The determinant of real matrices

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Definition 1/3

Definition. Associated to any vectors $v_1, \ldots, v_N \in \mathbb{R}^N$ is the volume

$$\det^+(v_1 \ldots v_N) = vol < v_1, \ldots, v_N >$$

of the parallelepiped made by these vectors.

Remark. This notion is useful, for instance because $v_1, \ldots, v_N$ are linearly dependent precisely when $\det^+(v_1 \ldots v_N) = 0$.

Definition 2/3

Theorem. In 2 dimensions we have the formula

$$\det^+ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = |ad - bc|$$

valid for any two vectors $\binom{a}{c}, \binom{b}{d} \in \mathbb{R}^2$.

Proof. We must show that the area of the parrallelogram formed by the vectors $\binom{a}{c}, \binom{b}{d}$ equals the quantity $|ad - bc|$.

But this latter quantity is a difference of areas of two rectangles, and this can be done in "puzzle" style.

Comment. This is nice, but with $ad - bc$ as "answer", which is linear in a, b, c, d, it would be even nicer.

Definition 3/3

Convention. A system of vectors $v_1, \ldots, v_N \in \mathbb{R}^N$ is called:

(1) Oriented (+), if one can pass from the standard basis to it.

(2) Unoriented (-), otherwise.

Definition. Associated to $v_1, \ldots, v_N \in \mathbb{R}^N$ is the signed volume

$$\det(v_1 \ldots v_N) = vol^{\pm} < v_1, \ldots, v_N >$$

of the parallelepiped made by these vectors.

Remark. We have $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$, which is nice.

Properties 1/4

Notation. Given a matrix $A \in M_N(\mathbb{R})$, we write $\det A$, or just $|A|$, for the determinant of the system of column vectors of A.

Notation. Given a linear map, written as $f(v) = Av$, we call the number $\det A$ the "inflation coefficient" of f.

Remark. The inflation coefficient of f is the signed volume of the image $f(\square_N)$ of the unit cube $\square_N \in \mathbb{R}^N$.

Properties 2/4

Theorem. The determinant $\det A$ of the matrices $A \in M_N(\mathbb{R})$ has the following properties:

(1) It is a linear function of the columns of A.

(2) We have $\det(AB) = \det A \cdot \det B$.

(3) We have $\det(AB) = \det(BA)$.

Proof. (1) By doing some geometry, we obtain indeed:

$$\det(u + v, \{w_i\}) = \det(u, \{w_i\}) + \det(v, \{w_i\})$$

$$\det(\lambda u, \{w_i\}) = \lambda \det(u, \{w_i\})$$

(2) This follows from $f_{AB} = f_A f_B$, by looking at "inflation".

(3) Follows from (2), both quantities being $\det A \cdot \det B$.

Properties 3/4

Theorem. Assuming that a matrix $A \in M_N(\mathbb{R})$ is diagonalizable, with eigenvalues $\lambda_1, \ldots, \lambda_N$, we have:

$$\det A = \lambda_1 \ldots \lambda_N$$

Proof. This is clear from the "inflation" viewpoint, because in the basis formed by the eigenvectors $v_1, \ldots, v_N$, we have:

$$f_A(v_i) = \lambda_i v_i$$

Alternatively, $A = PDP^{-1}$ with $D = diag(\lambda_1, \ldots, \lambda_N)$, so

$$\det(A) = \det(PDP^{-1}) = \det(DP^{-1} \cdot P) = \det(D)$$

and by linearity $\det(D) = \lambda_1 \ldots \lambda_N \cdot \det(1_N) = \lambda_1 \ldots \lambda_N$.

Properties 4/4

Theorem. We have the following formula, for any $\lambda \in \mathbb{R}$:

$$\det(u, v, \{w_i\}_i) = \det(u - \lambda v, v, \{w_i\}_i)$$

Theorem. For an upper triangular matrix we have

$$\begin{vmatrix} \lambda_1 & & * \\ & \ddots & \\ & & \lambda_N \end{vmatrix} = \lambda_1 \ldots \lambda_N$$

and a similar result holds for the lower triangular matrices.

