind:

$$f_k(t)=\sum_{p=0}^\infty\frac{t^{|k|+2p}}{(|k|+p)!p!}$$

The measure in the statement is given by the following formula:

$$\beta_t=e^{-t}\sum_{k=-\infty}^\infty\delta_k\,f_k(t/2)$$

Thus its Fourier transform is given by:

$$F\beta_t(y)=e^{-t}\sum_{k=-\infty}^\infty e^{ky}\,f_k(t/2)$$

We compute now the derivative with respect to t:

$$F\beta_t(y)'=-F\beta_t(y)+\frac{e^{-t}}{2}\sum_{k=-\infty}^\infty e^{ky}\,f_k'(t/2)$$

On the other hand, the derivative of f_k with $k\geq 1$ is given by:

$$\begin{aligned}
f_k'(t) &= \sum_{p=0}^\infty\frac{(k+2p)t^{k+2p-1}}{(k+p)!p!}\\
&= \sum_{p=0}^\infty\frac{(k+p)t^{k+2p-1}}{(k+p)!p!}+\sum_{p=0}^\infty\frac{p\,t^{k+2p-1}}{(k+p)!p!}\\
&= \sum_{p=0}^\infty\frac{t^{k+2p-1}}{(k+p-1)!p!}+\sum_{p=1}^\infty\frac{t^{k+2p-1}}{(k+p)!(p-1)!}\\
&= \sum_{p=0}^\infty\frac{t^{(k-1)+2p}}{((k-1)+p)!p!}+\sum_{p=1}^\infty\frac{t^{(k+1)+2(p-1)}}{((k+1)+(p-1))!(p-1)!}\\
&= f_{k-1}(t)+f_{k+1}(t)
\end{aligned}$$

This computation works in fact for any k, so we get:

$$\begin{aligned}
F\beta_t(y)' &= -F\beta_t(y)+\frac{e^{-t}}{2}\sum_{k=-\infty}^\infty e^{ky}(f_{k-1}(t/2)+f_{k+1}(t/2))\\
&= -F\beta_t(y)+\frac{e^{-t}}{2}\sum_{k=-\infty}^\infty e^{(k+1)y}f_k(t/2)+e^{(k-1)y}f_k(t/2)\\
&= -F\beta_t(y)+\frac{e^y+e^{-y}}{2}\,F\beta_t(y)\\
&= \left(\frac{e^y+e^{-y}}{2}-1\right)F\beta_t(y)
\end{aligned}$$

Thus the log of the Fourier transform is linear in t, and we get the assertion. $\square$

3. Quantum automorphisms

We present now a first approach to the noncommutative analogue of H_n. The idea is to use the fact that H_n is the symmetry group of the cube.

We begin with a discussion involving finite graphs and finite metric spaces:

(1) A finite graph, with vertices labelled $1, \ldots, n$, is uniquely determined by its adjacency matrix $d \in M_n(0,1)$, given by $d_{ij} = 1$ if i, j are connected, and $d_{ij} = 0$ if not. This matrix has to be symmetric, and null on the diagonal.

(2) A finite metric space, with points labelled $1, \ldots, n$, is uniquely determined by its distance matrix $d \in M_n(\mathbb{R}_+)$. This matrix has to be symmetric, 0 on the diagonal and > 0 outside, and its entries must satisfy the triangle inequality.

We see that in both cases, the object under consideration is determined by a scalar matrix $d \in M_n$. In both cases this matrix is real, symmetric, and null on the diagonal, but for our purposes we will just regard it as an arbitrary complex matrix.

Definition 3.1. *Let X be a finite graph, or finite metric space, with matrix $d \in M_n$.*

(1) *The quantum symmetry algebra of X is given by $A_X = A_s(n)/I$, where I is the ideal generated by the n^2 relations coming from the equality $du = ud$.*

(2) *A Hopf algebra coacting on X is a quotient of A_X. In other words, it is a quotient of $A_s(n)$ in which the equality $du = ud$ holds.*

As a first example, when X is a set, or a complete graph, or a simplex, we have $A_X = A_s(n)$. This is because in all these examples, the non-diagonal entries of d are all equal. In other words, we have $d = \lambda(F - 1)$, where λ is a scalar and F is the matrix filled with 1's. But the magic unitarity condition gives $uF = Fu$, hence $I = (0)$.

The basic example of coaction is that of the algebra $C(G_X)$, where G_X is the usual symmetry group of X. In fact, this is the biggest commutative algebra coacting on X. Or, in other words, it is the biggest commutative quotient of A_X. See [4].

Recall now from previous section that the cube space K_n and the cube graph K_n have the same symmetry group, because of a general "drawing principle". Our first result is that this principle holds as well for quantum symmetry groups.

Proposition 3.2. *The cube space K_n and the cube graph K_n have the same quantum symmetry algebra.*

Proof. The distance matrix of the cube is of the following form:

$$d = d_1 + \sqrt{2}\, d_2 + \sqrt{3}\, d_3 + \ldots + \sqrt{n}\, d_n$$

Here for $l = 1, \ldots, n$ the matrix d_l is obtained from d by replacing all lengths different from $\sqrt{l}$ by the number 0, and all lengths equal to $\sqrt{l}$ by the number 1. This is the same as the adjacency matrix of the graph X_l, having as vertices the vertices of the cube, and with edges drawn between pairs of vertices at distance $\sqrt{l}$.

Observe that we have $X_1 = K_n$. Now the powers of d_1 are computed by counting loops on the cube, and we get formulae of the following type:

$$\begin{aligned} d_1^2 &= x_{21}\, 1_n + x_{22}\, d_2 \\ d_1^3 &= x_{31}\, 1_n + x_{32}\, d_2 + x_{33}\, d_3 \end{aligned}$$

$$\begin{aligned} & \dots \\ d_1^n &= x_{n1}\,1_n + x_{n2}\,d_2 + x_{n3}\,d_3 + \ldots + x_{nn}\,d_n \end{aligned}$$

Here x_{ij} are some positive integers. Now these formulae, together with the above one for d, show that we have an equality of algebras $< d >=< d_1 >$. In particular u commutes with d if and only if it commutes with d_1, and this gives the result. □

We have to find the universal Hopf algebra coacting on the cube. For this purpose, we use the cube graph K_n, and we regard it as a Cayley graph. Recall from previous section that the direct product of n copies of Z_2 is denoted $L_n =< \tau_1, \ldots, \tau_n >$.

Lemma 3.3. *K_n is the Cayley graph of the group $L_n =< \tau_1, \ldots, \tau_n >$. Moreover, the eigenvectors and eigenvalues of K_n are given by*

$$\begin{aligned} v_{i_1\ldots i_n} &= \sum_{j_1\ldots j_n} (-1)^{i_1 j_1 + \ldots + i_n j_n} \tau_1^{j_1} \ldots \tau_n^{j_n} \\ \lambda_{i_1\ldots i_n} &= (-1)^{i_1} + \ldots + (-1)^{i_n} \end{aligned}$$

when the vector space spanned by vertices of K_n is identified with $C^*(L_n)$.

Proof. For the first assertion, consider the Cayley graph of L_n. The vertices are the elements of L_n, which are products of the following form:

$$g = \tau_1^{i_1} \ldots \tau_n^{i_n}$$

The sequence of $0-1$ exponents defining such an element determines a point of $\mathbb{R}^n$, which is a vertex of the cube. Thus the vertices of the Cayley graph are the vertices of the cube. Now regarding edges, in the Cayley graph these are drawn between elements g, h having the property $g = h\tau_i$ for some i. In terms of coordinates, the operation $h \to h\tau_i$ means to switch the sign of the i-th coordinate, and to keep the other coordinates fixed. In other words, we get in this way the edges of the cube.

