A NOTE ON FREE QUANTUM GROUPS

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Abstract. We study the free complexification operation for compact quantum groups, $G \to G^c$. We prove that, with suitable definitions, this induces a one-to-one correspondence between free orthogonal quantum groups of infinite level, and free unitary quantum groups satisfying $G = G^c$.

Introduction

In this paper we present some advances on the notion of free quantum group, introduced in [3]. We first discuss in detail a result mentioned there, namely that the free complexification operation $G \to G^c$ studied in [2] produces free unitary quantum groups out of free orthogonal ones. Then we work out the injectivity and surjectivity properties of $G \to G^c$, and this leads to the correspondence announced in the abstract. This correspondence should be regarded as being a first general ingredient for the classification of free quantum groups.

We include in our study a number of general facts regarding the operation $G \to G^c$, by improving some previous work in [2]. The point is that now we can use general diagrammatic techniques from [4], new examples, and the notion of free quantum group [3], none of them available at the time of writing [2].

The paper is organized as follows: 1 contains some basic facts about the operation $G \to G^c$, and in 2-5 we discuss the applications to free quantum groups.

1. Free complexification

A fundamental result of Voiculescu [6] states if $(s_1, \ldots, s_n)$ is a semicircular system, and z is a Haar unitary free from it, then $(zs_1, \ldots, zs_n)$ is a circular system. This makes appear the notion of free multiplication by a Haar unitary, $a \to za$, that we call here free complexification. This operation has been intensively studied since then. See Nica and Speicher [5].

This operation appears as well in the context of Wang's free quantum groups [7], [8]. The main result in [1] is that the universal free biunitary matrix is the free complexification of the free orthogonal matrix. In other words, the passage

2000 *Mathematics Subject Classification.* 16W30.

Key words and phrases. Free quantum group.

Work supported by the CNRS and by the Fields Institute.

$O_n^+ \to U_n^+$ is nothing but a free complexification: $U_n^+ = O_n^{+c}$. Moreover, some generalizations of this fact are obtained, in an abstract setting, in [2].

In this section we discuss the basic properties of $A \to \tilde{A}$, the functional analytic version of $G \to G^c$. We use an adaptation of Woronowicz's axioms in [9].

Definition 1.1. *A finitely generated Hopf algebra is a pair (A, u), where A is a C^*-algebra and $u \in M_n(A)$ is a unitary whose entries generate A, such that*

$$\begin{aligned}\Delta(u_{ij}) &= \sum u_{ik} \otimes u_{kj} \\ \varepsilon(u_{ij}) &= \delta_{ij} \\ S(u_{ij}) &= u_{ji}^*\end{aligned}$$

define morphisms of C^*-algebras (called comultiplication, counit and antipode).

In other words, given (A, u), the morphisms Δ, ε, S can exist or not. If they exist, they are uniquely determined, and we say that we have a Hopf algebra.

The basic examples are as follows:

1. The algebra of functions $A = C(G)$, with the matrix $u = (u_{ij})$ given by $g = (u_{ij}(g))$, where $G \subset U_n$ is a compact group.
2. The group algebra $A = C^*(\Gamma)$, with the matrix $u = \text{diag}(g_1, \ldots, g_n)$, where $\Gamma =< g_1, \ldots, g_n >$ is a finitely generated group.

Let $\mathbb{T}$ be the unit circle, and let $z : \mathbb{T} \to \mathbb{C}$ be the identity function, $z(x) = x$. Observe that $(C(\mathbb{T}), z)$ is a finitely generated Hopf algebra, corresponding to the compact group $\mathbb{T} \subset U_1$, or, via the Fourier transform, to the group $\mathbb{Z} =< 1 >$.

Definition 1.2. *Associated to (A, u) is the pair $(\tilde{A}, \tilde{u})$, where $\tilde{A} \subset C(\mathbb{T}) * A$ is the C^*-algebra generated by the entries of the matrix $\tilde{u} = zu$.*

It follows from the general results of Wang in [7] that $(\tilde{A}, \tilde{u})$ is indeed a finitely generated Hopf algebra. Moreover, $\tilde{u}$ is the free complexification of u in the free probabilistic sense, i.e. with respect to the Haar functional. See [2].

A morphism between two finitely generated Hopf algebras $f : (A, u) \to (B, v)$ is by definition a morphism of $*$-algebras $A_s \to B_s$ mapping $u_{ij} \to v_{ij}$, where $A_s \subset A$ and $B_s \subset B$ are the dense $*$-subalgebras generated by the elements u_{ij}, respectively v_{ij}. Observe that in order for a such a morphism to exist, u, v must have the same size, and that if such a morphism exists, it is unique. See [2].

