Introduction to operator algebras

Teo Banica

Hilbert spaces and operators, Spectral theory, C^*-algebras, von Neumann algebras, Quantum algebra, Spectral measures

07/20

Foreword

This is an introduction to operator algebras, from a quantum algebra and quantum physics perspective.

We discuss the foundational aspects of the theory, C^*-algebras and von Neumann algebras, and then more specialized topics.

These lecture notes consist of slides written in the Summer 2020. Presentations available at my Youtube channel.

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Hilbert spaces and operators

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Definition. The determinant of a system of N vectors in $\mathbb{R}^N$ is the signed volume of the associated parallelepiped

$$\det(V_1, \ldots, V_N) = \pm vol < V_1, \ldots, V_N >$$

with the sign being $+$ if you can pass from the standard basis of $\mathbb{R}^N$ to the system of vectors in V, and being $-$ otherwise.

Comment. This is the correct definition.

Calculus 2

Definition. The integral of a continuous function $f : [a, b] \to \mathbb{R}$ is the signed area below its graph. This can be computed as

$$\int_a^b f(x)dx = (b-a) \times \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(x_i)$$

where in order to compute the average height of the function, the points $x_1, \ldots, x_N \in [a, b]$ are chosen at random.

Comment. This is the best definition.

Linear algebra 1

Definition. We say that $M \in M_N(\mathbb{C})$ has eigenvector $v \in \mathbb{C}^N$ with eigenvalue $\lambda \in \mathbb{C}$ when M dilates by λ in the v direction:

$$Mv = \lambda v$$

When M has a basis of eigenvectors $\{v_i\}$, we call it diagonalizable, and we write $M = PDP^{-1}$, with $D = diag(v_i)$.

Examples. A diagonalizable matrix, and a non-diagonalizable one:

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad P = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

This follows indeed from $M\binom{x}{y} = \binom{y}{x}$, and from $P\binom{x}{y} = \binom{y}{0}$.

Linear algebra 2

Theorem. The following happen:

(1) The eigenvalues of M are the roots of $f(x) = \det(M - x1_N)$.

(2) If we have N distinct eigenvalues, M is diagonalizable.

(3) If M is normal, $MM^* = M^*M$, it is diagonalizable.

(4) The diagonalizable matrices are dense inside $M_N(\mathbb{C})$.

Proof. The idea is as follows:

(1) Follows from the eigenvalue equation, $(M - \lambda 1_N)v = 0$.

(2) Vectors with different eigenvalues are linearly independent.

(3) Generalizes "$M \in M_N(\mathbb{R})$ symmetric $\implies$ diagonalizable".

(4) Because the matrices with distinct eigenvalues are dense.

Rotations 1

Definition. A scalar product $<,>$ on $\mathbb{C}^N$ must satisfy:

(1) $< x, y >$ is linear in x, antilinear in y.
(2) $\overline{< x, y >} =< y, x >$, for any x, y.
(3) $< x, x >> 0$, for any $x \neq 0$.

Examples. The usual scalar product, $< x, y >= \sum_i x_i \bar{y}_i$, that we can further complicate by adding weights, and so on.

Theorem. For a matrix $U \in M_N(\mathbb{C})$, the following are equivalent:

(1) U preserves the scalar product, $< Ux, Uy >=< x, y >$.
(2) U preserves the norm, $||Ux|| = ||x||$, where $||x|| = \sqrt{< x, x >}$.
(3) U is unitary, in the sense that $U^* = U^{-1}$, where $(U^*)_{ij} = \bar{U}_{ji}$.

Proof. All this follows from $< Mx, y >=< x, M^*y >$.

Rotations 2

Theorem 1. The unitaries in $M_2(\mathbb{C})$ of determinant 1 are

$$U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

with $a, b \in \mathbb{C}$ satisfying $|a|^2 + |b|^2 = 1$.

Theorem 2. The unitaries in $M_3(\mathbb{R})$ of determinant 1 are

$$U = \begin{pmatrix} x^2+y^2-z^2-t^2 & 2(yz-xt) & 2(xz+yt) \\ 2(xt+yz) & x^2+z^2-y^2-t^2 & 2(zt-xy) \\ 2(yt-xz) & 2(xy+zt) & x^2+t^2-y^2-z^2 \end{pmatrix}$$

with $x, y, z, t \in \mathbb{R}$ satisfying $x^2 + y^2 + z^2 + t^2 = 1$.

Proofs. 1 follows from $U^* = U^{-1}$, and 2 follows from 1.

Hilbert spaces

Definition. A Hilbert space is a complex vector space H, typically infinite dimensional, with a scalar product $< x, y >$, satisfying:

(1) $< x, y >$ is linear in x, antilinear in y.
(2) $\overline{< x, y >} = < y, x >$, for any x, y.
(3) $< x, x > > 0$, for any $x \neq 0$.
(4) H is complete with respect to $||x|| = \sqrt{< x, x >}$.

Remark. Here (4) is based on Cauchy-Schwarz. Basic examples:

(1) $H = \mathbb{C}^N$, with $< x, y > = \sum_i x_i \bar{y}_i$.
(2) $H = l^2(\mathbb{N})$, with $< x, y > = \sum_i x_i \bar{y}_i$.
(3) $H = L^2(X)$, with $< f, g > = \int_X f(x)\overline{g(x)}dx$.

Gram-Schmidt

<u>Theorem</u>. Any basis $\{f_i\}_{i\in I}$, in a dense sense, can be turned into an orthonormal basis $\{e_i\}_{i\in I}$, by using the Gram-Schmidt procedure.

