Groups and Weingarten integration

Teo Banica

"Introduction to free probability", 2/6

07/20

Plan

1. Group theory

2. Reflection groups

3. Compact groups

4. Weingarten integration

Finite groups

We are interested in the closed subgroups $G \subset U_N$. These can be finite or continuous. In the finite case, we have:

Theorem 1. Any finite group appears as a subgroup $G \subset S_N$.

Proof. This is Cayley's theorem. We have indeed a group embedding $G \subset S_{|G|}$ given by $\sigma_g(h) = gh$.

Theorem 2. Any finite group appears as a subgroup $G \subset U_N$.

Proof. It is enough to do this for $G = S_N$. But we have

$$S_N \subset O_N \subset U_N$$

by making permutations $\sigma \in S_N$ act on the coordinate axes of $\mathbb{R}^N$.

Basic examples

(1) Cyclic groups $\mathbb{Z}_N$. These appear as subgroups $\mathbb{Z}_N \subset O_N$, by cyclically permuting the coordinate axes of $\mathbb{R}^N$.

(2) Dihedral groups D_N. We have $D_N \subset O_N$, because the "1" points on the N coordinate axes form a regular N-gon.

(3) Permutation groups S_N. Here $S_N \subset O_N$ comes via permutation matrices, permuting the coordinate axes of $\mathbb{R}^N$.

(4) Hyperoctahedral groups H_N. Here $H_N \subset O_N$ is by definition the symmetry group of the N-hypercube.

(5) Complex reflection groups K_N. Here $K_N \subset U_N$ consists by definition of permutation-like matrices with entries in $\mathbb{T}$.

Observe that we have $\mathbb{Z}_N \subset D_N \subset S_N \subset H_N \subset K_N$.

Compact groups

In the continuous case now, there are many examples of closed subgroups $G \subset U_N$. Besides K_N, the main examples are:

(1) O_N, U_N. The orthogonal and unitary groups.

(2) SO_N, SU_N. The subgroups defined by $\det U = 1$.

(3) $Sp_N \subset U_N$. The symplectic group, defined for N even.

These are all smooth. In fact, the closed subgroups $G \subset U_N$ are exactly the compact Lie groups. We will be back to them later.

Characters

Definition. Given a closed subgroup $G \subset U_N$, the variable

$$\chi(U) = Tr(U)$$

with respect to the uniform measure, is called main character.

We will see later that the computation of $law(\chi)$ is the "main problem" regarding G. For the moment, let us record:

Theorem. For the symmetric group $S_N \subset U_N$ we have

$$\chi(\sigma) = \#\left\{i \middle| \sigma(i) = i\right\}$$

and $law(\chi) = p_1$ in the $N \to \infty$ limit.

Truncated characters

Definition. Given a closed subgroup $G \subset U_N$, the variable

$$\chi_t(U) = \sum_{i=1}^{[tN]} U_{ii}$$

is called truncated character, of parameter $t \in (0,1]$.

Theorem. For the symmetric group $S_N \subset U_N$ we have

$$\chi(\sigma) = \#\left\{i \in \{1,\ldots,[tN]\}\Big|\sigma(i) = i\right\}$$

and $law(\chi_t) = p_t$ in the $N \to \infty$ limit.

Proofs

(1) By using inclusion-exclusion. Indeed, we obtain

$$\mathbb{P}(\chi = 0) = \sum_{k=0}^{N} \frac{(-1)^k}{k!} \simeq \frac{1}{e}$$

and the computation of $\mathbb{P}(\chi = k)$, and of $\mathbb{P}(\chi_t = k)$, is similar.

(2) Via the moment method. We can use indeed the formula

$$\int_{S_N} U_{i_1 j_1} \ldots U_{i_k j_k} \, dU = \begin{cases} \frac{(N - |\ker i|)!}{N!} & \text{if } \ker i = \ker j \\ 0 & \text{otherwise} \end{cases}$$

where $\ker i$ is the partition of $\{1, \ldots, k\}$ whose blocks collect the equal indices of i, and where $|.|$ is the number of blocks.