Proposition 1.3. *The operation $A \to \tilde{A}$ has the following properties:*

1. *We have a morphism $(\tilde{A}, \tilde{u}) \to (A, u)$.*
2. *A morphism $(A, u) \to (B, v)$ produces a morphism $(\tilde{A}, \tilde{u}) \to (\tilde{B}, \tilde{v})$.*
3. *We have an isomorphism $(\tilde{\tilde{A}}, \tilde{\tilde{u}}) = (\tilde{A}, \tilde{u})$.*

Proof. All the assertions are clear from definitions, see [2] for details. ☐

Theorem 1.4. *If $\Gamma =< g_1, \ldots, g_n >$ is a finitely generated group then $\tilde{C}^*(\Gamma) \simeq C^*(\mathbb{Z} * \Lambda)$, where $\Lambda =< g_i^{-1} g_j \mid i, j = 1, \ldots, n >$.*

Proof. By using the Fourier transform isomorphism $C(\mathbb{T}) \simeq C^*(\mathbb{Z})$ we obtain $\tilde{C}^*(\Gamma) = C^*(\tilde{\Gamma})$, with $\tilde{\Gamma} \subset \mathbb{Z} * \Gamma$. Then, a careful examination of generators gives the isomorphism $\tilde{\Gamma} \simeq \mathbb{Z} * \Lambda$. See [2] for details. □

At the dual level, we have the following question: what is the compact quantum group G^c defined by $C(G^c) = \tilde{C}(G)$? There is no simple answer to this question, unless in the abelian case, where we have the following result.

Theorem 1.5. *If $G \subset U_n$ is a compact abelian group then $\tilde{C}(G) = C^*(\mathbb{Z} * \widehat{L})$, where L is the image of G in the projective unitary group PU_n.*

Proof. The embedding $G \subset U_n$, viewed as a representation, must come from a generating system $\widehat{G} =< g_1, \ldots, g_n >$. It routine to check that the subgroup $\Lambda \subset \widehat{G}$ constructed in Theorem 1.4 is the dual of L, and this gives the result. □

2. Free quantum groups

Consider the groups $S_n \subset O_n \subset U_n$, with the elements of S_n viewed as permutation matrices. Consider also the following subgroups of U_n:

(1) $S_n' = \mathbb{Z}_2 \times S_n$, the permutation matrices multiplied by ± 1.
(2) $H_n = \mathbb{Z}_2 \wr S_n$, the permutation matrices with $\pm$ coefficients.
(3) $P_n = \mathbb{T} \times S_n$, the permutation matrices multiplied by scalars in $\mathbb{T}$.
(4) $K_n = \mathbb{T} \wr S_n$, the permutation matrices with coefficients in $\mathbb{T}$.

Observe that H_n is the hyperoctahedral group. It is convenient to collect the above definitions into a single one, in the following way.

Definition 2.1. *We use the diagram of compact groups*

$$\begin{array}{ccccc} U_n & \supset & K_n & \supset & P_n \\ \cup & & \cup & & \cup \\ O_n & \supset & H_n & \supset & S_n^* \end{array}$$

where S^* denotes at the same time S and S'.

In what follows we describe the free analogues of these 7 groups. For this purpose, we recall that a square matrix $u \in M_n(A)$ is called:

(1) Orthogonal, if $u = \bar{u}$ and $u^t = u^{-1}$.
(2) Cubic, if it is orthogonal, and $ab = 0$ on rows and columns.
(3) Magic', if it is cubic, and the sum on rows and columns is the same.
(4) Magic, if it is cubic, formed of projections ($a^2 = a = a^*$).

(5) Biunitary, if both u and u^t are unitaries.
(6) Cubik, if it is biunitary, and $ab^* = a^*b = 0$ on rows and columns.
(7) Magik, if it is cubik, and the sum on rows and columns is the same.

Here the equalities of type $ab = 0$ refer to distinct entries on the same row, or on the same column. The notions (1, 2, 4, 5) are from [7, 3, 8, 7], and (3, 6, 7) are new. The terminology is of course temporary: we have only 7 examples of free quantum groups, so we don't know exactly what the names name.

Theorem 2.2. *$C(G_n)$ with $G = OHS^*UKP$ is the universal commutative C^*-algebra generated by the entries of a $n \times n$ orthogonal, cubic, magic*, biunitary, cubik, magik matrix.*

Proof. The case $G = OHSU$ is discussed in [7, 3, 8, 7], and the case $G = S'KP$ follows from it, by identifying the corresponding subgroups. □

We proceed with liberation: definitions will become theorems and vice versa.

