Symmetry and reflection groups

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"Introduction to matrix groups", 2/6

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Finite groups 1/3

Theorem. Any finite group is a permutation group.

Proof. Given a finite group G, we have an embedding as follows:

$$G \subset S_G \quad , \quad \sigma_g(h) = gh$$

In other words, we have $G \subset S_N$, with $N = |G|$.

Finite groups 2/3

Theorem. Any finite group appears as group of orthogonal matrices.

Proof. This is true for S_N, which can be regarded as being the permutation group of the N coordinate axes of $\mathbb{R}^N$:

$$S_N \subset O_N$$

Thus, given a group G of finite order $N < \infty$, we have:

$$G \subset S_N \subset O_N$$

Finite groups 3/3

Conclusion. The following are the same thing:

(1) The finite groups.

(2) The subgroups $G \subset S_N$.

(3) The finite subgroups $G \subset O_N$.

(4) The finite subgroups $G \subset U_N$.

Problem. Given a finite group G, what is the "best" embedding of type $G \subset U_N$, say with $N \in \mathbb{N}$ being smallest possible?

Comment. This is a "representation theory" problem.

Dihedral groups 1/4

Theorem. Consider the cyclic group $\mathbb{Z}_N$.

(1) We have an embedding $\mathbb{Z}_N \subset O_2$, given by:

$$k \to \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \quad , \quad t = \frac{2k\pi}{N}$$

(2) We have an embedding $\mathbb{Z}_N \subset O_N$, given by:

$$k \to [e_i \to e_{i+k}]$$

(3) We have an embedding $\mathbb{Z}_N \subset U_1$, given by:

$$k \to (w^k) \quad , \quad w = e^{2\pi i/N}$$

Comment. (2) is nicer than (1), and (3) beats everything.

Dihedral groups 2/4

Definition. The dihedral group D_N is the group of symmetries of a regular N-gon.

Examples.

(1) At $N = 3$ we have 3 symmetries, with respect to the 3 medians of $\triangle$, as well as 3 rotations, of angles $0^\circ, 120^\circ, 240^\circ$.

(2) At $N = 4$ we have 4 symmetries, with respect to Ox, Oy and the diagonals of $\square$, and 4 rotations, of angles $0^\circ, 90^\circ, 180^\circ, 270^\circ$.

Dihedral groups 3/4

Theorem. The dihedral group D_N has $2N$ elements, as follows:

(1) N rotations, of angles $2k\pi/N$, with $k = 0, 1, \ldots, N-1$. These form a copy $\mathbb{Z}_N \subset D_N$ of the cyclic group $\mathbb{Z}_N$.

(2) N symmetries, with respect to the N medians when N is odd, and to the $N/2 + N/2$ symmetry axes, when N is even.

In addition, we have a formula of type $D_N = \mathbb{Z}_N \rtimes \mathbb{Z}_2$.

Proof. (1) and (2) are clear. Regarding the last part, D_N has the same number of elements as $\mathbb{Z}_N \times \mathbb{Z}_2$, but is not abelian. Thus, we must "twist" the product of $\mathbb{Z}_N \times \mathbb{Z}_2$ in order to obtain D_N.

Dihedral groups 4/4

Theorem. Consider the dihedral group D_N.

(1) We have an embedding $D_N \subset O_2$, given by the usual rotation and symmetry matrices:

$$R_k \to \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \quad , \quad t = \frac{2k\pi}{N}$$

$$S_k \to \begin{pmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{pmatrix} \quad , \quad t = \frac{2k\pi}{N}$$

(2) We have an embedding $D_N \subset O_N$, obtained by permuting the N-gon on the coordinate axes of $\mathbb{R}^N$, at distance 1 from 0:

$$\sigma \to [e_i \to e_{\sigma(i)}]$$

(3) We cannot have an embedding $D_N \subset U_1$, because the group U_1 is abelian, and D_N is not abelian.

Symmetric groups 1/4

Theorem. The permutation group S_N has $N!$ elements.

Proof. In order to construct a permutation $\sigma \in S_N$, we must:

(1) Choose $\sigma(1)$, and there are N choices here.

(2) Choose $\sigma(2)$, and there are $N-1$ choices left.

$\vdots$

$\vdots$

(N) Choose $\sigma(N)$, and there is 1 choice left.

Thus, we have a total of $N(N-1)\ldots 1 = N!$ choices.

Symmetric groups 2/4

Theorem. We have an embedding $S_N \subset O_N$, given by:

$$\sigma \to [e_i \to e_{\sigma(i)}]$$

By using the standard $e_{ij} : e_j \to e_i$ notation, the formula is:

$$\sigma \to \sum_i e_{\sigma(i)i}$$

In matrix notation, and with Kronecker symbols, the formula is:

$$\sigma \to [\delta_{i\sigma(j)}]_{ij}$$

Proof. The first assertion is clear, because the transformations $e_i \to e_{\sigma(i)}$ are isometries of $\mathbb{R}^N$, and the rest is clear too.

Symmetric groups 3/4

Theorem. The permutation matrices $S_N \subset O_N$ are precisely the 0-1 matrices having a 1 entry on each row and column.

Theorem. The trace of a permutation matrix $\sigma \in S_N \subset O_N$ is the number of its fixed points.

Proofs. Both these results are clear from definitions.

Symmetric groups 4/4

Theorem. The determinant of the permutation matrices

$$\det(\sigma) \in \{\pm 1\}$$

coinsides with the signature of the permutations,

$$\varepsilon(\sigma) = (-1)^c$$

where c is the number of inversions.

Proof. This is clear with any of the definitions of det.

Comment. Thus, $S_N \cap SO_N = A_N$, the alternating group.

Reflection groups 1/4

Definition. The hyperoctahedral group H_N is the symmetry group of the hypercube $\square_N \subset \mathbb{R}^N$.

Comment. Thus, we have by definition $H_N \subset S_{2^N}$.

Example. We have $H_2 = D_4$.

Problem. $|H_N| = ?$

Reflection groups 2/4

Theorem. The group H_N appears as well as the group of signed permutations of the coordinate axes of $\mathbb{R}^N$, so we have

$$H_N \subset O_N$$

with the image consisting of the $-1, 0, 1$ matrices having exactly one ± 1 entry on each row and each column. Thus we have:

$$|H_N| = 2^N N!$$

Comment. One can prove that $H_N = S_N \rtimes \mathbb{Z}_2^N$, which is also written as $H_N = \mathbb{Z}_2 \wr S_N$, wreath product.

Reflection groups 3/4

Definition. The reflection group H_N^s, depending on parameters

$$N \in \mathbb{N} \quad , \quad s \in \mathbb{N} \cup \{\infty\}$$

is the group of $N \times N$ matrices having entries in

$$\mathbb{Z}_s \cup \{0\}$$

having exactly one nonzero entry on each row and each column.

Examples. At $s = 1$ we obtain S_N, and at $s = 2$ we obtain H_N. In general, at $s < \infty$, we have a certain finite group $H_N^s \subset U_N$. At $s = \infty$ we have a group $K_N \subset U_N$, which is no longer finite.

Reflection groups 4/4

One can prove that the "complex reflection groups" are:

– The above groups $H_N^s = S_N \wr \mathbb{Z}_s$.

– Their subgroups H_N^{sd} given by $\det^d = 1$.

– And some exceptional examples.