Cyclic groups

Theorem. For the cyclic group $\mathbb{Z}_N \subset O_N$ we have

$$\chi(g) = N\delta_{g0}$$

and the corresponding distribution is a Bernoulli law:

$$law(\chi) = \left(1 - \frac{1}{N}\right)\delta_0 + \frac{1}{N}\delta_N$$

Proof. The cyclic matrices have 0 on the diagonal, and so trace 0, expect for the identity, having 1 on the diagonal, and trace N.

Remark. The truncated characters and the asymptotics are not interesting. We do not have convolution semigroups.

Dihedral groups

Theorem. For the dihedral group $D_N \subset S_N$ we have:

$$law(\chi) = \begin{cases} \left(1 - \frac{1}{2N}\right)\delta_0 + \frac{1}{2N}\delta_N & (N\ even) \\ \\ \left(\frac{1}{2} - \frac{1}{2N}\right)\delta_0 + \frac{1}{2}\delta_1 + \frac{1}{2N}\delta_N & (N\ odd) \end{cases}$$

Proof. The dihedral group D_N consists of:

(1) N symmetries, having $0, 1$ fixed points, depending on N.

(2) N rotations, having 0 fixed points, except for the identity.

Remark. The truncations and asymptotics are not interesting.

Reflections

Definition. The hyperoctahedral group $H_N \subset O_N$ is:

(1) The symmetry group of the unit hypercube $\square_N \subset \mathbb{R}^N$.

(2) The group of symmetries of the N coordinate axes of $\mathbb{R}^N$.

(3) The group of permutation-like matrices with ± 1 entries.

Theorem. The laws of truncated characters for H_N are

$$\mathrm{law}(\chi_t) \simeq e^{-t} \sum_{k=-\infty}^{\infty} \delta_k \sum_{p=0}^{\infty} \frac{(t/2)^{|k|+2p}}{(|k|+p)!p!}$$

for any $t \in (0,1]$, in the $N \to \infty$ limit.

Limiting laws

<u>Remark</u>. The limiting truncated character law for H_N is

$$b_t = e^{-t} \sum_{k \in \mathbb{Z}} \delta_k \, f_k(t/2)$$

where f_k is the Bessel function of the first kind:

$$f_k(t) = \sum_{p=0}^{\infty} \frac{t^{|k|+2p}}{(|k|+p)!p!}$$

<u>Theorem</u>. The Bessel laws b_t have the semigroup property

$$b_s * b_t = b_{s+t}$$

with respect to the usual convolution of real measures.

Bessel laws

Theorem. The following limit converges, for any measure ν,

$$p_t^\nu = \lim_{n\to\infty} \left(\left(1-\frac{t}{n}\right)\delta_0 + \frac{t}{n}\nu\right)^{*n}$$

and the limiting measure p_t^ν is called compound Poisson law.

Definition. The measures $b_t^s = p_t^{\varepsilon_s}$, with ε_s being the uniform measure on the s-roots of unity, are called Bessel laws.

Examples. At $s = 1$ we obtain the Poisson law p_t. At $s = 2$ we obtain the real Bessel law b_t. At $s = \infty$ we obtain a law B_t.

Complex reflections

Theorem. For the complex reflection group $H_N^s \subset U_N$, consisting of permutation-like matrices with entries in $\mathbb{Z}_s$,

$$H_N^s = \mathbb{Z}_s \wr S_N$$

the truncated characters become Bessel with $N \to \infty$:

$$\mathrm{law}(\chi_t) \simeq b_t^s$$

Examples. At $s = 1$ we obtain the Poisson result for S_N. At $s = 2$ we obtain the real Bessel result for H_N. At $s = \infty$ we obtain the complex Bessel law B_t, for the full reflection group $K_N = \mathbb{T} \wr S_N$.

Compact groups

In the continuous case, the main examples of closed subgroups $G \subset U_N$ are as follows:

(1) O_N, U_N. The orthogonal and unitary groups.

(2) SO_N, SU_N. The subgroups defined by $\det U = 1$.

(3) $Sp_N \subset U_N$. The symplectic group, defined for N even.

Observe that these groups are all smooth. In fact, the closed subgroups $G \subset U_N$ are exactly the compact Lie groups.

