Quantum algebra and free Bessel laws

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"Introduction to free probability", 5/6

08/20

Plan

1. Quantum groups

2. Laws of characters

3. Free Bessel laws

4. Quantum algebra

Quantum groups

Definition. A Woronowicz algebra is a C^*-algebra A, given with a unitary matrix $u \in M_N(A)$ whose entries generate A, such that:

– $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ defines a morphism $\Delta : A \to A \otimes A$.

– $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : A \to \mathbb{C}$.

– $S(u_{ij}) = u_{ji}^*$ defines a morphism $S : A \to A^{opp}$.

Notation. Given a Woronowicz algebra A we write

$$A = C(G) = C^*(\Gamma)$$

and call G, Γ compact and discrete quantum groups.

Basic examples

Example 1. Given a compact Lie group $G \subset U_N$, we have

$$A = C(G) \quad , \quad u_{ij}(g) = g_{ij}$$

with $\Delta = m^T, \varepsilon = u^T, S = i^T$ being the transposes of m, u, i.

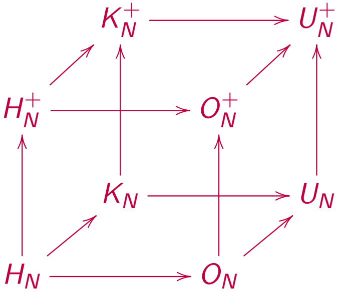
Example 2. Given a discrete group $\Gamma = < g_1, \ldots, g_N >$, we have

$$A = C^*(\Gamma) \quad , \quad u = diag(g_i)$$

with $\Delta(g) = g \otimes g, \varepsilon(g) = 1, S(g) = g^{-1}$ on group elements.

Rotations, reflections

<u>Theorem</u>. We have quantum unitary and reflection groups



obtained by "liberating" the classical unitary/reflection groups.

Theory 1/3

Theorem. Any Woronowicz algebra has a Haar integration,

$$\left(\int_G \otimes id\right)\Delta = \left(id \otimes \int_G\right)\Delta = \int_G (.)1$$

constructed by starting with $\varphi \in A^*$ unital positive, and setting

$$\int_G = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \varphi^{*k}$$

with the convolution operation being $\phi * \psi = (\phi \otimes \psi)\Delta$.

Examples. For compact Lie groups $G \subset U_N$ this is the usual integration. For group duals $G = \widehat{\Gamma}$ we have $\int_G g = \delta_{g1}$.

Theory 2/3

Definition. A corepresentation of a Woronowicz algebra A is a unitary matrix $v \in M_n(\mathcal{A})$ satisfying

$$\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj} \quad , \quad \varepsilon(v_{ij}) = \delta_{ij} \quad , \quad S(v_{ij}) = v_{ji}^*$$

where $\mathcal{A} =< u_{ij} >$ is the dense $*$-algebra of "smooth elements".

Theorem. The following Peter-Weyl type results hold:

(1) Any corepresentation decomposes as a sum of irreducibles.

(2) The irreducibles appear inside $u^{\otimes k}$, with $k =$ colored integer.

(3) We have $\mathcal{A} = \oplus_{r \in Irr(A)} B(H_r)$, $*$-coalgebra isomorphism, $\perp$.

(4) The characters of irreps form an orthonormal basis of $\mathcal{A}_{central}$.

Theory 3/3

Theorem. Given a corepresentation $v \in M_n(\mathcal{A})$, the integrals

$$P_{ij} = \int_G v_{ij}$$

form altogether the orthogonal projection $P = (P_{ij})$ onto $Fix(v)$.

Theorem. The Haar integration over G is given by

$$\int_G u_{i_1 j_1}^{s_1} \ldots u_{i_k j_k}^{s_k} = \sum_{\pi, \sigma \in D_k} \delta_\pi(i) \delta_\sigma(j) W_k(\pi, \sigma)$$

where D_k is a basis of $Fix(u^{\otimes k})$, $\delta_\pi(i) =< \pi, e_{i_1} \otimes \ldots \otimes e_{i_k} >$, and $W_k = G_k^{-1}$ is the inverse of $G_k(\pi, \sigma) =< \pi, \sigma >$.

Tannaka

Definition. The Tannakian category of a Woronowicz algebra (A, u) is the following collection $C = (C(k, l))$ of vector spaces:

$$C(k, l) = Hom(u^{\otimes k}, u^{\otimes l})$$

Definition. The Woronowicz algebra associated to a Tannakian category $C = (C(k, l))$ is constructed as follows:

$$A = C^* \left((u_{ij})_{i,j=1\ldots N} \middle| T \in Hom(u^{\otimes k}, u^{\otimes l}), \forall T \in C(k, l) \right)$$

Theorem. These operations produce a bijection $A \leftrightarrow C$, between Woronowicz algebras, and Tannakian categories.

