Probability on compact groups

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"Introduction to matrix groups", 6/6

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Characters 1/3

Problem. Given a closed subgroup $G \subset U_N$, what is the law of

$$\chi : G \to \mathbb{C} \quad , \quad g \to Tr(g)$$

with respect to the uniform integration over G?

Characters 2/3

Motivation. The moments of χ are the dimensions

$$M_k = \dim(Fix(\pi^{\otimes k}))$$

of the fixed point spaces of tensor powers of $\pi : G \subset U_N$.

Comment. We are mostly interested in the Tannakian category

$$C_{kl} = Hom(\pi^{\otimes k}, \pi^{\otimes l})$$

and by Frobenius, we have identifications as follows:

$$Hom(\pi^{\otimes k}, \pi^{\otimes l}) = Fix(\pi^{\otimes \bar{k}l})$$

Thus, the moments of χ count the dimensions $\dim(C_{kl})$.

Characters 3/3

Version. More generally, we are interested in the truncations

$$\chi_t : G \to \mathbb{C} \quad , \quad g \to \sum_{i=1}^{[tN]} g_{ii}$$

with $t \in (0,1]$ of the main character $\chi = \chi_1$.

Example. For the symmetric group $S_N \subset O_N$ we have

$$\chi \sim p_1$$

Poisson, and more generally $\chi_t \sim p_t$ for any t, with $N \to \infty$.

Finite groups 1/4

Theorem. For the cyclic group $\mathbb{Z}_N \subset O_N$ we have

$$\chi(g) = N\delta_{g0}$$

and the corresponding distribution is a Bernoulli law:

$$law(\chi) = \left(1 - \frac{1}{N}\right)\delta_0 + \frac{1}{N}\delta_N$$

Proof. The cyclic matrices have 0 on the diagonal, and so trace 0, expect for the identity, having 1 on the diagonal, and trace N.

Remark. The truncated characters and the asymptotics are not interesting. We do not have convolution semigroups.

Finite groups 2/4

Theorem. For the dihedral group $D_N \subset S_N$ we have:

$$law(\chi) = \begin{cases} \left(\frac{3}{4} - \frac{1}{2N}\right)\delta_0 + \frac{1}{4}\delta_2 + \frac{1}{2N}\delta_N & (N\ even) \\ \\ \left(\frac{1}{2} - \frac{1}{2N}\right)\delta_0 + \frac{1}{2}\delta_1 + \frac{1}{2N}\delta_N & (N\ odd) \end{cases}$$

Proof. The dihedral group D_N consists of:

(1) N symmetries, having 1 fixed point when N is odd, and having 0 or 2 fixed points, $50-50$, when N is even.

(2) N rotations, having 0 fixed points, except for the identity, which has N fixed points.

Remark. The truncations and asymptotics are not interesting.

Finite groups 3/4

Theorem. For the symmetric group $S_N \subset O_N$ we have

$$\chi_t(\sigma) = \left\{ i \in \{1, \ldots, [tN]\} \middle| \sigma(i) = i \right\}$$

and we have $law(\chi_t) \simeq p_t$, Poisson laws, with $N \to \infty$.

Proof. By using the inclusion-exclusion principle, we have:

$$P(\chi = 0) = 1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + \frac{(-1)^N}{N!} \simeq \frac{1}{e}$$

The same method gives succesively, by generalizing,

$$P(\chi = k) \simeq \frac{1}{e} \cdot \frac{1}{k!} \quad , \quad P(\chi_t = k) \simeq \frac{1}{e^t} \cdot \frac{t^k}{k!}$$

so we obtain in the $N \to \infty$ limit the Poisson laws p_t.

Finite groups 4/4

<u>Theorem</u>. For the complex reflection groups

$$H_N^s = \mathbb{Z}_s \wr S_N$$

we have $law(\chi_t) \simeq b_t^s$, Bessel laws, with $N \to \infty$.

<u>Proof</u>. The elements of H_N^s being usual permutations $\sigma \in S_N$ "decorated" with signs $\varepsilon \in \mathbb{Z}_s^N$, we can use the same method as before, inclusion-exclusion, and with $N \to \infty$ we are led to

$$b_t^s = \pi_{t\varepsilon_s}$$

compound Poisson laws, with ε_s being the uniform measure on $\mathbb{Z}_s$, which are called Bessel laws, due to the fact that at $s = 2$ the density is the Bessel function of the first kind.

Lie groups 1/4

Definition. The normal law of parameter 1 is:

$$g_1 = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

More generally, the normal law of parameter $t > 0$ is:

$$g_t = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t} dx$$

These laws appear via the Central Limit Theorem (CLT).

Lie groups 2/4

Theorem. The moments of the normal laws are

$$M_k(g_t) = t^{k/2} \times k!!$$

where $k!! = 1.3.5\dots(k-1)$, with $k!! = 0$ when k is odd.