Proofs. The first theorem follows from linearity, because we have $\det(v, v, \{w_i\}_i) = 0$, and the second theorem follows from it.

Examples 1/4

Theorem. In 2 dimensions, the determinant is given by:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Proof. This is something that we already know, but that we can recover by using the general theory developed above:

$$\begin{aligned} \begin{vmatrix} a & b \\ c & d \end{vmatrix} &= \begin{vmatrix} a - b \cdot c/d & b \\ c - d \cdot c/d & d \end{vmatrix} \\ &= \begin{vmatrix} a - bc/d & b \\ 0 & d \end{vmatrix} \\ &= (a - bc/d)d \end{aligned}$$

Thus, we obtain the formula in the statement.

Examples 2/4

Theorem. In 3 dimensions, the determinant is given by

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

and this can be memorized by using Sarrus' triangle method.

Proof. This follows a bit as in 2 dimensions, by using the "Gauss method". We will be back later with a more conceptual proof.

Examples 3/4

<u>Theorem</u>. The determinant of a projection is always 0, unless the projection is the identity, and the determinant is 1.

<u>Proof</u>. This is clear with the "inflation" viewpoint. Alternatively, P is diagonalizable, with 1 eigenvalues on the image, and 0 outside:

$$P \sim \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 0 \end{pmatrix}$$

By making the product we obtain $\det P = 1 \ldots 1 \cdot 0 \ldots 0$, with at least one 0 in the case $P \neq 1_N$, as claimed.

Examples 4/4

Example. For the symmetry with respect to $x = y$, we have:

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 \cdot 0 - 1 \cdot 1 = -1$$

Example. For the rotation of angle $t \in \mathbb{R}$, we have:

$$\begin{vmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1$$

These formulae follow as well without computations, by "inflation".

Remark. The "basic" matrices tend to have determinant $-1, 0, 1$.

Theory 1/4

<u>Theorem</u>. The determinant can be fully computed by using the Gauss method, namely:

(1) Multiplying row by scalars.

(2) Substracting rows.

<u>Theorem</u>. The determinant function

$$\det : \mathbb{R}^N \times \ldots \times \mathbb{R}^N \to \mathbb{R}$$

is multilinear, alternate and unital, and unique with these properties.

<u>Proofs</u>. The first theorem is something that we already know, and the second theorem follows from it, by uniqueness.

Theory 2/4

Definition. A permutation of $\{1,\ldots,N\}$ is a bijection, as follows:

$$\sigma:\{1,\ldots,N\}\to\{1,\ldots,N\}$$

The set of such permutations is denoted S_N.

Theorem. There are $N!=1.2.3\ldots N$ such permutations.

Proof. We have N choices for $\sigma(1)$, then $N-1$ choices for $\sigma(2)$, and so on, up to 1 choice for $\sigma(N)$.

Definition. The signature of a permutation $\varepsilon(\sigma)\in\{\pm 1\}$ is the number of inversions, $i<j$ with $\sigma(i)>\sigma(j)$.

Theory 3/4

Theorem. The determinant is given by the formula

$$\det A = \sum_{\sigma \in S_N} \varepsilon(\sigma) A_{1\sigma(1)} \ldots A_{N\sigma(N)}$$

with the signature function being the one introduced above.

Proof. This follows either by using the Gauss method, or by using the abstract characterization of the determinant.

Remark. At $N = 3$ we obtain in this way the Sarrus formula.

Theory 4/4

<u>Theorem</u>. The eigenvalues of a matrix $A \in M_N(\mathbb{R})$ must satisfy

$$P_A(\lambda) = 0$$

where $P_A = \det(A - \lambda 1_N)$ is the characteristic polynomial.

<u>Proof</u>. Given a vector $v \in \mathbb{R}^N$ and a number $\lambda \in \mathbb{R}$, we have:

$$Av = \lambda v \iff (A - \lambda 1_N)v = 0$$

But this latter equation has nonzero solutions when

$$B = \det(A - \lambda 1_N)$$

is not invertible, and so when $\det B = 0$.