Representations of compact groups

Teo Banica

"Introduction to matrix groups", 5/6

08/20

Representations 1/3

Definition. Given a closed subgroup $G \subset U_N$, its representations are the continuous morphisms into unitary groups:

$$\rho : G \to U_n$$

As a basic example, we have the embedding $G \subset U_N$, called fundamental representation, and denoted π.

Comment. We will assume that our representations are "smooth", in the sense that their coefficients are polynomials of g_{ij}.

Representations 2/3

Definition. The representations of G are subject to:

(1) Making sums: $\rho + \nu : g \to diag(\rho(g), \nu(g))$.

(2) Making products: $\rho \otimes \nu : g \to \rho(g) \otimes \nu(g)$.

(3) Taking conjugates: $\bar{\rho} : g \to \overline{\rho(g)}$.

Definition. Given $G \subset_\pi U_N$, its Peter-Weyl representations

$$\pi^{\otimes k} \quad , \quad k = \circ \bullet \bullet \circ \ldots$$

are the representations obtained by tensoring $\pi, \bar{\pi}$.

Representations 3/3

Definition. Given $\rho : G \to U_n$ and $\nu : G \to U_m$, we set:

$$Hom(\rho,\nu) = \left\{ T \in M_{m\times n}(\mathbb{C}) \middle| T\rho(g) = \nu(g)T \right\}$$

and we use the following conventions:

(1) $Fix(\rho) = Hom(1,\rho)$ and $End(\rho) = Hom(\rho,\rho)$.

(2) $\rho \sim \nu$ when $Hom(\rho,\nu)$ contains an invertible element.

(3) ρ is called irreducible, $\rho \in Irr(G)$, when $End(\rho) = \mathbb{C}1$.

Definition. Given $G \subset_\pi U_N$, the collection of vector spaces

$$C_{kl} = Hom(\pi^{\otimes k}, \pi^{\otimes l})$$

with $k, l = \circ\bullet\bullet\circ\ldots$ is called Tannakian category of G.

Peter-Weyl 1/7

Theorem (PW1). Any representation $\rho : G \to U_n$ decomposes as

$$\rho = \rho_1 + \ldots + \rho_k$$

direct sum of irreducible representations.

Proof. Consider the intertwiner algebra of our representation:

$$A = End(\rho) \subset M_n(\mathbb{C})$$

By writing its unit as $1 = q_1 + \ldots + q_k$, with q_i being minimal projections, we obtain a decomposition as follows:

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

We can now define a subrepresentation ρ_i by restricting ρ to the space $Im(q_i)$, which is invariant, and the result follows.

Peter-Weyl 2/7

Theorem (PW2). Any irreducible representation $\rho : G \to U_n$ appears inside a certain Peter-Weyl representation $\pi^{\otimes k}$.

Proof. Given a representation $\rho : G \to U_n$, consider its space of coefficients, $C_\rho = span(g \to \rho(g)_{ij})$. Then $\rho \to C_\rho$ is functorial, mapping subrepresentations into subspaces. We have:

$$< C_\pi >= \sum_k C_{\pi^{\otimes k}}$$

By smoothness, $C_\rho \subset < C_\pi >$, for certain exponents $k_1, \ldots, k_p$:

$$C_\rho \subset C_{\pi^{\otimes k_1} \oplus \ldots \oplus \pi^{\otimes k_p}}$$

Thus we have $\rho \subset \pi^{\otimes k_1} \oplus \ldots \oplus \pi^{\otimes k_p}$, and PW1 gives the result.

Peter-Weyl 3/7

Theorem. Any closed subgroup $G \subset U_N$ has a Haar measure

$$\mu(gE) = \mu(Eg) = \mu(E)$$

which can be constructed by starting with any probability measure ν, and taking the following Cesàro limit:

$$\mu = \lim_{r \to \infty} \frac{1}{r} \sum_{k=1}^{r} \nu^{*k}$$

Moreover, for any representation $\rho : G \to U_n$, the matrix

$$P = \left(\int_G \rho(g)_{ij} \, dg \right)_{ij}$$

is the projection onto $Fix(\rho) = \{\xi \in \mathbb{C}^n | \rho(g)\xi = \xi\}$.

Peter-Weyl 4/7

Proof. Our first claim is that given any positive mass 1 measure ν on our group G, not necessarily strictly positive, the limit

$$\int_G^\nu f = \lim_{r\to\infty}\frac{1}{r}\sum_{k=1}^r\int_G f(g)d\nu^{*k}(g)$$

exists, and for any representation $\rho : G \to U_n$, the matrix

$$P = \left(\int_G^\nu \rho(g)_{ij}\,dg\right)_{ij}$$

is the projection onto the 1-eigenspace of the matrix:

$$M = \left(\int_G \rho(g)_{ij}d\nu(g)\right)_{ij}$$

This is indeed standard algebra, on the coefficient space C_ρ.