<u>Theorem</u>. Any Hilbert space has an orthonormal basis $\{e_i\}_{i\in I}$. In other words, we have $H \simeq l^2(I)$, for some index set I.

<u>Definition</u>. When I is countable, and usually not finite, H is called separable. This is the same as saying that $H \simeq l^2(\mathbb{N})$.

<u>Example</u>. The space $H = L^2[0,1]$ is separable, because we can apply Gram-Schmidt to the basis $f_i = x^i$, with $i \in \mathbb{N}$.

Linear operators

Theorem. Let H be a Hilbert space, with basis $\{e_i\}_{i\in I}$. We have

$$\mathcal{L}(H) \subset M_I(\mathbb{C})$$

with $T : H \to H$ linear corresponding to the following matrix:

$$M_{ij} = < Te_j, e_i >$$

– When $\dim(H) = N < \infty$, we obtain $\mathcal{L}(H) \simeq M_N(\mathbb{C})$.
– In the infinite separable case, we obtain $\mathcal{L}(H) \subset M_\infty(\mathbb{C})$.

Proof. The correspondence $T \to M$ is indeed linear and injective.

Comment. However, $H = L^2[0,1]$ suggests not to use all this.

Bounded operators

Theorem. Given a Hilbert space H, the linear operators $T : H \to H$ which are bounded, in the sense that

$$||T|| = \sup_{||x|| \leq 1} ||Tx||$$

is finite, form a complex algebra $B(H)$, which:

(1) Is complete with respect to $||.||$ (Banach algebra).

(2) Has an involution $T \to T^*$, $< Tx, y >=< x, T^* y >$.

The norm and involution are related by $||TT^*|| = ||T||^2$.

Proof. Complex algebra is clear, given $\{T_n\}$ Cauchy we can set $Tx = \lim_{n\to\infty} T_n x$, the involution comes from $\varphi(x) =< Tx, y >$ which is linear, and $||TT^*|| = ||T||^2$ is by double inequality.

Operator algebras

Definition. A C^*-algebra is an algebra $A \subset B(H)$, which:

(1) Is norm closed: $T_n \in A,\ T_n \to T \implies T \in A$.

(2) Is stable under the involution: $T \in A \implies T^* \in A$.

Definition. A von Neumann algebra is an algebra $A \subset B(H)$, which:

(1) Is weakly closed: $T_n \in A,\ T_n x \to Tx, \forall x \implies T \in A$.

(2) Is stable under the involution: $T \in A \implies T^* \in A$.

Examples. We have $C(X)$ and $L^\infty(X)$, acting by multiplication on $L^2(X)$. We will see that when $T \in B(H)$ is normal, the algebras that it generates are of this form ("Spectral theorem").

Basic spectral theory

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Framework

Definition. An abstract C^*-algebra is a complex algebra A, with:

(1) A norm $a \to ||a||$, making it a Banach algebra.

(2) An involution $a \to a^*$, such that $||aa^*|| = ||a||^2$, $\forall a \in A$.

– We know that $B(H)$ is a C^*-algebra in the above sense.

– And so are all the norm closed $*$-subalgebras $A \subset B(H)$.

– We'll see later that any abstract C^*-algebra is of this form.

$\implies$ However, useful formalism, because we can construct many examples of C^*-algebras with generators and relations.

Spectra

Definition. The spectrum of an element $a \in A$ is the set

$$\sigma(a) = \left\{\lambda \in \mathbb{C} \middle| a - \lambda \notin A^{-1}\right\}$$

where $A^{-1} \subset A$ is the set of invertible elements.

Remark. For the usual matrices we obtain the eigenvalues,

$$\begin{aligned} Mv = \lambda v \quad &\iff \quad (M - \lambda)v = 0 \\ &\iff \quad M - \lambda \notin M_N(\mathbb{C})^{-1} \end{aligned}$$

or rather the eigenvalue set, with no multiplicities.

Basics

Theorem. The spectrum of any element $a \in A$ is:

(1) A compact subset of $\mathbb{C}$.

(2) Contained in the disk $D(0, ||a||)$.

Proof. The spectrum of a norm 1 element is in the unit disk. This comes from the following formula, valid for any $||a|| < 1$:

$$\frac{1}{1-a} = 1 + a + a^2 + \ldots$$

But this gives (2) by dilation, and shows as well that A^{-1} is open, and so that $\sigma(a) \subset \mathbb{C}$ is closed, and so we get (1) as well.

Nonzero

Theorem. The spectrum of any element $a \in A$ is non-empty.

Proof. Assume $\sigma(a) = \emptyset$. Pick a linear form $\varphi \in A^*$ and set:

$$f(\lambda) = \varphi\left(\frac{1}{\lambda - a}\right)$$

Then f is differentiable, so holomorphic. For $\lambda >> 0$ we have

$$\begin{aligned} \left|\left|\frac{1}{\lambda - a}\right|\right| &= \frac{1}{|\lambda|} \times \left|\left|1 + \frac{a}{\lambda} + \frac{a^2}{\lambda^2} + \ldots\right|\right| \\ &\leq \frac{1}{|\lambda| - ||a||} \end{aligned}$$

so $f(\lambda) \to 0$ with $\lambda \to \infty$. By Liouville $f = 0$, contradiction.

Products

Theorem. For any two elements $a, b \in A$ we have

$$\sigma(ab) = \sigma(ba)$$

outside $\{0\}$. Non-equality at 0 can happen.