For the second part, recall first that the eigenspaces and eigenvalues of a graph X are by definition those of its adjacency matrix d, and that the action of d on functions on vertices is by summing over neighbors:

$$df(p) = \sum_{q\sim p} f(q)$$

Now by identifying vertices with elements of L_n, hence functions on vertices with elements of the group algebra $C^*(L_n)$, we get the following formula:

$$dv = \tau_1 v + \ldots + \tau_n v$$

For the vector $v_{i_1\ldots i_n}$ in the statement, we have the following computation:

$$\begin{aligned} dv_{i_1\ldots i_n} &= \sum_s \tau_s \sum_{j_1\ldots j_n} (-1)^{i_1 j_1 + \ldots + i_n j_n} \tau_1^{j_1} \ldots \tau_n^{j_n} \\ &= \sum_s \sum_{j_1\ldots j_n} (-1)^{i_1 j_1 + \ldots + i_n j_n} \tau_1^{j_1} \ldots \tau_s^{j_s+1} \ldots \tau_n^{j_n} \\ &= \sum_s \sum_{j_1\ldots j_n} (-1)^{i_s} (-1)^{i_1 j_1 + \ldots + i_n j_n} \tau_1^{j_1} \ldots \tau_s^{j_s} \ldots \tau_n^{j_n} \end{aligned}$$

$$\begin{aligned} &= \sum_s (-1)^{i_s} \sum_{j_1\ldots j_n} (-1)^{i_s}(-1)^{i_1j_1+\ldots+i_nj_n}\tau_1^{j_1}\ldots\tau_n^{j_n} \\ &= \lambda_{i_1\ldots i_n} v_{i_1\ldots i_n} \end{aligned}$$

Together with the fact that the vectors $v_{i_1\ldots i_n}$ are linearly independent, and that there are 2^n of them, this gives the result. □

The above result shows that the eigenvalues of K_n appear as sums of ± 1 terms, with n terms in each sum. Thus the eigenvalues are the numbers $s \in \{-n,\ldots,n\}$. The corresponding eigenspaces are denoted E_s.

4. Twisted orthogonality

We know from previous section that the cube has a certain quantum symmetry algebra. This is by definition a quotient of $A_s(2^n)$, the relations being those expressing edge preservation, in a metric space or graph-theoretic sense.

Moreover, we have a preference for the cube regarded as a graph, because this is a familiar object, namely the Cayley graph of the group $L_n = Z_2^n$.

The quantum symmetry algebras of Cayley graphs are not computed in general. However, for the group L_n, the study leads to the following algebra:

Definition 4.1. *$C(O_n^{-1})$ is the quotient of $A_o(n)$ by the following relations:*
(1) *$u_{ij}u_{ik} = -u_{ik}u_{ij}$, $u_{ji}u_{ki} = -u_{ki}u_{ji}$, for $i \neq j$.*
(2) *$u_{ij}u_{kl} = u_{kl}u_{ij}$ for $i \neq k$, $j \neq l$.*

The relations in the statement are the skew-commutation relations for the quantum group GL_n^{-1}. They define a Hopf ideal, so we have indeed a Hopf algebra.

We should mention that this is not exactly the algebra of continuous fuctions on the quantum group O_n^{-1}, in the sense of classical texts like [21], [24], [25]. However, our definition is natural as well, and is further justified by the following result:

Theorem 4.2. *$C(O_n^{-1})$ is the quantum symmetry algebra of K_n.*

Proof. We have to show first that the quantum group O_n^{-1} acts on the cube. This means that the associated Hopf algebra $C(O_n^{-1})$ coacts on the cube, in the sense explained in the beginning of this section. Now recall from [4] that this is the same as asking for the existence of a coaction map:

$$\alpha : C(V_n) \to C(V_n) \otimes C(O_n^{-1})$$

Here V_n is the set of vertices of K_n, and the coaction α should be thought of as being the functional analytic transpose of an action map $a : (v, g) \to g(v)$.

With notations from previous proof, we have to construct a map as follows:

$$\alpha : C^*(L_n) \to C^*(L_n) \otimes C(O_n^{-1})$$

We claim that the following formula provides the answer:

$$\alpha(\tau_i) = \sum_j \tau_j \otimes u_{ji}$$

Indeed, the elements τ_i generate the algebra on the left, with the relations $\tau_i^* = \tau_i$, $\tau_i^2 = 1$ and $\tau_i\tau_j = \tau_j\tau_i$. These relations are satisfied as well by the elements $\alpha(\tau_i)$, so we have a morphism of algebras. Now since u is a corepresentation, α is coassociative and counital. It remains to check that α preserves the adjacency matrix of K_n.

It is routine to check that for $i_1 \neq i_2 \neq \ldots \neq i_l$ we have:

$$\alpha(\tau_{i_1}\ldots\tau_{i_l}) = \sum_{j_1\neq\ldots\neq j_l} \tau_{j_1}\ldots\tau_{j_l} \otimes u_{j_1i_1}\ldots u_{j_li_l}$$

In terms of eigenspaces E_s of the adjacency matrix, this gives:

$$\alpha(E_s) \subset E_s \otimes C(O_n^{-1})$$

It follows that α preserves the adjacency matrix of K_n, so it is a coaction on K_n as claimed. See [4] for unexplained terminology in this proof.

So far, we know that $C(O_n^{-1})$ coacts on the cube. We have to show that the canonical map $B \to C(O_n^{-1})$ is an isomorphism, where $B = A_{K_n}$ is the quantum symmetry algebra of the cube. For this purpose, consider the universal coaction on the cube:

$$\beta : C^*(L_n) \to C^*(L_n) \otimes B$$

By eigenspace preservation we get elements x_{ij} such that:

$$\beta(\tau_i) = \sum_j \tau_j \otimes x_{ji}$$

By applying β to the relation $\tau_i\tau_j = \tau_j\tau_i$ we get $x^tx = 1$, so the matrix $x = (x_{ij})$ is orthogonal. By applying β to the relation $\tau_i^2 = 1$ we get:

$$1 \otimes \sum_k x_{ki}^2 + \sum_{k<l} \tau_k\tau_l \otimes (x_{ki}x_{li} + x_{li}x_{ki}) = 1 \otimes 1$$

This gives $x_{ki}x_{li} = -x_{li}x_{ki}$ for $i \neq j$, $k \neq l$, and by using the antipode we get $x_{ik}x_{il} = -x_{il}x_{ik}$ for $k \neq l$. Also, by applying β to $\tau_i\tau_j = \tau_j\tau_i$ with $i \neq j$ we get:

$$\sum_{k<l} \tau_k\tau_l \otimes (x_{ki}x_{lj} + x_{li}x_{kj}) = \sum_{k<l} \tau_k\tau_l \otimes (x_{kj}x_{li} + x_{lj}x_{ki})$$

It follows that for $i \neq j$ and $k \neq l$, we have:

$$x_{ki}x_{lj} + x_{li}x_{kj} = x_{kj}x_{li} + x_{lj}x_{ki}$$

In other words, we have $[x_{ki}, x_{lj}] = [x_{kj}, x_{li}]$. By using the antipode we get $[x_{jl}, x_{ik}] = [x_{il}, x_{jk}]$. Now by combining these relations we get:

$$[x_{il}, x_{jk}] = [x_{ik}, x_{jl}] = [x_{jk}, x_{il}] = -[x_{il}, x_{jk}]$$

This gives $[x_{il}, x_{jk}] = 0$, so the elements x_{ij} satisfy the relations for $C(O_n^{-1})$. This gives a map $C(O_n^{-1}) \to B$, which is inverse to the canonical map $B \to C(O_n^{-1})$. □

We discuss now representations of O_n^{-1}. There are many techniques that can be applied, see e.g. [24]. We present here a simplified proof of the tensor equivalence between the categories of finite dimensional representations of O_n and O_n^{-1}, by using Morita equivalence type techniques from [10], [12], [26].

Theorem 4.3. *The category of corepresentations of $C(O_n^{-1})$ is tensor equivalent to the category of representations of O_n.*

Proof. Let A be the algebra of polynomial functions on O_n, the standard matrix of generators being denoted a_{ij}. This is a dense Hopf $*$-subalgebra of $C(O_n)$.

Let also B be the universal $*$-algebra defined by the relations of $C(O_n^{-1})$, with generators denoted b_{ij}. This embeds as a dense Hopf $*$-algebra of $C(O_n^{-1})$.