Easiness

Definition. A compact quantum group G is called easy when

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

for a certain category of partitions $D \subset P$, where

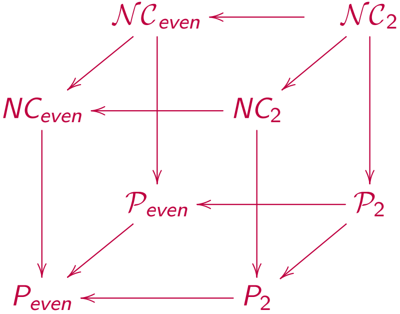
$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with $\delta_\pi \in \{0,1\}$ depending on whether the indices fit or not.

Examples. The Brauer theorem says that O_N, U_N are easy, coming from $P_2, \mathcal{P}_2$: the pairings, and the matching pairings.

Basic examples

Theorem. The main unitary/reflection groups are all easy,



being the corresponding categories of partitions $D \subset P$.

Weingarten

Theorem. For an easy quantum group $G \subset U_N^+$, coming from a category of partitions $D = (D(k,l))$, we have

$$\int_G u_{i_1 j_1}^{s_1} \ldots u_{i_k j_k}^{s_k} = \sum_{\pi,\sigma \in D(k)} \delta_\pi(i)\delta_\sigma(j) W_{kN}(\pi,\sigma)$$

for any colored integer k, where:

(1) $D(k) = D(\emptyset, k)$.

(2) δ are usual Kronecker symbols.

(3) $W_{kN} = G_{kN}^{-1}$ is the inverse of $G_{kN}(\pi,\sigma) = N^{|\pi \vee \sigma|}$.

Comment. This generalizes the Weingarten formula for easy groups $G \subset U_N$. There are "twisted" and "super-easy" versions of it.

Truncated characters

Theorem. Consider an easy quantum group $G = (G_N)$, coming from a category of partitions $D = (D(k,l))$.

(1) The asymptotic moments of the main character are:

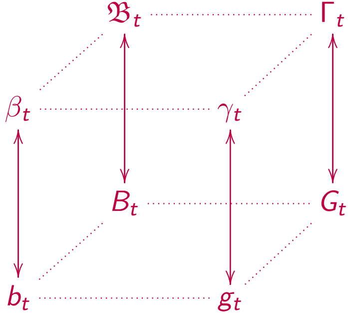
$$\int_{G_N} \left(\sum_i u_{ii} \right)^k \simeq |D(0,k)|$$

(2) The asymptotic moments of the truncated characters are:

$$\int_{G_N} \left(\sum_{i=1}^{[tN]} u_{ii} \right)^k \simeq \sum_{\pi \in D(0,k)} t^{|\pi|}$$

Laws of characters

Theorem. For the main unitary and reflection groups, the truncated characters $\chi_t = \sum_{i=1}^{[tN]} u_{ii}$ follow with $N \to \infty$ the laws



which are the main laws in classical/free probability, modulo the fact that the Poisson laws are replaced by real Bessel laws.

Bessel laws 1/6

Definition. The classical and free Bessel laws are

$$b_t^s = p_{t\varepsilon_s} \quad , \quad \beta_t^s = \pi_{t\varepsilon_s}$$

with $s \in \mathbb{N} \cup \{\infty\}$ and $t > 0$. That is, we have

$$b_t^s = \lim_{n\to\infty} \left(\left(1 - \frac{t}{n}\right)\delta_0 + \frac{t}{n}\varepsilon_s\right)^{*n}$$

$$\beta_t^s = \lim_{n\to\infty} \left(\left(1 - \frac{t}{n}\right)\delta_0 + \frac{t}{n}\varepsilon_s\right)^{\boxplus n}$$

with ε_s being the uniform measure on the s-roots of unity.

Bessel laws 2/6

Case $s = 1$. Here the limiting result is the PLT, and we obtain the Poisson law p_t, and the Marchenko-Pastur law π_t.

Case $s = 2$. Here we obtain the real Bessel law b_t, given by

$$b_t = e^{-t} \sum_{k \in \mathbb{Z}} \delta_k \, f_k(t/2)$$

where f_k is the Bessel function of the first kind,

$$f_k(t) = \sum_{p=0}^{\infty} \frac{t^{|k|+2p}}{(|k|+p)!p!}$$

and the free real Bessel law β_t, also called Fuss-Catalan law.

Case $s = \infty$. Here we obtain the complex Bessel laws $B_t, \mathfrak{B}_t$.

Bessel laws 3/6

Theorem. The Bessel laws b_t^s and free Bessel laws β_t^s with $s \in \mathbb{N}$ and $t > 0$ can be defined alternatively as

$$b_t^s/\beta_t^s = \mathrm{law}\left(\sum_{r=1}^{s} w^r \alpha_i\right)$$

where $w = e^{2\pi i/s}$, and where α_i are Poisson/free Poisson (t) variables, which are independent/free.

Bessel laws 4/6

Theorem. We have the Fourier transform formula for b_t^s,

$$F_{b_t^s}(y) = \exp\left(t\sum_{r=1}^{s}(e^{iw^r y} - 1)\right)$$

as well as the following R-transform formula for β_t^s,

$$R_{\beta_t^s}(y) = t\sum_{r=1}^{s}\frac{w^r}{1 - w^r y}$$

valid for any $s \in \mathbb{N}$ and $t > 0$, where $w = e^{2\pi i/s}$.