Proof. For the equality, by dilation it is enough to prove that $\sigma(ab) = \sigma(ba)$ at $\lambda = 1$. But this follows from:

$$c = (1 - ab)^{-1} \implies 1 + cba = (1 - ba)^{-1}$$

Consider the shift $S : l^2(\mathbb{N}) \to l^2(\mathbb{N})$, $\delta_i \to \delta_{i+1}$. Then

$$S^*S = 1 \quad , \quad SS^* = Proj_{e_0^\perp}$$

and so $0 \notin \sigma(S^*S)$, but $0 \in \sigma(SS^*)$.

Rational functions

Definition. Given $a \in A$, and a rational function $f = P/Q$ having poles outside $\sigma(a)$, we set $f(a) = P(a)Q(a)^{-1}$. We write:

$$f(a) = \frac{P(a)}{Q(a)}$$

Theorem. We have the "rational functional calculus" formula

$$\sigma(f(a)) = f(\sigma(a))$$

valid for any $f \in \mathbb{C}(X)$ having poles outside $\sigma(a)$.

Proof

Case $f \in \mathbb{C}[X]$. With $f(X) - \lambda = c(X - r_1)\dots(X - r_n)$:

$$\begin{aligned} \lambda \notin \sigma(f(a)) &\iff c(a - r_1)\dots(a - r_n) \in A^{-1} \\ &\iff a - r_1, \dots, a - r_n \in A^{-1} \\ &\iff r_1, \dots, r_n \notin \sigma(a) \\ &\iff \lambda \notin f(\sigma(a)) \end{aligned}$$

Case $f \in \mathbb{C}(X)$. With $f = P/Q$ and $F = P - \lambda Q$:

$$\begin{aligned} \lambda \in \sigma(f(a)) &\iff 0 \in \sigma(F(a)) \\ &\iff 0 \in F(\sigma(a)) \\ &\iff \exists \mu \in \sigma(a), F(\mu) = 0 \\ &\iff \lambda \in f(\sigma(a)) \end{aligned}$$

Unitaries

Theorem. The spectrum of a unitary element,

$$a^* = a^{-1}$$

is on the unit circle $\mathbb{T} \subset \mathbb{C}$.

Proof. This follows by using $f(z) = z^{-1}$. Indeed, we have:

$$\sigma(a)^{-1} = \sigma(a^{-1}) = \sigma(a^*) = \overline{\sigma(a)}$$

Thus $\sigma(a)$ consists of numbers satisfying $\lambda^{-1} = \bar{\lambda}$.

Self-adjoints

<u>Theorem</u>. The spectrum of a self-adjoint element,

$$a = a^*$$

consists of real numbers.

<u>Proof</u>. This follows by using $f(z) = (z + it)/(z - it)$, with $t \in \mathbb{R}$. Indeed, for $t >> 0$ the element $f(a)$ is well-defined, and:

$$\left(\frac{a+it}{a-it}\right)^* = \frac{a-it}{a+it} = \left(\frac{a+it}{a-it}\right)^{-1}$$

Thus $f(a)$ is unitary, with spectrum contained in $\mathbb{T}$. We conclude that $f(\sigma(a)) = \sigma(f(a)) \subset \mathbb{T}$, and so $\sigma(a) \subset f^{-1}(\mathbb{T}) = \mathbb{R}$.

Spectral radius 1/2

Definition. Given an element $a \in A$, its spectral radius $\rho(a)$ is the radius of the smallest disk centered at 0 containing $\sigma(a)$.

Theorem. The spectral radius of a normal element,

$$aa^* = a^*a$$

equals its norm.

Proof. We already know that $\rho(a) \leq ||a||$, for any $a \in A$.

Spectral radius 2/2

For the converse, if we fix $\rho > \rho(a)$, we have:

$$\int_{|z|=\rho} \frac{z^n}{z-a}\, dz = \sum_{k=0}^{\infty} \left(\int_{|z|=\rho} z^{n-k-1} dz \right) a^k = a^{n-1}$$

By applying the norm and taking n-th roots we obtain:

$$\rho \geq \lim_{n \to \infty} ||a^n||^{1/n}$$

(1) In the case $a = a^*$ we have $||a^n|| = ||a||^n$ for any exponent of the form $n = 2^k$, and by taking n-th roots we get $\rho \geq ||a||$.

(2) In general we have $a^n(a^n)^* = (aa^*)^n$, so $\rho(a)^2 = \rho(aa^*)$. Now since aa^* is self-adjoint, $\rho(aa^*) = ||a||^2$, and we are done.

C*-algebra basics

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C^*-algebras

Definition. A C^*-algebra is a complex algebra A, with:

(1) A norm $a \to ||a||$, making it a Banach algebra.

(2) An involution $a \to a^*$, satisfying $||aa^*|| = ||a||^2$.

Examples. The closed $*$-subalgebras $A \subset B(H)$. We'll see later that any C^*-algebra is of this form (GNS theorem).

Remark. We have seen that $\rho(a) = ||a||$ for $a \in A$ normal. Thus

$$||a|| = \sqrt{||aa^*||} = \sqrt{\rho(aa^*)} \quad , \quad \forall a \in A$$

so the norm is uniquely determined by the $*$-algebra structure.

Gelfand

Theorem. The commutative C^*-algebras are the algebras of the form $C(X)$, with X being a compact space.

Proof. If X is compact, $C(X)$ is indeed a C^*-algebra. Conversely, given A commutative, consider the space of characters

$$X = \{\chi : A \to \mathbb{C}\}$$

with topology making continuous each $ev_a : \chi \to \chi(a)$. Then X is compact, and $a \to ev_a$ is a morphism of algebras $ev : A \to C(X)$.