The two categories in the statement are those of comodules over A and B. By [26], what we have to do is to construct an $A-B$ Hopf bi-Galois extension.

Let Z be the universal algebra generated by the entries z_{ij} of an orthogonal $n \times n$ matrix, subject to the following relations:

$$z_{ij}z_{kl} = -z_{kl}z_{ij} \text{ for } j \neq l, \quad z_{ij}z_{kj} = z_{kj}z_{ij}$$

We can make Z into an $A-B$ bicomodule algebra, by constructing coactions $\alpha : Z \to A \otimes Z$ and $\beta : Z \to Z \otimes B$ as follows:

$$\begin{aligned}\alpha(z_{ij}) &= \sum_k a_{ik} \otimes z_{kj} \\ \beta(z_{ij}) &= \sum_k z_{ik} \otimes b_{kj}\end{aligned}$$

We have to check first that Z is nonzero. Consider the group $L_n =< \tau_1, \ldots, \tau_n >$. We have a Hopf algebra morphism $\pi : A \to C^*(L_n)$, given by $a_{ij} \to \delta_{ij}\tau_i$. We define a bicharacter of L_n as follows:

$$\sigma(\tau_i, \tau_j) = \begin{cases} -1 \text{ if } i < j \\ +1 \text{ if } i \geq j \end{cases}$$

Since $\rho = \sigma(\pi \otimes \pi)$ is a Hopf 2-cocycle on A, we can construct as in [10] the right twisted algebra A_ρ. The map $Z \to A_\rho$ given by $z_{ij} \to a_{ij}$ shows that Z is nonzero.

Consider now the universal algebra T generated by entries of an orthogonal $n \times n$ matrix t, subject to the following relations:

$$t_{ij}t_{kl} = -t_{kl}t_{ij} \text{ for } i \neq k, \quad t_{ij}t_{il} = t_{il}t_{ij}$$

We define now algebra maps $\gamma : A \to Z \otimes T$ and $\delta : B \to T \otimes Z$ as follows:

$$\begin{aligned}\gamma(a_{ij}) &= \sum_k z_{ik} \otimes t_{kj} \\ \delta(b_{ij}) &= \sum_k t_{ik} \otimes z_{kj}\end{aligned}$$

Together with the map $s : T \to Z$ given by $t_{ij} \to z_{ji}$, these make (A, B, Z, T) into a Hopf-Galois system in the sense of [10]. Thus Z is an $A-B$ bi-Galois object, and by [26] this gives the desired equivalence of categories. Moreover, our proof extends in a straightforward manner to the C^*-algebraic setting of [12], so the tensor category equivalence we have constructed is a tensor C^*-category equivalence. $\square$

The above result shows that integration over O_n^{-1} appears as a kind of twisting of integration over O_n. In particular, the integral geometry of O_n^{-1} is basically governed by independence, so we don't have freeness.

5. Sudoku unitaries

We construct in this section the free version of H_n. The idea is to use the fact that H_n is the symmetry group of I_n, the graph formed by n segments.

Definition 5.1. *A sudoku unitary is a magic unitary of the form*

$$w = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

where both a, b are square matrices.

In this definition the matrices a, b are of course of same size. The terminology comes from a vague analogy with sudoku squares.

As a first example, any 2×2 magic unitary is a sudoku unitary:

$$w = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

Here is an example of 4×4 sudoku unitary, with p, q being projections:

$$w = \begin{pmatrix} p & 0 & 1-p & 0 \\ 0 & q & 0 & 1-q \\ 1-p & 0 & p & 0 \\ 0 & 1-q & 0 & q \end{pmatrix}$$

The third example comes from the hyperoctahedral group. When regarding H_n as symmetry group of I_n, hence as subgroup of $S_{2n} \subset O_{2n}$, the matrix coordinates form a sudoku unitary. Indeed, the characteristic functions a_{ij}, b_{ij} are those of group elements mapping $I^i \to I^j$ by fixing orientations, respectively by changing them.

Definition 5.2. *$A_h(n)$ is the universal algebra generated by the entries of a $2n \times 2n$ sudoku unitary.*

It follows from the above discussion that we have $A_h(1) = C(Z_2)$, and that the algebra $A_h(2)$ is not commutative, and infinite dimensional. The noncommutativity and infinite dimensionality of $A_h(n)$ hold in fact for any $n \geq 2$, because we have surjective morphisms $A_h(n+1) \to A_h(n)$. These are given by the fact that if $w = \left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$ is sudoku, then the following matrix is sudoku as well:

$$w' = \begin{pmatrix} a & & b & \\ & 1 & & 0 \\ b & & a & \\ & 0 & & 1 \end{pmatrix}$$

Here the blank spaces are filled with row or column vectors consisting of zeroes.

A first result motivating our definition is as follows.

Proposition 5.3. *$A_h(n)$ is the quantum symmetry algebra of I_n.*

Proof. The quantum symmetry algebra of I_n is by definition the quotient of $A_s(2n)$ by the relations coming from commutation of the magic unitary matrix, say w, with the adjacency matrix of I_n. This adjacency matrix is given by:

$$e = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$$

We can write as well w as a 2×2 matrix, with $n \times n$ matrix entries:

$$w = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The relation $ew = we$ is equivalent to $a = d, b = c$, and this gives the result. □

We proceed now with the study of $A_h(n)$, in analogy with what we know about H_n. Our first goal is to extend the wreath product decomposition $H_n = Z_2 \wr S_n$. We use the notion of free wreath product, introduced in [11].

Consider two quantum permutation algebras (A, u) and (B, v). This means that we have surjective Hopf algebra morphisms $A_s(n) \to A$ and $A_s(m) \to B$ for certain numbers n, m, and the fundamental corepresentations of A, B are denoted u, v.

Definition 5.4. *The free wreath product of (A, u) and (B, v) is given by*

$$A *_w B = (A^{*n} * B) / < [u_{ij}^{(a)}, v_{ab}] = 0 >$$

where n is the size of v, and has magic unitary matrix $w_{ia,jb} = u_{ij}^{(a)} v_{ab}$.

This definition is justified by formulae of the following type, where G, A denote classical symmetry groups, respectively quantum symmetry algebras:

$$\begin{aligned} G(X * Y) &= G(X) \wr G(Y) \\ A(X * Y) &= A(X) *_w A(Y) \end{aligned}$$

There are several such formulae, depending on the types of graphs and free products considered. See [4], [11]. The formula we are interested in is:

$$\begin{aligned} G(I \dots I) &= G(I) \wr G(\circ \dots \circ) \\ A(I \dots I) &= A(I) *_w A(\circ \dots \circ) \end{aligned}$$

Here I is a segment, $\circ$ is a point, and the dots mean n-fold disjoint union. We recognize at left the graph I_n. The upper formula is nothing but the isomorphism $H_n = Z_2 \wr S_n$ we started with. As for the lower formula, this is what we need:

Theorem 5.5. *We have $A_h(n) = C(Z_2) *_w A_s(n)$.*

This follows from the above discussion, see [11], [4]. As a first application, we can make a comparison between $A_h(n)$ and $C(O_n^{-1})$. Recall that both algebras appear as noncommutative versions of $C(H_n)$, in the sense that their maximal commutative quotient if $C(H_n)$. The problem is whether these algebras are isomorphic or not.

Proposition 5.6. *We have the following isomorphisms:*

(1) $A_h(2) = C(Z_2) *_w C(Z_2)$.
(2) $A_h(3) = C(Z_2) *_w C(S_3)$.

(3) $C(O_2^{-1}) = C(Z_2) *_w C(Z_2)$.
(4) $C(O_3^{-1}) = C(Z_2) \otimes A_s(4)$.

Proof. The first two assertions follow from the above theorem and from the exceptional isomorphisms $A_s(2) = C(Z_2)$ and $A_s(3) = C(S_3)$.

The last two assertions follow by using the exceptional isomorphisms $K_2 = I^c$ and $K_3 = I \times T$, where I is the segment and T is the tetrahedron. See [5] for details. $\square$

This result shows that $A_h(n)$ and $C(O_n^{-1})$ are equal for $n = 2$, and different for $n = 3$. Actually these algebras are different for any $n \geq 3$, as one can see by comparing representation theory invariants: the fusion rules for $C(O_n^{-1})$ are the same as those for O_n, and in particular they are commutative, while for $A_h(n)$ with $n \geq 3$ we will see that the fusion rules are not commutative.

