Special matrices and matrix tricks

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Fourier 1/3

Theorem. We have the Vandermonde formula:

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1 & x_2 & \dots & x_N \\ \vdots & \vdots & & \vdots \\ x_1^{N-1} & x_2^{N-1} & \dots & x_N^{N-1} \end{vmatrix} = \prod_{i>j}(x_i - x_j)$$

Proof. The determinant D is a polynomial in $x_1, \dots, x_N$, of degree $N-1$ in each variable. Since $x_i = x_j$ makes $D = 0$, we obtain:

$$D = c\prod_{i>j}(x_i - x_j)$$

The constant $c \in \mathbb{R}$ can be computed by recurrence, we get $c = 1$.

Fourier 2/3

Definition. The Fourier matrix F_N is given by:

$$F_N = (w^{ij})_{ij} \quad , \quad w = e^{2\pi i/N}$$

With matrices indices $i, j = 0, 1, \ldots, N-1$, we have:

$$F_N = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & w & w^2 & \ldots & w^{N-1} \\ 1 & w^2 & w^4 & \ldots & w^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & w^{N-1} & w^{(2N-1)} & \ldots & w^{(N-1)^2} \end{pmatrix}$$

This is a Vandermonde matrix, with $x_i = w^i$.

Fourier 3/3

Theorem. The rescaled matrix $\mathcal{F}_N = \frac{1}{\sqrt{N}}(w^{ij})_{ij}$ is unitary.

Proof. We have the following computation:

$$\begin{aligned}(F_N F_N^*)_{ij} &= \sum_k (F_N)_{ik}(\bar{F}_N)_{jk} \\ &= \sum_k w^{ik} \cdot w^{-jk} \\ &= \sum_k (w^{i-j})^k \\ &= N\delta_{ij}\end{aligned}$$

Thus the rescaled matrix $\mathcal{F}_N = F_N/\sqrt{N}$ is unitary.

Special matrices 1/4

Theorem. For a matrix $H \in M_N(\mathbb{C})$, the following are equivalent,

(1) H is circulant, $H_{ij} = \xi_{j-i}$ for some $\xi \in \mathbb{C}^N$.

(2) H is Fourier-diagonal, $H = \mathcal{F}Q\mathcal{F}^*$ with Q diagonal.

where $\mathcal{F} = \mathcal{F}_N$. In addition, the first row vector of H is

$$\xi = \mathcal{F}q/\sqrt{N}$$

where $q_i = Q_{ii}$ is the vector formed by the diagonal entries of Q.

Special matrices 2/4

Proof. If $H_{ij} = \xi_{j-i}$ is circulant then $Q = \mathcal{F}^* H \mathcal{F}$ is diagonal:

$$Q_{ij} = \frac{1}{N} \sum_{kl} w^{jl-ik} \xi_{l-k} = \delta_{ij} \sum_r w^{jr} \xi_r$$

Also, if $Q = diag(q)$ is diagonal then $H = \mathcal{F} Q \mathcal{F}^*$ is circulant:

$$H_{ij} = \sum_k \mathcal{F}_{ik} Q_{kk} \bar{\mathcal{F}}_{jk} = \frac{1}{N} \sum_k w^{(i-j)k} q_k$$

This formula proves as well the last assertion, $\xi = \mathcal{F} q / \sqrt{N}$.

Special matrices 3/4

Theorem. The various sets of circulant matrices are as follows,

(1) $M_N(\mathbb{C})^{circ} = \{\mathcal{F}Q\mathcal{F}^* | q \in \mathbb{C}^N\}$.

(2) $U_N^{circ} = \{\mathcal{F}Q\mathcal{F}^* | q \in \mathbb{T}^N\}$.

(3) $O_N^{circ} = \{\mathcal{F}Q\mathcal{F}^* | q \in \mathbb{T}^N, \bar{q}_i = q_{-i}, \forall i\}$.

with the convention $Q = diag(q)$, for $q \in \mathbb{C}^N$.

Proof. (1) This is something that we already know.

(2) This is because the eigenvalues must be on the unit circle $\mathbb{T}$.

(3) For $q \in \mathbb{C}^N$ we have $\overline{\mathcal{F}q} = \mathcal{F}\tilde{q}$, with $\tilde{q}_i = \bar{q}_{-i}$, and so $\xi = \mathcal{F}q$ is real if and only if $\bar{q}_i = q_{-i}$ for any i. This gives the result.

Special matrices 4/4

Theorem. The groups $B_N \subset O_N$ and $C_N \subset U_N$ of bistochastic matrices (sum 1 on each row and column) are given by:

$$B_N \simeq O_{N-1} \quad , \quad C_N \simeq U_{N-1}$$

Proof. The all-1 vector ξ being equal to $\sqrt{N}\mathcal{F}e_0$, we have:

$$\begin{aligned} U\xi = \xi \quad &\iff \quad U\mathcal{F}e_0 = \mathcal{F}e_0 \\ &\iff \quad \mathcal{F}^*U\mathcal{F}e_0 = e_0 \\ &\iff \quad \mathcal{F}^*U\mathcal{F} = diag(1, w) \end{aligned}$$

Thus we have isomorphisms as in the statement.