(1) ev involutive. Using real + imaginary parts, we must prove that $ev_{a^*} = ev_a^*$ when $a = a^*$. But this follows from $\sigma(a) \subset \mathbb{R}$.

(2) ev isometric. Follows from $||ev_a|| = \rho(a) = ||a||$.

(3) ev surjective. Follows from Stone-Weierstrass.

Normal elements

Theorem. Assume that $a \in A$ is normal.

(1) We have $< a >= C(\sigma(a))$.

(2) For $f \in C(\sigma(a))$ we can define $f(a) \in A$.

(3) We have the formula $\sigma(f(a)) = f(\sigma(a))$.

Proof. Since a is normal, the algebra $< a >$ is commutative, and the Gelfand theorem gives $< a >= C(X)$, with:

$$X = \{\chi :< a > \to \mathbb{C}\}$$

The map $X \to \sigma(a)$ given by evaluation at a being bijective, we have $X = \sigma(a)$. Thus we get (1), and (2,3) follow as well.

Remarks. This extends the rational calculus, in the normal case. Also, it applies to any $T \in B(H)$ normal (spectral theorem).

Embeddings

We want to prove that any C^*-algebra appears as $A \subset B(H)$.

Theorem. Assume that A is commutative, $A = C(X)$, and let μ be a positive measure on X. We have then an embedding

$$A \subset B(H)$$

where $H = L^2(X)$, with $f \in A$ corresponding to $T_f : g \to fg$.

Proof. T_f is well-defined, and bounded as well, because:

$$||fg||_2 = \sqrt{\int_X |f(x)|^2 |g(x)|^2 d\mu(x)} \leq ||f||_\infty ||g||_2$$

We obtain in this way $A \subset B(H)$, as claimed.

Positivity

Theorem. For an element $a \in A$, the following are equivalent:

(1) a is positive, in the sense that $\sigma(a) \subset [0, \infty)$.

(2) $a = b^2$, for some $b \in A$ satisfying $b = b^*$.

(3) $a = cc^*$, for some $c \in A$.

(1) $\implies$ (2): $\sigma(a) \subset \mathbb{R}$ implies $a = a^*$, so $< a >$ is commutative, and by using the Gelfand theorem, we can set $b = \sqrt{a}$.

(2) $\implies$ (3): this is trivial, because we can set $c = b$.

(3) $\implies$ (1): by contradition. By multiplying c by a suitable element of $< cc^* >$, we are led to the existence of an element $d \neq 0$ satisfying $-dd^* \geq 0$. With $d = x + iy$ we have:

$$dd^* + d^*d = 2(x^2 + y^2) \geq 0$$

Thus $d^*d \geq 0$, contradicting $\sigma(dd^*) = \sigma(d^*d)$ outside $\{0\}$.

Forms

<u>Definition</u>. Consider a linear map $\varphi : A \to \mathbb{C}$.

(1) φ is called positive when $a \geq 0 \implies \varphi(a) \geq 0$.

(2) φ is called faithful and positive if $a \geq 0, a \neq 0 \implies \varphi(a) > 0$.

<u>Theorem</u>. Let $\varphi : A \to \mathbb{C}$ be a positive linear form.

(1) $< a, b >= \varphi(ab^*)$ defines a generalized scalar product on A.

(2) By separating and completing we obtain a Hilbert space H.

(3) $\pi(a) : b \to ab$ defines a representation $\pi : A \to B(H)$.

(4) If φ is faithful in the above sense, then π is faithful.

<u>Proof</u>. Everything here is straightforward, and the last assertion follows from $a \neq 0 \implies \pi(aa^*) \neq 0 \implies \pi(a) \neq 0$.

GNS theorem

Theorem. Let A be a C^*-algebra.

(1) A appears as $A \subset B(H)$, for some Hilbert space H.

(2) When A is separable, H can be chosen to be separable.

(3) When A is FD, the space H can be chosen to be FD.

Proof. We just need a faithful positive linear form $\varphi : A \to \mathbb{C}$, and this can be constructed as in the classical case, as follows:

(1) Any positive linear form $\varphi : A \to \mathbb{C}$ is continuous.

(2) φ is positive iff there is a norm one $h \in A_+$, $||\varphi|| = \varphi(h)$.

(3) $\forall a \in A$ there exists φ positive of norm 1, $\varphi(aa^*) = ||a||^2$.

(4) There exists a faithful positive linear form $\varphi : A \to \mathbb{C}$.

Noncommutative spaces

Definition. Given an arbitrary C^*-algebra A, we write

$$A = C(X)$$

and call X a "noncommutative compact space".

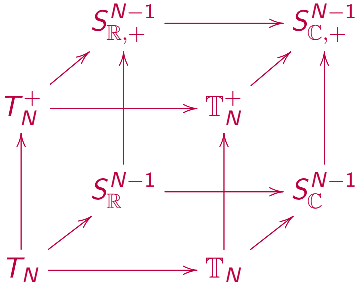
Equivalently, the category of noncommutative compact spaces is the category of C^*-algebras, with the arrows reversed.

Example 1. Given a morphism $\Phi : A \to B$, we write $A = C(X)$, $B = C(Y)$, and speak of the morphism $\phi : Y \to X$.

Example 2. Given a product $A = B \otimes C$, we write $A = C(X)$, $B = C(Y)$, $C = C(Z)$, and speak of $X = Y \times Z$.