6. Cubic unitaries

In this section we present an alternative approach to $A_h(n)$. The idea is to use the fact that H_n is a subgroup of O_n, hence the matrix of coordinates has size n. The relevant condition on the entries of this matrix is as follows.

Definition 6.1. *A cubic unitary is an orthogonal matrix u satisfying*

$$u_{ij}u_{ik} = u_{ji}u_{ki} = 0$$

for $j \neq k$. In other words, on rows and columns, distinct entries satisfy $ab = ba = 0$.

The basic example comes indeed from the hyperoctahedral group $H_n \subset O_n$. In terms of the action on the space $I_n = \{I^1, \ldots, I^n\}$ formed by the $[-1, 1]$ segments on each coordinate axis, the matrix coordinates are given by:

$$u_{ij}(g) = \begin{cases} 0 & \text{if } g(I^j) \neq I^i \\ +1 & \text{if } g(I^j) = I^i \\ -1 & \text{if } g(I^j) = I^i \text{ returned} \end{cases}$$

Now when i is fixed and j varies, or vice versa, the supports of u_{ij} form partitions of H_n. Thus the matrix of coordinates u is cubic in the above sense.

Theorem 6.2. *$A_h(n)$ is the universal algebra generated by the entries of a $n \times n$ cubic unitary matrix.*

Proof. Consider the universal algebra in the statement:

$$A_c(n) = C^* \left(u_{ij} \;\middle|\; u = n \times n \text{ cubic unitary} \right)$$

We have to compare it with the following universal algebra:

$$\begin{aligned} A_h(n) &= C^* \left(w_{ij} \;\middle|\; w = 2n \times 2n \text{ sudoku unitary} \right) \\ &= C^* \left(a_{ij}, b_{ij} \;\middle|\; \begin{pmatrix} a & b \\ b & a \end{pmatrix} = 2n \times 2n \text{ magic unitary} \right) \end{aligned}$$

We construct first the arrow $A_c(n)\to A_h(n)$. Consider the matrix $a-b$, having as entries the elements $a_{ij}-b_{ij}$. The elements a_{ij},b_{ij} being self-adoint, their difference is self-adjoint as well. Thus $a-b$ is a matrix of self-adjoint elements. We have the following formula for products on columns of $a-b$:

$$\begin{aligned}(a-b)_{ik}(a-b)_{jk}&=a_{ik}a_{jk}-a_{ik}b_{jk}-b_{ik}a_{jk}+b_{ik}b_{jk}\\&=\begin{cases}0-0-0+0\text{ for } i\neq j\\ a_{ik}-0-0+b_{ik}\text{ for } i=j\end{cases}\\&=\begin{cases}0\text{ for } i\neq j\\ a_{ik}+b_{ik}\text{ for } i=j\end{cases}\end{aligned}$$

In the $i=j$ case it follows from the sudoku condition that the elements $a_{ik}+b_{ik}$ sum up to 1. Thus the columns of $a-b$ are orthogonal. A similar computation works for rows, and we conclude that $a-b$ is an orthogonal matrix.

Now by using the $i\neq j$ computation, along with its row analogue, we conclude that $a-b$ is cubic. Thus we can define our morphism by the following formula:

$$\varphi(u_{ij})=a_{ij}-b_{ij}$$

We construct now the inverse morphism. Consider the following elements:

$$\begin{aligned}\alpha_{ij}&=\frac{u_{ij}^2+u_{ij}}{2}\\ \beta_{ij}&=\frac{u_{ij}^2-u_{ij}}{2}\end{aligned}$$

It follows from the cubic condition that these elements are projections, and that the following matrix is a sudoku unitary:

$$M=\begin{pmatrix}(\alpha_{ij})&(\beta_{ij})\\(\beta_{ij})&(\alpha_{ij})\end{pmatrix}$$

Thus we can define our morphism by the following formula:

$$\begin{aligned}\psi(a_{ij})&=\frac{u_{ij}^2+u_{ij}}{2}\\ \psi(b_{ij})&=\frac{u_{ij}^2-u_{ij}}{2}\end{aligned}$$

We check now the fact that ψ,φ are indeed inverse morphisms:

$$\begin{aligned}\psi\varphi(u_{ij})&=\psi(a_{ij}-b_{ij})\\&=\frac{u_{ij}^2+u_{ij}}{2}-\frac{u_{ij}^2-u_{ij}}{2}\\&=u_{ij}\end{aligned}$$

As for the other composition, we have the following computation:

$$\varphi\psi(a_{ij})=\varphi\left(\frac{u_{ij}^2+u_{ij}}{2}\right)$$

$$\begin{aligned} &= \frac{(a_{ij}-b_{ij})^2+(a_{ij}-b_{ij})}{2} \\ &= a_{ij} \end{aligned}$$

A similar computation gives $\varphi\psi(b_{ij}) = b_{ij}$, which completes the proof. □

We are interested now in quantum analogues of the embeddings $S_n \subset H_n \subset O_n$. It is convenient to state the corresponding result in the following way.

Proposition 6.3. *We have the following commutative diagram:*

$$\begin{array}{ccccc} A_o(n) & \to & A_h(n) & \to & A_s(n) \\ \downarrow & & \downarrow & & \downarrow \\ C(O_n) & \to & C(H_n) & \to & C(S_n) \end{array}$$

Moreover, both horizontal compositions are the canonical quotient maps.

Proof. The upper left arrow $A_o(n) \to A_h(n)$ is by definition the one coming from previous theorem. We construct now the arrow $A_h(n) \to A_s(n)$. With sudoku notations for $A_h(n)$, consider the ideal I generated by the relations $b_{ij} = 0$. We have:

$$\begin{aligned} A_h(n)/I &= C^*\left(a_{ij} \,\middle|\, \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = 2n \times 2n \text{ magic unitary}\right) \\ &= C^*\left(a_{ij} \,\middle|\, a = n \times n \text{ magic unitary}\right) \end{aligned}$$

The algebra on the right is $A_s(n)$, and this gives a map $A_h(n) \to A_s(n)$. The fact that both squares of the diagram commute is clear from definitions.

It remains to check the last assertion. With the above notations, the upper horizontal composition is given by $u_{ij} \to a_{ij} - b_{ij} \to a_{ij}$, so it is the canonical quotient map $A_o(n) \to A_s(n)$. As for the lower horizontal composition, the fact that this is the transpose of the canonical embedding $S_n \subset O_n$ follows from definitions. □

One can prove that the quotient map $A_h(n) \to A_s(n)$ has a canonical section, which is a quantum analogue for the transpose of the canonical quotient map $H_n \to S_n$.

The general idea is that the quantum analogues of various properties of H_n are shared by the quantum groups O_n^{-1} and H_n^+, the latter being the spectrum of $A_h(n)$.

7. Representation theory

For the rest of this paper, we make the assumption $n \geq 4$.

In this section we discuss representation theory problems. The idea is that the presentation relations for various quantum algebras have graphical interpretations, which lead via Tannakian duality to Temperley-Lieb type diagrams.

The fact that free wreath products of type $A_h(n)$ correspond to the Fuss-Catalan algebra of Bisch and Jones [13] is known from some time. See [4] for a general statement in this sense, and for the complete story and state-of-art of the subject.

In this paper we have some special requirements in this sense:

(1) In order to get the good integration formulae, the fundamental corepresentation here must be the $n \times n$ cubic unitary. This is a bit different from what can be found in the litterature, where the $2n \times 2n$ sudoku unitary is used.
(2) In order to use various free probability formulae, we need results in terms of colored non-crossing partitions, rather than in terms of colored Temperley-Lieb diagrams (or Fuss-Catalan diagrams) used in the litterature.
(3) The whole thing is to be extended in next sections, for Hopf algebras satisfying $A_o(n) \to A \to A_s(n)$. In other words, the results for $A_o(n), A_s(n)$ must be explained as well in detail, at some point of the paper.