Proof. We have the following computation:

$$\begin{aligned} M_k &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} x^k e^{-x^2/2t}dx \\ &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} (tx^{k-1})\left(-e^{-x^2/2t}\right)' dx \\ &= \frac{1}{\sqrt{2\pi t}}\int_{\mathbb{R}} t(k-1)x^{k-2}e^{-x^2/2t}dx \end{aligned}$$

We obtain $M_k = t(k-1)M_{k-2}$, which gives the result.

Lie groups 3/4

Theorem. For the orthogonal group O_N we have

$$law(\chi) \simeq g_1$$

with $N \to \infty$.

Proof. By using the Brauer easiness result, we have:

$$\begin{aligned} M_k(\chi) &= \dim(Fix(\pi^{\otimes k})) \\ &= \dim(span(T_\pi | \pi \in P_2(k))) \\ &\simeq |P_2(k)| \\ &= k!! \end{aligned}$$

Thus, the main character χ has the same moments as g_1.

Lie groups 4/4

The other classical Lie groups can be investigated by using the same method, and the asymptotic law of χ is as follows:

(1) For U_N we obtain the complex Gaussian law G_1. The proof is similar, by using $M_k(G_1) = |\mathcal{P}_2(k)|$.

(2) For the bistochastic groups $B_N \subset O_N$ and $C_N \subset U_N$ we obtain shifted versions of g_1, G_1.

(3) The symplectic group $Sp_N \subset U_N$ is not exactly easy, but rather "super-easy", and we obtain the Gaussian law g_1.

Truncation 1/4

Theorem. The Haar integration over $G\subset_\pi U_N$ is given by

$$\int_G g_{i_1j_1}^{s_1}\ldots g_{i_kj_k}^{s_k}dg=\sum_{\sigma,\tau\in D_k}\delta_\sigma(i)\delta_\tau(j)W_k(\sigma,\tau)$$

where D_k is a basis of $Fix(\pi^{\otimes k})$, $\delta_\sigma(i)=<\sigma,e_{i_1}\otimes\ldots\otimes e_{i_k}>$, and $W_k=G_k^{-1}$ is the inverse of $G_k(\sigma,\tau)=<\sigma,\tau>$.

Proof. The integrals in the statement form the projection P onto $Fix(\pi^{\otimes k})=span(D_k)$. Consider the following linear map:

$$E(x)=\sum_{\sigma\in D_k}<x,\sigma>\sigma$$

By linear algebra we have $P=WE$, where W is the inverse on $span(D_k)$ of the restriction of E, and this gives the result.

Truncation 2/4

Theorem. For an easy group $G_N \subset U_N$, coming from a category of partitions $D = (D(k,l))$, we have

$$\int_{G_N} g_{i_1 j_1}^{s_1} \ldots g_{i_k j_k}^{s_k} dg = \sum_{\sigma,\tau \in D(k)} \delta_\sigma(i)\delta_\tau(j) W_{kN}(\sigma,\tau)$$

where $D(k) = D(\emptyset, k)$, δ are usual Kronecker symbols, and $W_{kN} = G_{kN}^{-1}$ is the inverse of $G_{kN}(\sigma,\tau) = N^{|\sigma \vee \tau|}$.

Proof. The vectors associated to partitions are given by:

$$T_\sigma(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\sigma \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

Thus the Gram matrix and Kronecker symbols are those above.

Truncation 3/4

Application. We have the following computation,

$$\begin{aligned}
&\int_{G_N}(g_{11}+\ldots+g_{ss})^k\,dg\\
=&\sum_{i_1=1}^s\ldots\sum_{i_k=1}^s\int_{G_N}g_{i_1i_1}\ldots g_{i_ki_k}\,dg\\
=&\sum_{\sigma,\tau\in D(k)}W_{kN}(\sigma,\tau)\sum_{i_1=1}^s\ldots\sum_{i_k=1}^s\delta_\sigma(i)\delta_\tau(i)\\
=&\sum_{\sigma,\tau\in D(k)}W_{kN}(\sigma,\tau)G_{ks}(\tau,\sigma)\\
=&\ Tr(W_{kN}G_{ks})
\end{aligned}$$

and the $s=[tN]\to\infty$ asymptotics can be worked out.

Truncation 4/4

Theorem. The truncated characters χ_t for the main unitary and reflection groups are as follows, in the $N\to\infty$ limit,

$$\begin{matrix} O_N & \longrightarrow & U_N \\ \uparrow & & \uparrow \\ H_N & \longrightarrow & K_N \end{matrix} \qquad \sim \qquad \begin{matrix} g_t & \cdots\cdots & G_t \\ \vdots & & \vdots \\ b_t & \cdots\cdots & B_t \end{matrix}$$

and we have independence results as well, with $N\to\infty$.

Proof. In the discrete case, this is something that we already know. In general, this follows by using the above results.