Definition 2.3. *$A_g(n)$ with $g = ohs^*ukp$ is the universal C^*-algebra generated by the entries of a $n \times n$ orthogonal, cubic, magic*, biunitary, cubik, magik matrix.*

The $g = ohsu$ algebras are from [7, 3, 8, 7], and the $g = s'kp$ ones are new.

Theorem 2.4. *We have the diagram of Hopf algebras*

$$\begin{array}{ccccc} A_u(n) & \rightarrow & A_k(n) & \rightarrow & A_p(n) \\ \downarrow & & \downarrow & & \downarrow \\ A_o(n) & \rightarrow & A_h(n) & \rightarrow & A_{s^*}(n) \end{array}$$

where s^* denotes at the same time s and s'.

Proof. The morphisms in Definition 1.1 can be constructed by using the universal property of each of the algebras involved. For the algebras A_{ohsu} this is known from [7, 3, 8, 7], and for the algebras $A_{s'kp}$ the proof is similar. □

3. Diagrams

Let $F =< a, b >$ be the monoid of words on two letters a, b. For a given corepresentation u we let $u^a = u$, $u^b = \bar{u}$, then we define the tensor powers u^α with $\alpha \in F$ arbitrary, according to the rule $u^{\alpha\beta} = u^\alpha \otimes u^\beta$.

Definition 3.1. *Let (A, u) be a finitely generated Hopf algebra.*

(1) *CA is the collection of linear spaces $\{Hom(u^\alpha, u^\beta) | \alpha, \beta \in F\}$.*
(2) *In the case $u = \bar{u}$ we identify CA with $\{Hom(u^k, u^l) | k, l \in \mathbb{N}\}$.*

A morphism $(A,u)\to(B,v)$ produces inclusions $Hom(u^\alpha,u^\beta)\subset Hom(v^\alpha,v^\beta)$ for any $\alpha,\beta\in F$, so we have the following diagram:

$$\begin{array}{ccccc} CA_u(n) & \subset & CA_k(n) & \subset & CA_p(n)\\ \\ \cap & & \cap & & \cap\\ \\ CA_o(n) & \subset & CA_h(n) & \subset & CA_{s^*}(n)\end{array}$$

We recall that $CA_s(n)$ is the category of Temperley-Lieb diagrams. That is, $Hom(u^k,u^l)$ is isomorphic to the abstract vector space spanned by the diagrams between an upper row of $2k$ points, and a lower row of $2l$ points. See [3].

In order to distinguish between various meanings of the same diagram, we attach words to it. For instance $\Cap_{ab},\Cap_{ba}$ are respectively in $D_s(\emptyset,ab),D_s(\emptyset,ba)$.

Lemma 3.2. *The categories for $A_g(n)$ with $g=ohsukp$ are as follows:*

(1) $CA_o(n)=<\Cap>$.
(2) $CA_h(n)=<\Cap,|{}^{\cup}_{\cap}|>$.
(3) $CA_{s'}(n)=<\Cap,|{}^{\cup}_{\cap}|,\ {}^{\cup}_{\cap}>$.
(4) $CA_s(n)=<\Cap,|{}^{\cup}_{\cap}|,\cap>$.
(5) $CA_u(n)=<\Cap_{ab},\Cap_{ba}>$.
(6) $CA_k(n)=<\Cap_{ab},\Cap_{ba},|{}^{\cup}_{\cap}|^{ab}_{ab},|{}^{\cup}_{\cap}|^{ba}_{ba}>$.
(7) $CA_p(n)=<\Cap_{ab},\Cap_{ba},|{}^{\cup}_{\cap}|^{ab}_{ab},|{}^{\cup}_{\cap}|^{ba}_{ba},\ {}^{\cup a}_{\cap a},\ {}^{\cup b}_{\cap b}>$.

Proof. The case $g=ohs$ is discussed in [3], and the case $g=u$ is discussed in [4]. In the case $g=s'kp$ we can use the following formulae:

$$\begin{aligned} {}^{\cup}_{\cap} &= \sum_{ij} e_{ij}\\ |{}^{\cup}_{\cap}| &= \sum_i e_{ii}\otimes e_{ii}\end{aligned}$$

The commutation conditions $|{}^{\cup}_{\cap}|\in End(u\otimes\bar{u})$ and $|{}^{\cup}_{\cap}|\in End(\bar{u}\otimes u)$ correspond to the cubik condition, and the extra relations ${}^{\cup}_{\cap}\in End(u)$ and ${}^{\cup}_{\cap}\in End(\bar{u})$ correspond to the magik condition. Together with the fact that orthogonal plus magik means magic', this gives all the $g=s'kp$ assertions. □

We can color the diagrams in several ways: either by putting the sequence $xyyxxyyx\ldots$ on both rows of points, or by putting α,β on both rows, then by replacing $a\to xy, b\to yx$. We say that the diagram is colored if all the strings match, and half-colored, if there is an even number of unmatches.