Spheres and tori

Definition. We have noncommutative spheres and cubes/tori,



with the free complex sphere being defined by the formula

$$C(S^{N-1}_{\mathbb{C},+}) = C^*\left(x_1,\ldots,x_N\Big|\sum_i x_ix_i^* = \sum_i x_i^*x_i = 1\right)$$

and $S^{N-1}_{\mathbb{R},+}$, $\mathbb{T}_N^+$, T_N^+ being obtained via $x_i = x_i^*$, $x_ix_i^* = x_i^*x_i = 1/N$.

Group duals

Definition. The group algebra $C^*(\Gamma)$ of a discrete group Γ is the enveloping C^*-algebra of $\mathbb{C}[\Gamma]$, with involution $g^* = g^{-1}$.

Theorem. When Γ is abelian, we have an identification

$$C^*(\Gamma) = C(G)$$

where $G = \widehat{\Gamma}$ is the group formed by the characters $\chi : \Gamma \to \mathbb{T}$.

Proof. This follows from Gelfand, because the algebra characters $\chi : C^*(\Gamma) \to \mathbb{C}$ must come from group characters $\chi : \Gamma \to \mathbb{T}$.

Definition. Given a discrete group Γ, the space G given by

$$C(G) = C^*(\Gamma)$$

is called abstract dual of Γ, and is denoted $G = \widehat{\Gamma}$.

Cubes and tori

Theorem. The basic cubes and tori are all group duals,

$$\begin{array}{ccccccc}T_N^+ & \to & \mathbb{T}_N^+ & & \widehat{\mathbb{Z}_2^{*N}} & \to & \widehat{F_N}\\ \uparrow & & \uparrow & = & \uparrow & & \uparrow\\ T_N & \to & \mathbb{T}_N & & \mathbb{Z}_2^N & \to & \mathbb{T}^N\end{array}$$

where F_N is the free group, and $*$ is a free product.

Proof. The various algebras $C(T)$ are generated by unitaries, with certain relations between them, and this gives the result.

Von Neumann algebras

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Basics

Definition. A von Neumann algebra is a $*$-algebra of operators $A \subset B(H)$ which is closed under the weak topology:

$$T_n \in A, T_n x \to Tx \implies T \in A$$

Examples. The usual C^*-algebras, in finite dimensions. Also, the algebras $L^\infty(X) \subset B(L^2(X))$, which are commutative.

Theorem. The commutative von Neumann algebras are those of the form $L^\infty(X)$, with X being a measured space.

Proof. Basic functional analysis and operator theory. The full statement involves as well a multiplicity, in regards with H.

Theory

Theorem. For a $*$-algebra of operators $A \subset B(H)$, the following conditions are equivalent:

(1) A is weakly closed, i.e. is a von Neumann algebra.

(2) A is equal to its algebraic bicommutant, $A = A''$.

This is von Neumann's "bicommutant theorem". As a consequence, the von Neumann algebras appear as commutants, $A = P'$.

Comments. Von Neumann $\implies$ C^*. Conversely, the von Neumann algebras are the C^*-algebras having separable predual. Also,

$$L^\infty(X) = C(\widehat{X})$$

by Gelfand, with $\widehat{X}$ being the Stone-Čech compactification of X.

Finite dimensions

Theorem. Let $A \subset M_N(\mathbb{C})$ be a $*$-algebra.

(1) We have $1 = p_1 + \ldots + p_k$, with $p_i \in A$ minimal projections.

(2) The spaces $A_i = p_i A p_i$ are non-unital $*$-subalgebras of A.

(3) We have a non-unital $*$-algebra sum $A = A_1 \oplus \ldots \oplus A_k$.

(4) Unital $*$-algebra isomorphisms $A_i \simeq M_{N_i}(\mathbb{C})$, $N_i = rank(p_i)$.

(5) Thus, we can decompose $A \simeq M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$.

Proof. (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5).

Reduction theory

Theorem. When writing the center of the algebra as

$$Z(A) = L^\infty(X)$$

wihh X measured space, the algebra decomposes as

$$A = \int_X A_x \, dx$$

with the summands being "factors", $Z(A_x) = \mathbb{C}$.

Example. In finite dimensions the algebra must be

$$A = M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$$

and this is its decomposition as a sum of factors.

Factors

Theorem. The factors, $Z(A) = \mathbb{C}$, fall into 3 classes:

(1) Type I. These are the usual matrix algebras $M_N(\mathbb{C})$ (type I_N), and the algebra $B(H)$, with H separable (type I_∞).

(2) Type II. These are the ∞D factors having a trace $tr : A \to \mathbb{C}$ (type II_1) and their tensor products with $B(H)$ (type II_∞).

(3) Type III. These fall into several classes, III_λ with $\lambda \in [0, 1]$, and appear from II_1 factors, via crossed product type constructions.

Proof. This is heavy, due to Murray and von Neumann, and then Connes, based on ideas of Tomita, Takesaki and others.

$\implies$ The II_1 factors are the "building blocks" of the theory.

II_1 factors

Definition. A II_1 factor is a von Neumann algebra $A \subset B(H)$:

(1) Which is infinite dimensional, $\dim(A) = \infty$.

(2) Has trivial center, $Z(A) = \mathbb{C}$.

(3) And has a faithful positive unital trace, $tr : A \to \mathbb{C}$.

Theorem 1. The trace is unique.

Theorem 2. The trace of projections can take any value in $[0, 1]$.

$\implies$ This is very interesting, "continuous dimension".

The factor R

Theorem 1. The following limiting von Neumann algebra,

$$R=\lim_{k\to\infty}M_{N_k}(\mathbb{C})$$

is a II_1 factor, independent of the limiting procedure.

Theorem 2. R is the unique "hyperfinite" II_1 factor.