SU_2

Theorem. The group $SU_2 = \{U \in U_2 | \det U = 1\}$ is given by

$$SU_2 = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \middle| \, |a|^2 + |b|^2 = 1 \right\}$$

and the main character follows the Wigner semicircle law:

$$\gamma_1 = \frac{1}{2\pi}\sqrt{4 - x^2}dx$$

Proof. The matrices $U \in SU_2$ can be computed using $U^* = U^{-1}$ and the inversion formula for 2×2 matrices, with $\det U = 1$.

With $a = x + iy$, $b = z + it$ we must have $x^2 + y^2 + z^2 + t^2 = 1$, and so $SU_2 = S^3$, and $\chi = 2Re(a)$ follows to be semicircular.

SO_3

Theorem. The elements of $SO_3 = \{U \in O_3 | \det U = 1\}$ are

$$U = \begin{pmatrix} x^2+y^2-z^2-t^2 & 2(yz-xt) & 2(xz+yt) \\ 2(xt+yz) & x^2+z^2-y^2-t^2 & 2(zt-xy) \\ 2(yt-xz) & 2(xy+zt) & x^2+t^2-y^2-z^2 \end{pmatrix}$$

with $x, y, z, t \in \mathbb{R}$ satisfying $x^2+y^2+z^2+t^2=1$, and the main character of SO_3 follows the Marchenko-Pastur law:

$$\pi_1 = \frac{1}{2\pi}\sqrt{4x^{-1}-1}\,dx$$

Proof. This follows from the result for SU_2, by using the double cover map $SU_2 \to SO_3$. Indeed, we obtain the above formula for $U \in SO_3$, as well as the fact that χ is squared-semicircular.

Theory 1/3

Theorem. Any closed subgroup $G \subset U_N$ has a Haar measure,

$$\mu(gE) = \mu(Eg) = \mu(E)$$

constructed by starting with any measure ν, and setting:

$$\mu = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \nu^{*k}$$

Equivalently, for any representation $\rho : G \to U_n$, the quantities

$$P_{ij} = \int_G \rho_{ij}(U) dU$$

must be such that $P = (P_{ij})$ is the projection onto $Fix(\rho)$.

Theory 2/3

Theorem. The finite dimensional smooth unitary representations of $G \subset U_N$ are subject to the following Peter-Weyl results:

(1) Any such representation decomposes as a sum of irreducible representations, which are unique up to equivalence.

(2) The irreducibles appear inside the tensor products between the fundamental representation $\rho : G \to U_N$ and its adjoint.

(3) We have a decomposition $C^\infty(G) = \oplus_{r \in Irr(A)} B(H_r)$, as linear spaces, with the summands being pairwise orthogonal.

(4) The characters of the irreducible representations of G form an orthonormal basis of the algebra $C(G)_{central}$.

Theory 3/3

Theorem. Given a closed subgroup $G \subset U_N$, consider the character of the fundamental representation $\rho : G \to U_N$:

$$\chi = Tr(\rho)$$

The moments of $\chi : G \to \mathbb{C}$ are given then by the formula

$$\int_G \chi^k = \dim(Fix(\rho^{\otimes k}))$$

where $k \in \mathbb{N}$ in the case $G \subset O_N$, and $k \in \mathbb{N} \times \mathbb{N}$ in general.

Remark. This does not apply to individual coordinates $U \to U_{ij}$, or truncated characters χ_t, or other more complicated variables.

Integration formula

Theorem. The Haar integration over $G \subset_\rho U_N$ is given by

$$\int_G U_{i_1 j_1}^{s_1} \ldots U_{i_k j_k}^{s_k} dU = \sum_{\pi,\sigma \in D_k} \delta_\pi(i)\delta_\sigma(j) W_k(\pi,\sigma)$$

where D_k is a basis of $Fix(\rho^{\otimes k})$, $\delta_\pi(i) = <\pi, e_{i_1} \otimes \ldots \otimes e_{i_k}>$, and $W_k = G_k^{-1}$ is the inverse of $G_k(\pi,\sigma) = <\pi,\sigma>$.