Summarizing, we need a uniform treatment of the algebras $A_x(n)$ with $x = o, h, s$, with the fundamental corepresentation being the $n \times n$ one, and in terms of both diagrams and partitions. We use the group subscripts o, h, s for various types of diagrams.

Definition 7.1. *The set of Temperley-Lieb diagrams is*

$$TL_s = \left\{ \begin{matrix} \cdot\ \cdot\ \cdot\ \cdot & \leftarrow & 2l \text{ points} \\ W & \leftarrow & l+k \text{ strings} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \leftarrow & 2k \text{ points} \end{matrix} \right\}$$

where strings join pairs of points, and do not cross.

Given a Temperley-Lieb diagram, we can color both rows of points *abbaabba*.. with the pattern *abba* repeated as many time as needed, and with an extra *ab* block at the end, if needed. If strings can be colored as well, we say that the diagram is colored.

Definition 7.2. *The set of colored Temperley-Lieb diagrams is*

$$TL_h = \left\{ \begin{matrix} \cdot\ \cdot\ \cdot\ \cdot & \leftarrow & 2l \text{ points} \\ W & \leftarrow & l+k \text{ colored strings} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \leftarrow & 2k \text{ points} \end{matrix} \right\}$$

where both rows of points are colored abbaabba...

Given a Temperley-Lieb diagram, we can collapse pairs of consecutive neighbors, in both rows. If strings can be collapsed as well, we say that the diagram is double. Observe that a double Temperley-Lieb diagram is colored.

Definition 7.3. *The set of double Temperley-Lieb diagrams is*

$$TL_o = \left\{ \begin{matrix} \cdot\ \cdot\ \cdot\ \cdot & \leftarrow & 2l \text{ points} \\ W & \leftarrow & (l+k)/2 \text{ double strings} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \leftarrow & 2k \text{ points} \end{matrix} \right\}$$

where points and strings are taken, as usual, up to isotopy.

Probably most illustrating here is the following table, for small number of points. The Temperley-Lieb diagrams between $2l$ points and $2k$ points are called $l \to k$ diagrams.

	TL_o	TL_h	TL_s	$\#TL_s$
$0 \to 1$			$\cap$	1
$0 \to 2$	$\Cap$	$\Cap$	$\cap\cap$, $\Cap$	2
$1 \to 1$	$\vert\vert$	$\vert\vert$	$\vert\vert$, $\overset{\cup}{\cap}$	2
$1 \to 2$			$\vert\vert\cap$, $\cap\vert\vert$, $\vert\cap\vert$, $\overset{\cup}{\cap\cap}$, $\overset{\cup}{\Cap}$	5
$2 \to 2$	$\vert\vert\vert\vert$, $\overset{\Cup}{\Cap}$	$\vert\vert\vert\vert$, $\overset{\Cup}{\Cap}$, $\vert\overset{\cup}{\cap}\vert$	$\vert\vert\vert\vert$, $\overset{\Cup}{\Cap}$, $\vert\overset{\cup}{\cap}\vert$, $\overset{\Cup}{\cap\cap}$, ${}^{\cup}/_{\cap}$, ...	14

We recognize at right the Catalan numbers. The number of elements in the left column is also a Catalan number. As for the middle column, the diagrams here are counted by Fuss-Catalan numbers. This will be discussed in detail later on.

To any number n and any subscript $x \in \{o, h, s\}$ we associate a tensor category $TL_x(n)$, in the following way:

1. The objects are the positive integers: $0, 1, 2, \ldots$
2. The arrows $l \to k$ are linear combinations of $l \to k$ diagrams in TL_x.
3. The composition is by vertical concatenation, with the rule $\bigcirc = n$.
4. The tensor product is by horizontal concatenation.
5. The involution is by upside-down turning.

Temperley-Lieb diagrams act on tensors according to the following formula, where the middle symbol is 1 if all strings of p join pairs of equal indices, and is 0 if not:

$$p(e_{i_1} \otimes \ldots \otimes e_{i_l}) = \sum_{j_1 \ldots j_k} \begin{pmatrix} i_1\, i_1 \ldots i_l\, i_l \\ p \\ j_1\, j_1 \ldots j_k\, j_k \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_k}$$

By using the trace one can see that different diagrams produce different linear maps, and this action makes $TL_x(n)$ a subcategory of the category of Hilbert spaces.

Theorem 7.4. *We have the Tannakian duality diagram*

$$\begin{array}{ccccc} A_o(n) & \to & A_h(n) & \to & A_s(n) \\ & & & & \\ \updownarrow & & \updownarrow & & \updownarrow \\ & & & & \\ TL_o(n) & \subset & TL_h(n) & \subset & TL_s(n) \end{array}$$

where all horizontal arrows are the canonical ones.

The proof uses Woronowicz's Tannakian duality in [36]. The idea is that the presentations of the above algebras translate into presentations of associated tensor categories, which in turn lead to diagrams. Consider a unitary matrix u.

1. The fact that u is orthogonal is equivalent to $V \in Hom(1, u^{\otimes 2})$, where V is the canonical vector in $\mathbb{C}^n \otimes \mathbb{C}^n$. But the relations satisfied by V in a categorical sense are those satisfied by $v = \Cap$, which generates TL_o. See [7].
2. The fact that u is cubic is equivalent to $V \in Hom(1, u^{\otimes 2})$ and $E \in End(u^{\otimes 2})$, where E is the canonical projection. But the relations satisfied by V, E are those satisfied by $v = \Cap$ and $e = \vert\overset{\cup}{\cap}\vert$, which generate TL_h. See [3].

(3) The fact that u is magic is equivalent to $M \in Hom(u^{\otimes 2}, u)$ and $U \in Hom(1, u)$, where M, U are the multiplication and unit of $\mathbb{C}^n$. But M, U satisfy the same relations as $m = |\cup|$ and $u = \cap$, which generate TL_s. See [8].

These three remarks give the three vertical arrows, standing for Tannakian dualities. As for horizontal compatibility, this is clear from definitions.

8. The Weingarten formula

We can proceed now with the Weingarten general algorithm. Our first task is to translate the Tannakian result into a statement regarding fixed points.

Definition 8.1. *An arch is a Temperley-Lieb diagram having no upper points.*

The set of double, colored and usual arches are denoted AR_x with $x = o, h, s$. We have the following table, for small values of k.

	AR_o	AR_h	AR_s	$\#AR_s$
1			$\cap$	1
2	$\Cap$	$\Cap$	$\Cap$, $\cap\cap$	2
3			$\cap\Cap$, $\Cap\cap$, $\cap\cap\cap$, $\lceil\Cap\rceil$, $\lceil\cap\cap\rceil$	5
4	$\Cap\Cap$, $\lceil\Cap\rceil$	$\Cap\Cap$, $\lceil\Cap\rceil$, $\lceil\cap\cap\cap\rceil$	$\Cap\Cap$, $\lceil\Cap\rceil$, $\lceil\cap\cap\cap\rceil$, $\cap\Cap\cap$, ...	14

The Tannakian duality result in previous section has the following reformulation.

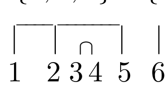
Proposition 8.2. *The space of fixed points of $u^{\otimes k}$ is as follows.*

(1) *For $A_o(n)$ this is the space spanned by $AR_o(k)$.*
(2) *For $A_h(n)$ this is the space spanned by $AR_h(k)$.*
(3) *For $A_s(n)$ this is the space spanned by $AR_s(k)$.*

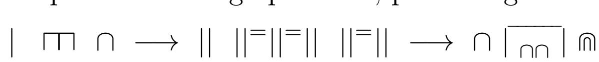
The arches can be regarded as non-crossing partitions, by collapsing pairs of consecutive neighbors. As an example, we have the following table:

arch	$\cap$	$\Cap$	$\cap\cap$	$\Cap\cap$	$\cap\Cap$	$\lceil\cap\cap\rceil$	$\cap\cap\cap$
partition	$\mid$	$\cap$	$\mid\mid$	$\cap\mid$	$\mid\cap$	$\sqcap\!\sqcap$	$\mid\mid\mid$

This construction probably deserves more explanations. First, we regard non-crossing partitions as diagrams, in the obvious way. Here is for instance the diagram corresponding to the partition $\{1,2,3,4,5,6\} = \{1,2,5\} \cup \{3,4\} \cup \{6\}$:



And here is an example of fattening operation, producing arches from partitions:



The sets of partitions coming from double arches, colored arches and usual arches are denoted $NC_x(k)$, with $x = o, h, s$. Here k is the number of points of the set which is partitioned, equal to half of the number of lower points of the arch.