Theorem 3. R is the unique "building block" for the whole hyperfinite von Neumann algebra theory.

These results, building on what has been said before, are heavy, due to Murray-von Neumann, Connes, and Connes-Haagerup.

Noncommutative geometry

Definition. The von Neumann algebra of a discrete group Γ is the weak closure of $\mathbb{C}[\Gamma]$ in the left regular representation:

$$L(\Gamma) \subset B(l^2(\Gamma))$$

Comment. When Γ is abelian, we obtain $L^\infty(\widehat{\Gamma})$. This is true in general, with $\widehat{\Gamma}$ being the NC space from the previous lecture:

$$L(\Gamma) = L^\infty(\widehat{\Gamma})$$

Theorem. The algebra $L(\Gamma)$ is a factor (of type II_1) when Γ has ICC. Also, $L(\Gamma) = R$ when Γ has ICC, and is amenable.

More. We can talk as well about $L^\infty(S)$ for the free spheres, but we need here free analogues of O_N, U_N, for integrating. Later.

Random matrices

Definition. A random matrix algebra is an algebra of type:

$$A = M_N(L^\infty(X))$$

The elements of A are called random matrices.

Theorem. The matrices $M \in A$ having i.i.d. normal entries, up to the constraint $M = M^*$, follow with $N \to \infty$ the semicircle law:

$$\gamma_t = \frac{1}{2\pi t}\sqrt{4t^2 - x^2}dx$$

Proof. Moment method. The Wick formula gives with $N \to \infty$ the Catalan numbers, which are the moments of the semicircle law.

Free probability

Definition. Two subalgebras $B, C \subset A$ are called:

(1) Independent, if $tr(b) = tr(c) = 0$ implies $tr(bc) = 0$.

(2) Free, if $tr(b_i) = tr(c_i) = 0$ implies $tr(b_1 c_1 b_2 c_2 \ldots) = 0$.

Theorem. We have the following results:

(1) $C^*(\Gamma), C^*(\Lambda)$ are independent inside $C^*(\Gamma \times \Lambda)$.

(2) $C^*(\Gamma), C^*(\Lambda)$ are free inside $C^*(\Gamma * \Lambda)$.

Theorem. Assuming that $x_1, x_2, x_3, \ldots \in A$ are i/f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim \mathcal{N}(0, t)/\gamma_t$$

where $\mathcal{N}(0, t)/\gamma_t$ are the normal/Wigner semicircle laws.

Subfactor theory

Definition. Consider an inclusion of II_1 factors $A \subset B$.

(1) Its index is the number $[B : A] = \dim_A B \in [1, \infty]$, defined as a Murray-von Neumann "continuous dimension" quantity.

(2) The "basic construction" is $A \subset B \subset C$, by "reflection", with $C =< B, e >$, where $e : B \to A$ is the orthogonal projection.

Theorem. Let $A_0 \subset A_1$ be a subfactor of finite index $N \in [1, \infty)$, and consider its Jones tower, obtained by basic construction:

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \ldots$$

The Jones projections $e_1, e_2, e_3, \ldots$ generate then a copy of the Temperley-Lieb algebra TL_N, inside the ambient algebra $B(H)$.

Quantum algebra explained

Teo Banica

"Introduction to operator algebras", 5/6

07/20

Landscape

There are basically 4 types of quantum algebra beasts:

1. Quantum groups
2. Tensor categories
3. Planar algebras
4. Subfactors

Quantum groups 1

Fact. Consider a compact group G, with structural maps:

$$m : G \times G \to G \quad , \quad u : \{.\} \to G \quad , \quad i : G \to G$$

By transposing m, u, i, we obtain certain morphisms Δ, ε, S.

Definition. A Hopf C^*-algebra is a C^*-algebra A, with morphisms

$$\Delta : A \to A \otimes A \quad , \quad \varepsilon : A \to \mathbb{C} \quad , \quad S : A \to A^{opp}$$

satisfying suitable axioms, as in the group case. We write

$$A = C(G) = C^*(\Gamma)$$

and call G compact quantum group, and Γ discrete quantum group.

Quantum groups 2

Examples. We have compact quantum groups defined via

$$\begin{aligned} C(O_N^+) &= C^*\left((u_{ij})_{i,j=1\ldots N}\middle| u=\bar{u}, u^t=u^{-1}\right) \\ C(U_N^+) &= C^*\left((u_{ij})_{i,j=1\ldots N}\middle| u^*=u^{-1}, u^t=\bar{u}^{-1}\right) \end{aligned}$$

called free orthogonal, and free unitary quantum groups.

Remark. These quantum groups act on the free spheres

$$O_N^+ \curvearrowright S^{N-1}_{\mathbb{R},+} \quad , \quad U_N^+ \curvearrowright S^{N-1}_{\mathbb{C},+}$$

so we have uniform integration, and von Neumann algebras.

Tensor categories 1

Definition. A corepresentation of A is a matrix $v \in M_n(A)$,

$$\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj} \quad , \quad \varepsilon(v_{ij}) = \delta_{ij} \quad , \quad S(v_{ij}) = v_{ji}^*$$

As basic examples, we have $u = (u_{ij})$, and its tensor powers $u^{\otimes k}$.

Theorem. The corepresentations are subject to a Peter-Weyl type theory. In particular, the irreducible ones appear by decomposing the tensor powers $u^{\otimes k}$ of the fundamental corepresentation u.