Proof. The integrals in the statement form the projection P onto $Fix(\rho^{\otimes k}) = span(D_k)$. Consider the following linear map:

$$E(x) = \sum_{\pi \in D_k} < x, \pi > \pi$$

By linear algebra we have $P = WE$, where W is the inverse on $span(D_k)$ of the restriction of E, and this gives the result.

Tannakian duality

<u>Definition</u>. The Tannakian category of a closed subgroup $G \subset_\rho U_N$ is the following collection $C = (C(k,l))$ of vector spaces:

$$C(k,l) = Hom(\rho^{\otimes k}, \rho^{\otimes l})$$

<u>Definition</u>. The closed subgroup $G \subset U_N$ associated to an abstract Tannakian category $C = (C(k,l))$ is constructed as follows:

$$G = \left\{ U \in U_N \middle| T \in Hom(U^{\otimes k}, U^{\otimes l}), \forall T \in C(k,l) \right\}$$

<u>Theorem</u>. These operations produce a bijection $G \leftrightarrow C$, between compact Lie groups G, and Tannakian categories C.

Easiness

Definition. A closed subgroup $G \subset_\rho U_N$ is called easy when

$$Hom(\rho^{\otimes k}, \rho^{\otimes l}) = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

for a certain category of partitions $D \subset P$, where

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with $\delta_\pi \in \{0, 1\}$ depending on whether the indices fit or not.

Examples

(1) The basic unitary and reflection groups are all easy, the corresponding categories of partitions being as follows:

$$\begin{matrix} K_N & \to & U_N \\ \uparrow & & \uparrow \\ H_N & \to & O_N \end{matrix} \quad : \quad \begin{matrix} \mathcal{P}_{even} & \leftarrow & \mathcal{P}_2 \\ \downarrow & & \downarrow \\ P_{even} & \leftarrow & P_2 \end{matrix}$$

(2) In relation with reflection groups, S_N is easy as well, coming from P itself. In fact all groups $H_N^s = \mathbb{Z}_s \wr S_N$ are easy.

(3) The symplectic group $Sp_N \subset U_N$, defined for N even, is not exactly easy, but rather "super-easy".

(4) The remaining groups, namely SO_N, SU_N, and H_N^{sd} as well, which all involve the determinant, are not easy.

Weingarten formula

<u>Theorem</u>. For an easy group $G_N \subset U_N$, coming from a category of partitions $D = (D(k,l))$, we have

$$\int_{G_N} U_{i_1 j_1}^{s_1} \ldots U_{i_k j_k}^{s_k} dU = \sum_{\pi,\sigma\in D(k)} \delta_\pi(i)\delta_\sigma(j) W_{kN}(\pi,\sigma)$$

where $D(k) = D(\emptyset, k)$, δ are usual Kronecker symbols, and $W_{kN} = G_{kN}^{-1}$ is the inverse of $G_{kN}(\pi,\sigma) = N^{|\pi\vee\sigma|}$.

<u>Proof</u>. The vectors associated to partitions are given by:

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

Thus the Gram matrix and Kronecker symbols are those above.

Applications

Theorem. The truncated characters χ_t for the main unitary and reflection groups are as follows, in the $N \to \infty$ limit,

$$\begin{matrix} K_N & \longrightarrow & U_N \\ \uparrow & & \uparrow \\ H_N & \longrightarrow & O_N \end{matrix} \qquad \sim \qquad \begin{matrix} B_t & \cdots\cdots & G_t \\ \vdots & & \vdots \\ b_t & \cdots\cdots & g_t \end{matrix}$$

and we have independence results as well, with $N \to \infty$.

Comment 1. In the discrete case we have more generally results for all groups H_N^s, but this is something that we already know.

Comment 2. In the continuous case, similar techniques apply to other easy groups (B_N, C_N) or super-easy (Sp_N).

Summary

We have seen that:

(1) The Peter-Weyl theory gives a theoretical formula for $\int_G$.

(2) In the easy case this is the Weingarten formula, very concrete.

(3) Easiness checks are non-trivial: Tannaka, Brauer, Schur-Weyl..

(4) Once we have easiness, everything follows from Weingarten.

Thanks

Next lecture: free probability.