Theorem 3.3. *For $g=ohs^*ukp$ we have $CA_g(n)=\mathrm{span}(D_g)$, where:*

(1) *$D_s(k,l)$ is the set of all diagrams between $2k$ points and $2l$ points.*
(2) *$D_{s'}(k,l)=D_s(k,l)$ for $k-l$ even, and $D_{s'}(k,l)=\emptyset$ for $k-l$ odd.*

(3) $D_h(k,l)$ *consists of diagrams which are colorable* $xyyxxyyx\ldots$.
(4) $D_o(k,l)$ *is the image of* $D_s(k/2,l/2)$ *by the doubling map.*
(5) $D_p(\alpha,\beta)$ *consists of diagrams half-colorable* $a\to xy, b\to yx$.
(6) $D_k(\alpha,\beta)$ *consists of diagrams colorable* $a\to xy, b\to yx$.
(7) $D_u(\alpha,\beta)$ *consists of double diagrams, colorable* $a\to xy, b\to yx$.

Proof. This is clear from the above lemma, by composing diagrams. The case $g=ohsu$ is discussed in [3, 4], and the case $g=s'kp$ is similar. □

Theorem 3.4. *We have the following isomorphisms:*
(1) $A_u(n)=\tilde{A}_o(n)$.
(2) $A_h(n)=\tilde{A}_k(n)$.
(3) $A_p(n)=\tilde{A}_{s^*}(n)$.

Proof. It follows from definitions that we have arrows from left to right. Now since by Theorem 3.3 the spaces $End(u\otimes\bar{u}\otimes u\otimes\ldots)$ are the same at right and at left, Theorem 5.1 in [2] applies, and gives the arrows from right to left. □

Observe that the assertion (1), known since [1], is nothing but the isomorphism $U_n^+=O_n^{+c}$ mentioned in the beginning of the first section.

4. Freeness, level, doubling

We use the notion of free Hopf algebra, introduced in [3]. Recall that a morphism $(A,u)\to(B,v)$ induces inclusions $Hom(u^\alpha,u^\beta)\subset Hom(v^\alpha,v^\beta)$.

Definition 4.1. *A finitely generated Hopf algebra* (A,u) *is called free if:*
(1) *The canonical map* $A_u(n)\to A_s(n)$ *factorizes through* A.
(2) *The spaces* $Hom(u^\alpha,u^\beta)\subset \mathrm{span}(D_s(\alpha,\beta))$ *are spanned by diagrams.*

It follows from Theorem 3.3 that the algebras A_{ohs^*ukp} are free.

In the orthogonal case $u=\bar{u}$ we say that A is free orthogonal, and in the general case, we also say that A is free unitary.

Theorem 4.2. *If* A *is free orthogonal then* $\tilde{A}$ *is free unitary.*

Proof. It is shown in [2] that the tensor category of $\tilde{A}$ is generated by the tensor category of A, embedded via alternating words, and this gives the result. □

Definition 4.3. *The level of a free orthogonal Hopf algebra* (A,u) *is the smallest number* $l\in\{0,1,\ldots,\infty\}$ *such that* $1\in u^{\otimes 2l+1}$.

As the level of examples, for $A_s(n)$ we have $l=0$, and for $A_{ohs'}(n)$ we have $l=\infty$. This follows indeed from Theorem 3.3.

Theorem 4.4. *If* $l<\infty$ *then* $\tilde{A}=C(\mathbb{T})*A$.

Proof. Let $< r >$ be the algebra generated by the coefficients of r. From $1 \in < u >$ we get $z \in < zu_{ij} >$, hence $< zu_{ij} > = < z, u_{ij} >$, and we are done. □

Corollary 4.5. *$A_p(n) = C(\mathbb{T}) * A_s(n)$.*

Proof. For $A_s(n)$ we have $1 \in u$, hence $l = 0$, and Theorem 4.4 applies. □

We can define a "doubling" operation $A \to A_2$ for free orthogonal algebras, by using Tannakian duality, in the following way: the spaces $Hom(u^k, u^l)$ with $k - l$ even remain by definition the same, and those with $k - l$ odd become by definition empty. The interest in this operation is that A_2 has infinite level.