Tensor categories 2

Definition. The Tannakian category of a Woronowicz algebra (A, u) is the following collection $C = (C(k, l))$ of vector spaces:

$$C(k, l) = Hom(u^{\otimes k}, u^{\otimes l})$$

Definition. The Woronowicz algebra associated to a Tannakian category $C = (C(k, l))$ is constructed as follows:

$$A = C^* \left((u_{ij})_{i,j=1\ldots N} \middle| T \in Hom(u^{\otimes k}, u^{\otimes l}), \forall T \in C(k, l) \right)$$

Theorem. These operations produce a bijection $A \leftrightarrow C$, between Woronowicz algebras, and Tannakian categories.

Planar algebras 1

Definition. A planar algebra is a collection of FD $*$-algebras

$$P = (P_k)_{k\in\mathbb{N}}$$

such that whenever we have a diagram consisting of

– a big circle, with k points on it

– containing small circles, with k_i points on them

– and with strings connecting the $k+\sum_i k_i$ points

we can put "input" elements of P_{k_i} on the small circles, and we obtain an "output" element of P_k on the big circle.

Examples. The Temperley-Lieb algebra, $P_k = TL_k$. The tensor planar algebra, $P_k = \mathcal{L}(H^{\otimes k})$, with $H = \mathbb{C}^N$.

Planar algebras 2

Theorem. Given a compact quantum group $G \subset U_N^+$, the sequence of finite dimensional algebras

$$P_k = End(u^{\otimes k})$$

form a subalgebra of the tensor planar algebra. Any subalgebra of the tensor planar algebra appears in this way.

Proof. By Tannakian duality we have a correspondence:

$$G \subset U_N^+ \quad \longleftrightarrow \quad C \subset \mathcal{L}(H^{\otimes k}, H^{\otimes l})_{kl}$$

By restricing to the diagonal (via Frobenius) we get the result.

Subfactors 1

Theorem. Let $A_0 \subset A_1$ be a subfactor of finite index, and consider its Jones tower, obtained by basic construction:

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \ldots$$

The Jones projections $e_1, e_2, e_3, \ldots$ generate then a copy of the Temperley-Lieb algebra TL, inside the ambient algebra $B(H)$.

Theorem. The sequence of higher relative commutants

$$P_k = A_0' \cap A_k$$

form a planar algebra, extending the algebra $< e_i >= TL$.

Subfactors 2

Theorem. Given a compact quantum group $G\subset U_N^+$, consider its adjoint action on the matrix algebra $M_N(\mathbb{C})$:

$$G\curvearrowright M_N(\mathbb{C})$$

Assume that G acts on a II_1 factor A, minimally, $(A^G)'\cap A=\mathbb{C}$. We have then an inclusion of II_1 factors, of index N^2,

$$A^G\subset(M_N(\mathbb{C})\otimes A)^G$$

whose planar algebra is the previously constructed one, namely:

$$P_k=End(u^{\otimes k})$$

As examples, O_N^+,U_N^+ produce Temperley-Lieb subfactors.

Summary

Very basic examples of the 4 quantum algebra beasts:

1. Quantum groups: $G \subset U_N^+$.
2. Tensor categories: $C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$.
3. Planar algebras: $P_k = End(u^{\otimes k})$.
4. Subfactors: $A^G \subset (M_N(\mathbb{C}) \otimes A)^G$.

An interesting variation, to be discussed next time:

1. Quantum groups: $G \subset S_N^+$.
2. Tensor categories: $C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$.
3. Planar algebras: $P_k = Fix(u^{\otimes k})$.
4. Subfactors: $A^G \subset (\mathbb{C}^N \otimes A)^G$.

Extensions

There are countless extensions and variations of all this:

(1) Quantum groups. Those studied here, $G \subset U_N^+$, are technically "compact quantum Lie groups of Kac type". Many other.

(2) Tensor categories. Those studied here are over $k = \mathbb{C}$, have an involution $*$, and importantly, are "semisimple". Many other.

(3) Planar algebras. Similar comments. In addition, the world of planar algebras, even "unmodified", is substantially bigger.

(4) Subfactors. Same situation as for planar algebras. Very good question here: what are the finite index subfactors of R?

Spectral measures and beyond

Teo Banica

"Introduction to operator algebras", 6/6

07/20

Quantum algebra 1

There are basically 4 types of quantum algebra beasts:

1. Quantum groups. Basic examples $G \subset U_N^+$.

2. Tensor categories. $C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$.

3. Planar algebras. Simplest $P_k = End(u^{\otimes k})$.

4. Subfactors. Simplest $A^G \subset (M_N(\mathbb{C}) \otimes A)^G$.

$\Longrightarrow$ For full fun, classify the finite index subfactors of R!

Quantum algebra 2

Further quantum beasts, more advanced/specialized:

1. Quantum permutation groups: $G \subset S_N^+$.

2. Tensor categories: $C_{kl} = Hom(u^{\otimes k}, u^{\otimes l})$.

3. Spin planar algebras: $P_k = Fix(u^{\otimes k})$.

4. Fixed point subfactors: $A^G \subset (\mathbb{C}^N \otimes A)^G$.

$\implies$ Also with $\{1, \ldots, N\}$ replaced by finite NC spaces.

Spectral measures

Definition. The spectral measure of $G \subset U_N^+$ is the law:

$$\mu = law(\chi) \quad , \quad \chi = Tr(u)$$

For a planar algebra P, this is the measure having as moments:

$$M_k = \dim(P_k)$$

Similar definitions for tensor categories and subfactors.

Comment. A popular invariant in "algebra" is the Poincaré series:

$$f(z) = \sum_{k=0}^{\infty} \dim(P_k) z^k$$

This is the Stieltjes transform of the spectral measure μ.