Peter-Weyl 5/7

End of proof. Assuming now that ν is strictly positive, we must prove that $M\xi = \xi$ implies $\xi \in Fix(\rho)$. Let us set:

$$f(g) = \sum_i \left(\sum_j \rho(g)_{ij}\xi_j - \xi_i \right) \overline{\left(\sum_k \rho(g)_{ik}\xi_k - \xi_i \right)}$$

We must prove that $f = 0$. Since $\rho(g)$ is unitary, we obtain:

$$f(g) = 2\Big(||\xi||^2 - Re(< \rho(g)\xi, \xi >) \Big)$$

By using now $M\xi = \xi$, we obtain from this, by integrating:

$$\int_G f(g) d\nu(g) = 0$$

Thus we have $f = 0$, and so $\xi \in Fix(\rho)$, as desired.

Peter-Weyl 6/7

Theorem (PW3). The space $\mathcal{C}(G) = < C_\pi >$ decomposes as

$$\mathcal{C}(G) = \bigoplus_{\rho \in Irr(A)} M_{\dim(\rho)}(\mathbb{C})$$

the summands being pairwise orthogonal with respect to $\int_G$.

Proof. We must prove that for $\rho, \nu \in Irr(G)$ we have:

$$\rho \not\sim \nu \implies C_\rho \perp C_\nu$$

The matrix P given by $P_{ia,jb} = \int_G \rho_{ij} \bar{\nu}_{ab}$ is the projection onto:

$$Fix(\rho \otimes \bar{\nu}) \simeq Hom(\rho, \nu) = \{0\}$$

Thus we have $P = 0$, and this gives the result.

Peter-Weyl 7/7

Theorem (PW4). The characters of irreducible representations

$$\chi_\rho : G \to \mathbb{C} \quad , \quad g \to Tr(\rho(g))$$

belong to the algebra of "smooth central functions"

$$\mathcal{C}(G)_{central} = \left\{ f \in \mathcal{C}(G) \middle| f(gh) = f(hg) \right\}$$

and form an orthonormal basis of it.

Proof. The only tricky assertion is the norm 1 one. But:

$$\int_G \chi_\rho \bar{\chi}_\rho = \sum_{ij} \int_G \rho_{ii} \bar{\rho}_{jj} = \sum_i \frac{1}{N} = 1$$

Here we have used the fact that the integrals $\int_G \rho_{ij} \bar{\rho}_{kl}$ form the orthogonal projection onto $Fix(\rho \otimes \bar{\rho}) \simeq End(\rho) = \mathbb{C}1$.

Easiness 1/3

Theorem. The closed subgroups $G \subset U_N$ are in correspondence with the Tannakian categories $C = (C_{kl})$, via the construction

$$C_{kl} = Hom(\pi^{\otimes k}, \pi^{\otimes l})$$

in one sense, and via the construction

$$G = \left\{ g \in U_N \middle| Tg^{\otimes k} = g^{\otimes l} T, \forall T \in C \right\}$$

in the other sense.

Proof. This is something quite technical, basically due to Tannaka and Krein, and heavily using the Peter-Weyl theory.

Easiness 2/3

Definition. A collection of subsets $D(k,l) \subset P(k,l)$ is called a category of partitions when it satisfies:

(1) Stability under the horizontal concatenation, $(\pi,\sigma) \to [\pi\sigma]$.

(2) Stability under vertical concatenation $(\pi,\sigma) \to [^{\sigma}_{\pi}]$ (matching).

(3) Stability under the upside-down turning $*$, with $\circ \leftrightarrow \bullet$.

(4) Each $P(k,k)$ contains the identity partition $|| \ldots ||$.

(5) Both $P(\emptyset, \circ\bullet)$ and $P(\emptyset, \bullet\circ)$ contain the semicircle $\cap$.

Easiness 3/3

Definition. A closed subgroup $G \subset U_N$ is called easy when

$$Hom(\pi^{\otimes k}, \pi^{\otimes l}) = span\left(T_\pi \middle| \pi \in D(k,l) \right)$$

for a certain category of partitions $D \subset P$, where

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with $\delta_\pi \in \{0,1\}$ depending on whether the indices fit or not.

Examples 1/2

Theorem. The basic unitary and reflection groups, namely

$$\begin{array}{ccc}
O_N & \longrightarrow & U_N \\
\uparrow & & \uparrow \\
H_N & \longrightarrow & K_N
\end{array}$$

are all easy, coming from the following categories of partitions:

$$\begin{array}{ccc}
P_2 & \longleftarrow & \mathcal{P}_2 \\
\downarrow & & \downarrow \\
P_{even} & \longleftarrow & \mathcal{P}_{even}
\end{array}$$

Proof. This result, due to Brauer, and also known as Schur-Weyl duality, comes from Tannaka, by working out the details.

Examples 2/2

In addition to the above, it is known that:

(1) In the continuous case, the bistochastic groups $B_N \subset O_N$ and $C_N \subset U_N$ are easy as well, coming from $P_{12}, \mathcal{P}_{12}$.

(2) In the discrete case, S_N is easy as well, coming from P itself. In fact, the reflection groups H_N^s are all easy, coming from P^s.

(3) Back to the continuous case, SU_2, SO_3 and $Sp_N \subset U_N$ are not easy. However, they are "super-easy" in a suitable sense.

(4) However, the general SO_N, SU_N, and other groups constructed using det, such as H_N^{sd}, are definitely not easy.