The above result has the following reformulation.

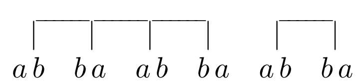
Proposition 8.3. *The space of fixed points of $u^{\otimes k}$ is as follows:*

1. *For $A_o(n)$ this is the space spanned by $NC_o(k)$.*
2. *For $A_h(n)$ this is the space spanned by $NC_h(k)$.*
3. *For $A_s(n)$ this is the space spanned by $NC_s(k)$.*

It is useful to give an intrinsic description of all these notions. Going from arches to non-crossing partitions leads to the following sets:

1. $NC_o(k)$ is the set of non-crossing partitions into pairs.
2. $NC_h(k)$ is the set of non-crossing partitions into even parts.
3. $NC_s(k)$ is the set of non-crossing partitions.

The elements of $NC_o(k)$ are also called non-crossing pair-partitions, and those of $NC_h(k)$, non-crossing colored partitions. This is because we have the following characterization of $NC_h(k)$, which makes the link with colored diagrams: when attaching alternative labels $ab, ba, ab, ba, \ldots$ to the legs of the partition, the condition is that when travelling on sides of strings, different labels do not meet. Here is an example:



Probably most illustrating here is the following table, for small values of k.

	NC_o	NC_h	NC_s	$\#NC_s$
1			$\mid$	1
2	$\cap$	$\cap$	$\cap$, $\mid\,\mid$	2
3			$\mid\cap$, $\cap\mid$, $\mid\,\mid\,\mid$, ⋒ , ⊓⊓	5
4	$\cap\cap$, ⋒	$\cap\cap$, ⋒ , ⊓⊓⊓	$\cap\cap$, ⋒ , ⊓⊓⊓ , $\mid\cap\mid$, $\ldots$	14

Now regarding fixed vectors, we can plug multi-indices $i = (i_1 \ldots i_k)$ into partitions of $\{1, \ldots, k\}$, then define coupling numbers δ_{pi} and vectors v_p, in the following way.

Proposition 8.4. *The vector created by a partition is given by*

$$v_p = \sum_i \delta_{pi}\, e_{i_1} \otimes \ldots \otimes e_{i_k}$$

where $\delta_{pi} = 0$ if some block of p contains two different indices of i, and $\delta_{pi} = 0$ if not.

This is indeed a reformulation of the action of Temperley-Lieb diagrams on tensors: the general formula makes arches create tensors, and by using the correspondence between arches and non-crossing partitions we get the above formula.

For a partition p we denote by $|p|$ the number of its blocks.

For two partitions p, q, we define $p \vee q$ to be the partition obtained as set-theoretic least upper bound of p, q. That is, $p \vee q$ is such that two elements which are in the same block of p, or in the same block of q, have to be in the same block of $p \vee q$.

Theorem 8.5. *We have the Weingarten formula*

$$\int u_{i_1j_1}\ldots u_{i_kj_k}=\sum_{pq}\delta_{pi}\delta_{qj}W_{kn}(p,q)$$

where u is the fundamental corepresentation of $A_x(n)$, and:

(1) *The diagrams are the partitions in $NC_x(k)$.*
(2) *The couplings with indices are by plugging indices into partitions.*
(3) *The Gram matrix is given by $G_{kn}(p,q)=n^{|p\vee q|}$.*
(4) *The Weingarten matrix is given by $W_{kn}=G_{kn}^{-1}$.*

Proof. The fact that G_{kn} is indeed the Gram matrix is checked as follows:

$$\begin{aligned}< v_p,v_q > &= \left\langle\sum_i\delta_{pi}\,e_{i_1}\otimes\ldots\otimes e_{i_k},\sum_i\delta_{qi}\,e_{i_1}\otimes\ldots\otimes e_{i_k}\right\rangle\\ &= \sum_i\delta_{pi}\delta_{qi}=\sum_i\delta_{p\vee q,i}=n^{|p\vee q|}\end{aligned}$$

The rest of the statement is clear from definitions and from general theory in [35], as explained in section 1 for classical groups. See [7] or [8] for details. □

Theorem 8.6. *We have the moment formula*

$$\int(u_{11}+\ldots+u_{ss})^k=Tr(G_{kn}^{-1}G_{ks})$$

where u is the fundamental corepresentation of $A_x(n)$.

Proof. We have the following computation:

$$\begin{aligned}\int(u_{11}+\ldots+u_{ss})^k &= \sum_{i_1=1}^s\ldots\sum_{i_k=1}^s\sum_{pq}\delta_{pi}\delta_{qi}G_{kn}^{-1}(p,q)\\ &= \sum_{pq}G_{kn}^{-1}(p,q)\sum_{i_1=1}^s\ldots\sum_{i_k=1}^s\delta_{pi}\delta_{qi}\\ &= \sum_{pq}G_{kn}^{-1}(p,q)G_{ks}(p,q)\end{aligned}$$

This gives the formula in the statement. □

9. Numeric results

We can compute now the low order moments of $u_{11}+\ldots+u_{ss}$. For $A_o(n)$ and $A_s(n)$ this is done in [7], [8]. For $A_h(n)$ the situation is as follows.

Theorem 9.1. *For the algebra $A_h(n)$ we have the moment formulae*

$$\begin{aligned}\int(u_{11}+\ldots+u_{ss})^2 &= \frac{s}{n}\\ \int(u_{11}+\ldots+u_{ss})^4 &= \frac{s}{n}\cdot\frac{n+2s-3}{n-1}\end{aligned}$$

$$\int (u_{11} + \ldots + u_{ss})^6 \quad = \quad \frac{s}{n} \cdot \frac{P_s(n)}{(n-1)^2(n-2)}$$

where $P_s(n) = n^3 + (6s - 10)n^2 + (5s^2 - 30s + 30)n - (8s^2 - 30s + 24)$.

Proof. At $l = 1$ there is just one diagram, namely $\cap$, having one block. Thus the Gram matrix is $G_{2n} = (n)$, and the computation of the second moment goes as follows:

$$Tr(G_{2n}^{-1} G_{2s}) = Tr\left((n^{-1})(s)\right) = Tr(n^{-1}s) = n^{-1}s$$

At $l = 2$ we have three diagrams. The $p \vee q$ diagrams are computed as follows:

	$\cap\cap$	$\Cap$	$\sqcap\sqcap\sqcap$
$\cap\cap$	$\cap\cap$	$\sqcap\sqcap\sqcap$	$\sqcap\sqcap\sqcap$
$\Cap$	$\sqcap\sqcap\sqcap$	$\Cap$	$\sqcap\sqcap\sqcap$
$\sqcap\sqcap\sqcap$	$\sqcap\sqcap\sqcap$	$\sqcap\sqcap\sqcap$	$\sqcap\sqcap\sqcap$

Thus the Gram and Weingarten matrices are:

$$G_{4n} = n \begin{pmatrix} n & 1 & 1 \\ 1 & n & 1 \\ 1 & 1 & 1 \end{pmatrix} \qquad G_{4n}^{-1} = \frac{1}{n(n-1)} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & n+1 \end{pmatrix}$$

This gives the formula of the fourth moment:

$$\begin{aligned} Tr(G_{4n}^{-1} G_{4s}) &= \frac{s}{n(n-1)} Tr\left(\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & n+1 \end{pmatrix} \begin{pmatrix} s & 1 & 1 \\ 1 & s & 1 \\ 1 & 1 & 1 \end{pmatrix}\right) \\ &= \frac{s}{n(n-1)} Tr \begin{pmatrix} s-1 & * & * \\ * & s-1 & * \\ * & * & n-1 \end{pmatrix} \\ &= \frac{s(n+2s-3)}{n(n-1)} \end{aligned}$$