Free probability 1/4

Definition. Two subalgebras $B, C \subset A$ are called:

(1) Independent, if $tr(b) = tr(c) = 0$ implies $tr(bc) = 0$.

(2) Free, if $tr(b_i) = tr(c_i) = 0$ implies $tr(b_1 c_1 b_2 c_2 \ldots) = 0$.

Theorem. We have the following results:

(1) $C^*(\Gamma), C^*(\Lambda)$ are independent inside $C^*(\Gamma \times \Lambda)$.

(2) $C^*(\Gamma), C^*(\Lambda)$ are free inside $C^*(\Gamma * \Lambda)$.

$\Longrightarrow$ Linearization of the usual/free convolution: $\log F/R$.

Free probability 2/4

<u>Theorem</u>. Assuming that $x_1, x_2, x_3, \ldots \in A$ are i.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim \mathcal{N}(0, t)$$

where $\mathcal{N}(0, t)$ is the normal law, having density $\frac{1}{\sqrt{2\pi t}} e^{-y^2/2t} dy$.

<u>Theorem</u>. Assuming that $x_1, x_2, x_3, \ldots \in A$ are f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim \gamma_t$$

where γ_t is the Wigner law, having density $\frac{1}{2\pi t} \sqrt{4t^2 - x^2} dx$.

Free probability 3/4

Theorem. We have the following convergence,

$$\left(\left(1-\frac{1}{n}\right)\delta_0+\frac{1}{n}\delta_t\right)^{*n}\to p_t$$

with p_t being the Poisson law of parameter $t>0$.

Theorem. We have the following convergence,

$$\left(\left(1-\frac{1}{n}\right)\delta_0+\frac{1}{n}\delta_t\right)^{\boxplus n}\to \pi_t$$

with π_t being the Marchenko-Pastur law of parameter $t>0$,

$$\pi_t=\max(1-t,0)\delta_0+\frac{\sqrt{4t-(x-1-t)^2}}{2\pi x}\,dx$$

also called free Poisson law of parameter $t>0$.

Free probability 4/4

Definition. Associated to any compactly supported positive measure ρ on $\mathbb{R}$, with mass $c = mass(\rho)$, are the probability measures

$$p_\rho = \lim_{n\to\infty}\left(\left(1-\frac{c}{n}\right)\delta_0 + \frac{1}{n}\rho\right)^{*n}$$

$$\pi_\rho = \lim_{n\to\infty}\left(\left(1-\frac{c}{n}\right)\delta_0 + \frac{1}{n}\rho\right)^{\boxplus n}$$

called compound Poisson and compound free Poisson laws.

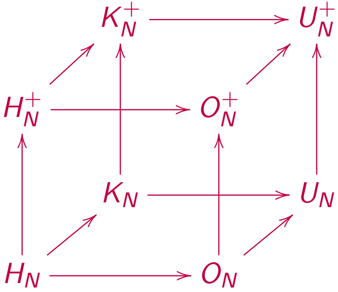
Definition. Let ε_s be the uniform mesure on the s-th roots of unity.

(1) With $\rho = t\varepsilon_s$, we get the classical and free Bessel laws.

(2) At $s = 2, \infty$, we call these laws "real" and "complex".

Core objects

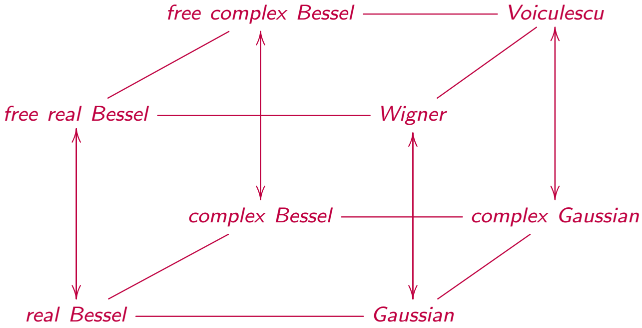
The main unitary and reflection groups, where $H_N = \mathbb{Z}_2 \wr S_N$, $K_N = \mathbb{T} \wr S_N$, and H_N^+, K_N^+ are their free analogues:



In planar algebra terms, the free objects correspond to TL and FC, depending on the correspondence which is chosen.

Core measures

The asymptotic laws of truncated characters $\chi_t = \sum_{i=1}^{[tN]} u_{ii}$ are



with the vertical arrows standing for the Bercovici-Pata bijection.

Beyond

These were very basic results, concerning the most basic quantum groups, and the most basic planar algebras. Problems:

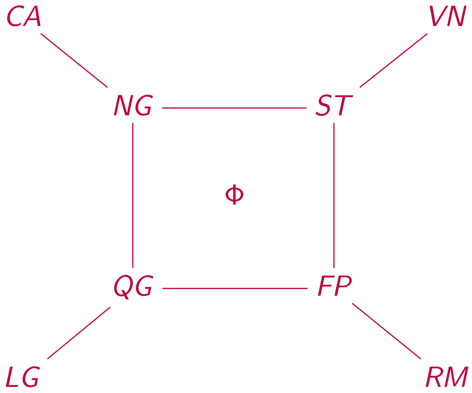
(1) Quantum groups. Easy quantum groups, and beyond. Meander determinants. Advanced probability, a la Diaconis.

(2) Subfactors. Poincaré series obstructions. Spectral measure blow-up. Big index subfactors. Also, what is $t > 0$.

(3) Free probability, NCG. The tori and other manifolds are subject to Meixner/free Meixner. Unification with Bercovici-Pata.

Conclusion

The picture of modern operator algebras is as follows,



with the hot stuff and physics being in the middle.

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