At the level of examples, the doublings are $A_{ohs^*}(n) \to A_{ohs'}(n)$.

Proposition 4.6. *For a free orthogonal algebra A, the following are equivalent:*

1. *A has infinite level.*
2. *The canonical map $A_2 \to A$ is an isomorphism.*
3. *The quotient map $A \to A_s(n)$ factorizes through $A_{s'}(n)$.*

Proof. The equivalence between (1) and (2) is clear from definitions, and the equivalence with (3) follows from Tannakian duality. □

5. The main result

We know from Theorem 3.4 that the two rows of the diagram formed by the algebras A_{ohs^*ukp} are related by the operation $A \to \tilde{A}$. Moreover, the results in the previous section suggest that the correct choice in the lower row is $s^* = s'$. The following general result shows that this is indeed the case.

Theorem 5.1. *The operation $A \to \tilde{A}$ induces a one-to-one correspondence between the following objects:*

1. *Free orthogonal algebras of infinite level.*
2. *Free unitary algebras satisfying $A = \tilde{A}$.*

Proof. We use the notations $\gamma_k = abab\ldots$ and $\delta_k = baba\ldots$ (k terms each).

We know from Theorem 4.2 that the operation $A \to \tilde{A}$ is well-defined, between the algebras in the statement. Moreover, since by Tannakian duality an orthogonal algebra of infinite level is determined by the spaces $Hom(u^k, u^l)$ with $k - l$ even, we get that $A \to \tilde{A}$ is injective, because these spaces are:

$$Hom(u^k, u^l) = Hom((zu)^{\gamma_k}, (zu)^{\gamma_l})$$

It remains to prove surjectivity. So, let A be free unitary satisfying $A = \tilde{A}$. We have $CA = \text{span}(D)$ for certain sets of diagrams $D(\alpha, \beta) \subset D_s(\alpha, \beta)$, so we can define a collection of sets $D_2(k, l) \subset D_s(k, l)$ in the following way:

1. For $k - l$ even we let $D_2(k, l) = D(\gamma_k, \gamma_l)$.
2. For $k - l$ odd we let $D_2(k, l) = \emptyset$.

It follows from definitions that $C_2 = \mathrm{span}(D_2)$ is a category, with duality and involution. We claim that C_2 is stable under $\otimes$. Indeed, for k, l even we have:

$$\begin{aligned} D_2(k,l) \otimes D_2(p,q) &= D(\gamma_k, \gamma_l) \otimes D(\gamma_p, \gamma_q) \\ &\subset D(\gamma_k\gamma_p, \gamma_l\gamma_q) \\ &= D(\gamma_{k+p}, \gamma_{l+q}) \\ &= D_2(k+p, l+q) \end{aligned}$$

For k, l odd and p, q even, we can use the canonical antilinear isomorphisms $Hom(u^{\gamma_K}, u^{\gamma_L}) \simeq Hom(u^{\delta_K}, u^{\delta_L})$, with K, L odd. At the level of diagrams we get equalities $D(\gamma_k, \gamma_L) = D(\delta_K, \delta_L)$, that can be used in the following way:

$$\begin{aligned} D_2(k,l) \otimes D_2(p,q) &= D(\delta_k, \delta_l) \otimes D(\gamma_p, \gamma_q) \\ &\subset D(\delta_k\gamma_p, \delta_l\gamma_q) \\ &= D(\delta_{k+p}, \delta_{l+q}) \\ &= D_2(k+p, l+q) \end{aligned}$$

Finally, for k, l odd and p, q odd, we can proceed as follows:

$$\begin{aligned} D_2(k,l) \otimes D_2(p,q) &= D(\gamma_k, \gamma_l) \otimes D(\delta_p, \delta_q) \\ &\subset D(\gamma_k\delta_p, \gamma_l\delta_q) \\ &= D(\gamma_{k+p}, \gamma_{l+q}) \\ &= D_2(k+p, l+q) \end{aligned}$$

Thus we have a Tannakian category, and by Woronowicz's results in [10] we get an algebra A_2. This algebra is free orthogonal, of infinite level. Moreover, the spaces $End(u \otimes \bar{u} \otimes u \otimes \ldots)$ being the same for A and A_2, Theorem 5.1 in [2] applies, and gives $\tilde{A}_2 = \tilde{A}$. Now since we have $A = \tilde{A}$, we are done. □

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