At $l = 3$ we have 12 diagrams, and after computing the corresponding $p \vee q$ diagrams we get the following Gram matrix, whose inverse will not be given here:

$$G_{6n} = n \begin{pmatrix} n^2 & n & n & n & n & 1 & n & 1 & n & 1 & 1 & 1 \\ n & n^2 & 1 & n & 1 & n & 1 & 1 & 1 & n & 1 & 1 \\ n & 1 & n^2 & 1 & n & n & 1 & 1 & 1 & 1 & n & 1 \\ n & n & 1 & n & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ n & 1 & n & 1 & n & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & n & n & 1 & 1 & n^2 & n & n & 1 & n & n & 1 \\ n & 1 & 1 & 1 & 1 & n & n^2 & n & n & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & n & n & n & 1 & 1 & 1 & 1 \\ n & 1 & 1 & 1 & 1 & 1 & n & 1 & n & 1 & 1 & 1 \\ 1 & n & 1 & 1 & 1 & n & 1 & 1 & 1 & n & 1 & 1 \\ 1 & 1 & n & 1 & 1 & n & 1 & 1 & 1 & 1 & n & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

By computing $Tr(G_{6n}^{-1}G_{6s})$ we get the formula in the statement. □

The $l=3$ computation was done with Maple. The question of inverting general Gram matrices is for the moment beyond our understanding.

In fact, the only conceptual result here is Di Francesco's formula in [20], for the determinant of the matrix (which appears as denominator when computing inverses) in the simplest case, namely when we have all Temperley-Lieb diagrams.

10. Asymptotic freeness

The results in previous section show that the whole subject belongs to analysis: some $0,1,2,3,\ldots$ hierarchy of problems is required, in order to make further advances.

In the general context of integration problems, this kind of revelation first appeared to Weingarten [34]. His idea was to regard the various formulae as power series in n^{-1}, the inverse of the dimension, and to look at small order coefficients. In our specific quantum group situation, it is known from [7], [8] that the reasonable quantity to look at is the asymptotic law of $u_{11}+\ldots+u_{ss}$, with $s\simeq tn$.

Proposition 10.1. *We have the asymptotic moment formula*

$$\lim_{n\to\infty}\int(u_{11}+\ldots+u_{ss})^k=\sum_p t^{|p|}$$

with $s=[tn]$, where the sum is over all partitions in $NC_h(k)$.

Proof. Consider the diagonal matrix Δ_{kn} formed by the diagonal entries of G_{kn}. We have the following formula:

$$\begin{array}{rcl}(\Delta_{kn}^{-1/2}G_{kn}\Delta_{kn}^{-1/2})(p,q)&=&\Delta_{kn}^{-1/2}(p,p)G_{kn}(p,q)\Delta_{kn}^{-1/2}(q,q)\\&=&n^{-|p|/2}n^{|p\vee q|}n^{-|q|/2}\\&=&n^{|p\vee q|-(|p|+|q|)/2}\end{array}$$

The exponent is negative for $p\neq q$, and zero for $p=q$. This gives:

$$\begin{array}{rcl}G_{kn}&=&\Delta_{kn}^{1/2}(Id+O(n^{-1/2}))\Delta_{kn}^{1/2}\\G_{kn}^{-1}&=&\Delta_{kn}^{-1/2}(Id+O(n^{-1/2}))\Delta_{kn}^{-1/2}\end{array}$$

We obtain the following moment estimate:

$$\begin{array}{rcl}\displaystyle\int(u_{11}+\ldots+u_{ss})^k&=&Tr(G_{kn}^{-1}G_{ks})\\&\simeq&Tr(\Delta_{kn}^{-1}\Delta_{ks})\\&=&\displaystyle\sum_p n^{-|p|}s^{|p|}\end{array}$$

With $s=[tn]$ we get the formula in the statement. □

Theorem 10.2. *The asymptotic measures*

$$\mu_t=\lim_{n\to\infty}\mathrm{law}(u_{11}+\ldots+u_{[tn][tn]})$$

form a one-parameter truncated semigroup with respect to free convolution.

Proof. The idea is to follow the proof from the classical case, namely for the group H_n, with the log of the Fourier transform replaced by Voiculescu's R-transform.

We denote by F_{kb} of number of partitions in $NC_h(2k)$ having b blocks. We set $F_{kb} = 0$ for other integer values of k, b. All sums will be over integer indices ≥ 0.

With these notations, the above result shows that the generating series of the moments of μ_t is given by $f(z) = g(z^2)$, where:

$$g = \sum_{kb} F_{kb} z^k t^b$$

For a partition in $NC_h(2k+2)$, consider the last two legs of the first block. These correspond to certain numbers in $\{1, 2, \ldots, 2k+1\}$, that we denote $2x_0 + 1$ and $2(x_0 + x_1) + 2$. With the notation $x_2 = k - x_0 - x_1$, we see that our partition is obtained from a partition in $NC_h(2x_0)$, a partition in $NC_h(2x_1)$, and a partition in $NC_h(2x_2)$.

Thus the numbers $F_k = \#NC_h(2k)$ satisfy the following relation:

$$F_{k+1} = \sum_{\Sigma x_i = k} F_{x_0} F_{x_1} F_{x_2}$$

In terms of the numbers F_{kb}, this can be written in the following way:

$$\sum_b F_{k+1,b} = \sum_{\Sigma x_i = k} \sum_{u_i} F_{x_0 u_0} F_{x_1 u_1} F_{x_2 u_2}$$

In this formula, each $F_{x_0 u_0} F_{x_1 u_1} F_{x_2 u_2}$ term contributes to $F_{k+1,b}$ with $b = \Sigma u_i$, except for $F_{00} F_{x_1 u_1} F_{x_2 u_2} = F_{x_1 u_1} F_{x_2 u_2}$, which contributes to $F_{k+1,b+1}$. We get:

$$\begin{aligned}
F_{k+1,b} &= \sum_{\Sigma x_i = k} \sum_{\Sigma u_i = b} F_{x_0 u_0} F_{x_1 u_1} F_{x_2 u_2} \\
&+ \sum_{\Sigma x_i = k} \sum_{\Sigma u_i = b-1} F_{x_1 u_1} F_{x_2 u_2} \\
&- \sum_{\Sigma x_i = k} \sum_{\Sigma u_i = b} F_{x_1 u_1} F_{x_2 u_2}
\end{aligned}$$

By multiplying by t^b and summing over b, this gives the following formula for the polynomials $P_k = \sum_b F_{kb} t^b$:

$$P_{k+1} = \sum_{\Sigma x_i = k} P_{x_0} P_{x_1} P_{x_2} + (t-1) \sum_{\Sigma x_i = k} P_{x_1} P_{x_2}$$

In terms of $g = \sum_k P_k z^k$, we have the following equation:

$$g - 1 = zg^3 + (t-1)zg^2$$

In terms of the series $f(z) = g(z^2)$, we have:

$$f - 1 = z^2 f^3 + (t-1) z^2 f^2$$

In other words, f satisfies the following equation:

$$f = 1 + (zf)^2 (f + t - 1)$$

We can turn now to the last part of the proof, which uses Voiculescu's R-transform, and general theory from free probability [31]. The R-transform of a real probability measure μ is constructed as follows:

(1) $f(z)$ is the generating series of the moments of μ.
(2) $G(\xi) = \xi^{-1} f(\xi^{-1})$ is the Cauchy transform.
(3) $K(z)$ is defined by $G(K(z)) = z$.
(4) $R(z) = K(z) - z^{-1}$.

In terms of the Cauchy transform, the equation of f translates as follows:

$$\xi G = 1 + \xi G^3 + (t-1)G^2$$

Thus the K-transform satisfies the following equation:

$$zK = 1 + z^3 K + (t-1)z^2$$

This gives the following equation for the R-transform:

$$1 + zR = 1 + tz^2 + z^3 R$$

We can solve this equation, with the following solution:

$$R = \frac{tz}{1-z^2}$$

Thus the R-transform is linear in t, and this gives the result. □

Observe that the above proof also reads that the k-th free cumulant of μ_t is t if k is even, and zero if k is odd. This fact can also be seen directly from combinatorial considerations together with the free moment-cumulant formula. So, it is also possible to write a proof of Theorem 10.2 of combinatorial flavour.