Hadamard matrices 1/4

Definition. A complex Hadamard matrix is a square matrix

$$H \in M_N(\mathbb{C})$$

whose entries are on the unit circle, $H_{ij} \in \mathbb{T}$, and whose rows are pairwise orthogonal, with respect to the scalar product of $\mathbb{C}^N$.

Example. For the Fourier matrix, $F_N = (w^{ij})$ with $w = e^{2\pi i/N}$, the scalar products between rows are:

$$< R_a, R_b >= \sum_j w^{aj} w^{-bj} = \sum_j w^{(a-b)j} = N\delta_{ab}$$

Thus the Fourier matrix F_N is Hadamard.

Hadamard matrices 2/4

Theorem. Given a finite abelian group G, with group dual

$$\widehat{G}=\{\chi:G\to\mathbb{T}\}$$

consider the Fourier coupling $G\times\widehat{G}\to\mathbb{T}$:

$$(i,\chi)\to\chi(i)$$

(1) Via the standard isomorphism $G\simeq\widehat{G}$, this Fourier coupling is a square matrix, $F_G\in M_G(\mathbb{T})$, which is complex Hadamard.

(2) For a cyclic group $G=\mathbb{Z}_N$ we obtain in this way, via the standard identification $\mathbb{Z}_N=\{1,\ldots,N\}$, the Fourier matrix F_N.

(3) In general, when using a decomposition $G=\mathbb{Z}_{N_1}\times\ldots\times\mathbb{Z}_{N_k}$, the corresponding Fourier matrix is $F_G=F_{N_1}\otimes\ldots\otimes F_{N_k}$.

Hadamard matrices 3/4

Examples. (1) For the cyclic group $\mathbb{Z}_2$ we obtain the Fourier matrix F_2, also denoted W_2, and called first Walsh matrix:

$$W_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(2) For the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$ we obtain the tensor product $W_4 = W_2 \otimes W_2$, called second Walsh matrix:

$$W_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

(3) In general, for the group $\mathbb{Z}_2^n$ we obtain the n-th Walsh matrix $W_N = W_2^{\otimes n}$, having size $N = 2^n$. Useful in radio, coding.

Hadamard matrices 4/4

Hadamard Conjecture. There is at least one real Hadamard matrix

$$H \in M_N(\pm 1)$$

for any integer $N \in 4\mathbb{N}$.

Comment. Verified so for up to $\mathfrak{N} = 666$.

Rotations 1/4

Theorem. For a matrix $U \in M_N(\mathbb{C})$, the following are equivalent:

(1) U preserves the scalar product, $< Ux, Uy > = < x, y >$.

(2) U preserves the norm, $||Ux|| = ||x||$, where $||x|| = \sqrt{< x, x >}$.

(3) U is unitary, in the sense that $U^* = U^{-1}$, where $(U^*)_{ij} = \bar{U}_{ji}$.

(4) U has its eigenvalues on the unit circle $\mathbb{T}$.

Proof. The equivalences (1) $\iff$ (2) $\iff$ (3) follow by using $< Mx, y > = < x, M^*y >$, and (4) is something that we know.

Rotations 2/4

Theorem. The unitaries in $M_2(\mathbb{C})$ of determinant 1 are

$$U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$$

with $a, b \in \mathbb{C}$ satisfying $|a|^2 + |b|^2 = 1$.

Proof. For $U = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ of determinant 1, $U^* = U^{-1}$ reads:

$$\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Thus $c = -\bar{b}, d = \bar{a}$. Finally, $\det U = 1$ gives $|a|^2 + |b|^2 = 1$.

Rotations 3/4

<u>Theorem</u>. The unitaries in $M_3(\mathbb{R})$ of determinant 1 are

$$O=\begin{pmatrix}x^2+y^2-z^2-t^2&2(yz-xt)&2(xz+yt)\\2(xt+yz)&x^2+z^2-y^2-t^2&2(zt-xy)\\2(yt-xz)&2(xy+zt)&x^2+t^2-y^2-z^2\end{pmatrix}$$

with $x,y,z,t\in\mathbb{R}$ satisfying $x^2+y^2+z^2+t^2=1$.

<u>Proof</u>. With $a=x+iy$, $b=z+it$, the previous formula reads:

$$U=\begin{pmatrix}x+iy&z+it\\-z+it&x-iy\end{pmatrix}$$

But we must have "$O+1=ad(U)$", and this gives the result.

Rotations 4/4

Conclusion. We can now:

- do some serious engineering
- or write 3D games software.