Bessel laws 5/6

Theorem. The Bessel laws for a convolution semigroup

$$b_t^s * b_{t'}^s = b_{t+t'}^s$$

and the free Bessel laws form a free convolution semigroup

$$\beta_t^s \boxplus \beta_{t'}^s = \beta_{t+t'}^s$$

and these semigroups are in Bercovici-Pata bijection.

Bessel laws 6/6

Theorem. The free Bessel laws have the following properties,

(1) They appear as free compressions of $\boxtimes$ powers of π:

$$\beta_t^s = \left(\pi^{\boxtimes s}\right)_t$$

(2) Alternatively, we have the following formula:

$$\beta_t^s = \pi^{\boxtimes s-1} \boxtimes \pi^{\boxplus t}$$

(3) In particular, we have the following formulae:

$$\begin{cases}\beta_1^s = \pi^{\boxtimes s}\\ \beta_t^1 = \pi^{\boxplus t}\end{cases}$$

in terms of the Marchenko-Pastur law $\pi = \pi_1$.

Classical groups

<u>Problem</u>. The laws b_t^s and β_t^s appear as asymptotic laws of truncated characters for the quantum reflection groups:

$$H_N^s = \mathbb{Z}_s \wr S_N \quad , \quad H_N^{s+} = \mathbb{Z}_s \wr_* S_N^+$$

In the classical case, the full series of reflection groups is

$$H_N^{sd} = \left\{ U \in H_N^s \middle| (\det U)^d = 1 \right\}$$

and the character fluctuations are still to be understood.

Quantum groups

Problem. Classify the quantum reflection groups. In the real case, the easy quantum groups $H_N \subset G \subset H_N^+$ are

$$H_N \subset H_N^\Gamma \subset H_N^{(r)} \subset H_N^+$$

with Γ being a uniform real reflection group, and with $r \in \mathbb{N}$ being a parameter. In the purely complex case, namely

$$K_N \subset G \subset K_N^+$$

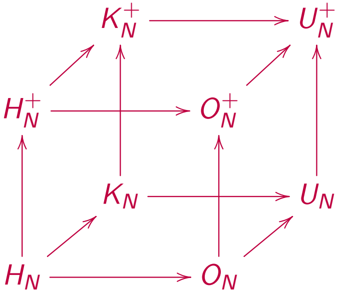
the situation is expected to be quite similar. In the general complex case, the classification of the easy quantum groups

$$H_N \subset G \subset K_N^+$$

is not known yet. In addition to all this, we have the question of computing and understanding the laws of truncated characters.

Noncommutative geometry

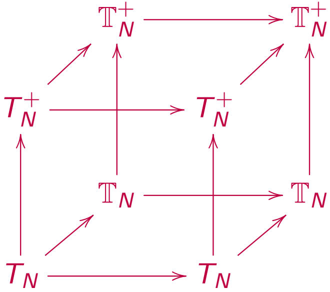
<u>Problem</u>. The main examples of quantum unitary groups and quantum reflection groups, namely



correspond to the 4 main geometries, real/complex, classical/free. In general, the classification problem is open.

Liberation theory

Problem. The main noncommutative tori, appearing as diagonal tori of the main quantum groups, are given by



where $T_N = \mathbb{Z}_2^N$, $\mathbb{T}_N = \mathbb{T}^N$ and $T_N^+ = \widehat{\mathbb{Z}_2^{*N}}$, $\mathbb{T}_N^+ = \widehat{F_N}$. At the character level we obtain Meixner/free Meixner instead of BP.

Subfactor theory

Problem. Associated to a finite index subfactor $A_0 \subset A_1$ is the Jones tower $A_0 \subset A_1 \subset A_2 \subset \ldots$, the planar algebra

$$P_k = A_0' \cap A_k$$

and the spectral measure μ, whose moments are given by:

$$M_k = \dim(P_k)$$

For TL subfactors we obtain free Poisson laws, for FC subfactors we obtain free Bessel laws. What is the meaning of $t > 0$ here?

Blowup questions

Problem. For subfactors of index $N \leq 4$, the Jones manipulation

$$\Theta(q) = q + \frac{1-q}{1+q} f\left(\frac{q}{(1+q)^2}\right)$$

on the associated Poincaré series, which is defined as

$$f(z) = \sum_{k=0}^{\infty} \dim(P_k) z^k$$

and which is the Stieltjes transform of the spectral measure μ,

$$f(x) = \int_{\mathbb{R}} \frac{1}{1-xz} \, d\mu(x)$$

blows up μ on $\mathbb{T}$, in a nice way. What about in general?

Summary

We have seen that:

(1) The free Bessel laws are central objects in free probability.

(2) And, more generally, in all branches of quantum algebra.

(3) The current research level is right above free Bessel.

Thanks

Next lecture: random matrices.