As a last comment, the set $NC_h(2k)$ is the $s = 2$ particular case of the set $NC^{(s)}(sk)$, consisting of partitions of $\{1, 2, \ldots, sk\}$ into blocks of size multiple of s. These sets, introduced by Edelman [22], have been studied in connection with several situations, and have many interesting properties. See e.g. Armstrong [1] and Stanley [29].

The general results about $NC^{(s)}(sk)$ can be used at $s = 2$ in order to get further information about the semigroup $\{\mu_t\}$. As an example, the numbers F_{kb} in the proof of the above theorem are known to be the Fuss-Narayana numbers, namely:

$$F_{kb} = \frac{1}{b}\binom{k-1}{b-1}\binom{2k}{b-1}$$

This gives the moment formula announced in the introduction.

It is not clear whether at $s \geq 3$ the combinatorics of $NC^{(s)}(sk)$ corresponds or not to some quantum group. The technical problem is actually at the level of probability measures, where positivity seems to be lacking at $s \neq 1, 2$. However, at a purely combinatorial level there is no problem, and a one-parameter semigroup of "free Bessel laws" seems to exist for any s, at least in some virtual sense. These questions are to be added to those raised in the end of the present paper.

11. Free quantum groups: the orthogonal case

We present here some generalizations of notions and results in previous sections. This will lead to a kind of axiomatization for free quantum groups, improving previous considerations from [8]. The starting point is the following conclusion.

Theorem 11.1. *Let A be one of the series $A_s(n), A_h(n), A_o(n)$. Let A_{com} be the series of maximal commutative quotients, and consider the following measures:*

$$\mu_t = \lim_{n\to\infty} \mathrm{law}(u_{11} + \ldots + u_{[tn][tn]})$$

1. *For A_{com} these form a truncated semigroup with respect to convolution.*
2. *For A these form a truncated semigroup with respect to free convolution.*

This statement was pointed out in [8] in the symmetric and orthogonal case, with the comment that this is a formulation of "freeness". In other words, what we have shown in this paper is that the series $A_h(n)$ is free in the sense of [8]. Actually, the story of this paper is that we wanted to construct a new example, and by starting with the above freeness notion we eventually ended up with the cube and with $A_h(n)$.

One thing to be done now is to look back at the proof of the above result, and try to understand where freeness comes from, and axiomatize it. A first notion that emerges from this kind of study is as follows.

Definition 11.2. *A Hopf algebra realizing a factorization*

$$A_o(n) \to A \to A_s(n)$$

of the canonical map from left to right is called homogeneous.

As a first remark, all three algebras $A_o(n), A_h(n), A_s(n)$ are homogeneous.

Consider now the following Tannakian duality diagram, where C symbols denote as usual categories generated by tensor powers of the fundamental corepresentation:

$$\begin{array}{ccccc} A_o(n) & \to & A & \to & A_s(n) \\ \updownarrow & & \updownarrow & & \updownarrow \\ CA_o(n) & \subset & CA & \subset & CA_s(n) \end{array}$$

In the lower right corner we have the category spanned by all Temperley-Lieb diagrams. Moreover, in the studied cases $A = A_x(n)$ with $x = o, h, s$ we know that CA is spanned by certain Temperley-Lieb diagrams. This suggests the following definition.

Definition 11.3. *An homogeneous Hopf algebra is called free if the corresponding tensor category is spanned by Temperley-Lieb diagrams.*

In other words, we ask for the existence of a set of diagrams $TL \subset TL_s$ such that the above Tannakian duality diagram becomes:

$$\begin{array}{ccccc} A_o(n) & \to & A & \to & A_s(n) \\ & & & & \\ \updownarrow & & \updownarrow & & \updownarrow \\ & & & & \\ TL_o(n) & \subset & TL & \subset & TL_s(n) \end{array}$$

We denote by NC the set of non-crossing partitions associated to arches in TL. The general techniques in previous sections apply, and give the following results.

(1) The Weingarten formula holds, with G_{kn} being the Gram matrix of NC.
(2) The trace formula for sums of diagonal coefficients holds as well.
(3) We don't have the asymptotic moment formula, nor asymptotic freeness.

The problem with asymptotic results is that we don't have any asymptotics at all. This would require series of free Hopf algebras $A_x(n)$ having the property that the corresponding sets of partitions are equal to a given set NC_x, for n big enough. For the moment, we don't have further examples in this sense.

12. Free quantum groups: the unitary case

The considerations in previous section can be further generalized. The idea is that the various notions extend to the unitary case, according to the following rules:

(1) The orthogonal matrices are replaced by biunitary matrices. These are square matrices satisfying $u^* = u^{-1}$ and $\bar{u} = (u^t)^{-1}$. See Wang [32].
(2) The orthogonal Hopf algebras are replaced by unitary Hopf algebras. The fundamental corepresentation u must be biunitary, the formulae for Δ, ε are the same, and the antipode is given by $S(u_{ij}) = u_{ji}^*$. See Woronowicz [35].
(3) The algebra $A_o(n)$ is replaced by the algebra $A_u(n)$, generated by entries of a $n \times n$ biunitary matrix. A Hopf algebra realizing a factorization $A_u(n) \to A \to A_s(n)$ of the canonical map from left to right is called homogeneous.
(4) The tensor categories generated by u are replaced by tensor categories generated by $u, \bar{u}$. The freeness condition is that $CA \subset CA_s(n) = TL_s(n)$ is spanned by Temperley-Lieb diagrams. We use here the new meaning of $TL_s(n)$.

In other words, we have the following definition in the unitary case.

Definition 12.1. *A free Hopf algebra is a quotient of $A_u(n)$, having $A_s(n)$ as quotient, and having the property that its tensor category is spanned by Temperley-Lieb diagrams.*

The fact that we have a Weingarten formula and a moment formula in this general situation follows from [7], where the case of $A_u(n)$ is investigated. That is, the results follow by combining the arguments in previous sections with those in [7].

As in the orthogonal case, the construction of examples is a quite delicate problem. However, we have here the following simple construction. Given an orthogonal Hopf algebra (A, u), we can define a unitary Hopf algebra $(\tilde{A}, \tilde{u})$ in the following way:

(1) $\tilde{A} \subset C^*(Z) * A$ is the subalgebra generated by entries of $\tilde{u}$.

(2) $\tilde{u} = zu$, where z is the canonical generator of $C^*(Z)$.

This notion is introduced in [2], with the remark that we have $A_u(n) = \tilde{A}_o(n)$. It follows from results in there that if A is free orthogonal, then $\tilde{A}$ is free unitary. Thus we have the following examples of free Hopf algebras:

$$\begin{array}{ccccc} A_u(n) & \to & \tilde{A}_h(n) & \to & \tilde{A}_s(n) \\ \\ \downarrow & & \downarrow & & \downarrow \\ \\ A_o(n) & \to & A_h(n) & \to & A_s(n) \end{array}$$

We don't know what is the precise structure of the upper right algebras. We don't have either free unitary examples not coming from free orthogonal ones.

Summarizing, we have very few examples of free Hopf algebras.

This suggests that free Hopf algebras might be classifiable by discrete data, say analogous to Coxeter-Dynkin diagrams classifying simple Lie groups. We don't know if it is so, but we intend to come back to this question in future work.

We should mention that such a project might look a bit too ambitious. It is most likely that a complete classification of free Hopf algebras, if any, would rather have the size of a book. Probably most illustrating here is the story of the table in the introduction: each column is based on a certain number of journal articles, and complexity grows with each column added. So, the hope would be that the list is not too long.

As a conclusion, the reasonable question that we would like to raise here is that of finding a 4-th free quantum group, in the orthogonal case. The answer would probably require some new ideas, even at level of various combinatorial ingredients.

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