Quantum isometries and reflections

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"Introduction to noncommutative geometry", 2/6

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Plan

1. There is no free $\mathbb{R}^N$, or free $\mathbb{C}^N$.

2. We want to axiomatize the quadruplets (S, T, U, K).

3. So far we have pairs (S, T), real/complex, classical/free.

$\implies$ We will complete with pairs (U, K), and with arrows:

$$\begin{array}{ccc} S & \longrightarrow & T \\ \downarrow & & \downarrow \\ U & \longrightarrow & K \end{array}$$

$\implies$ Later: missing 8 arrows, main 4 cases + axiomatization.

Quantum groups 1/6

Definition. A Woronowicz algebra is a C^*-algebra A, given with a biunitary $u \in M_N(A)$ whose entries generate A, such that:

– $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ defines a morphism $\Delta : A \to A \otimes A$.

– $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : A \to \mathbb{C}$.

– $S(u_{ij}) = u_{ji}^*$ defines a morphism $S : A \to A^{opp}$.

Notation. Given a Woronowicz algebra A we write

$$A = C(G) = C^*(\Gamma)$$

and call G, Γ compact and discrete quantum groups.

Examples. $A = C(G)$, with $u_{ij}(g) = g_{ij}$, for $G \subset U_N$. Also $A = C^*(\Gamma)$, with $u = diag(g_i)$, for $\Gamma =< g_1, \ldots, g_N >$.

Quantum groups 2/6

Theorem. The comultiplication Δ, counit ε and antipode S satisfy the following conditions,

(1) Coassociativity: $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$.

(2) Counitality: $(id \otimes \varepsilon)\Delta = (\varepsilon \otimes id)\Delta = id$.

(3) Coinversality: $m(id \otimes S)\Delta = m(S \otimes id)\Delta = \varepsilon(.)1$.

on the dense $*$-subalgebra $\mathcal{A} \subset A$ generated by the variables u_{ij}.

Proof. Clear on coordinates, and so on the $*$-algebra $\mathcal{A}$.

Remark. The square of the antipode is the identity, $S^2 = id$.

Quantum groups 3/6

Theorem. Any Woronowicz algebra has a Haar integration,

$$\left(\int_G \otimes id\right)\Delta = \left(id \otimes \int_G\right)\Delta = \int_G (.)1$$

constructed by starting with $\varphi \in A^*$ unital positive, and setting

$$\int_G = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \varphi^{*k}$$

where $\phi * \psi = (\phi \otimes \psi)\Delta$. Moreover, for any corepresentation v,

$$\left(id \otimes \int_G\right) v = P$$

where P is the projection onto $Fix(v) = \{\xi \in \mathbb{C}^n | v\xi = \xi\}$.

Quantum groups 4/6

Definition. A corepresentation of a Woronowicz algebra A is a biunitary matrix $v \in M_n(\mathcal{A})$ satisfying

$$\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj} \quad , \quad \varepsilon(v_{ij}) = \delta_{ij} \quad , \quad S(v_{ij}) = v_{ji}^*$$

where $\mathcal{A} \subset A$ is the dense $*$-subalgebra of "smooth elements".

Theorem. The following Peter-Weyl type results hold:

(1) Any corepresentation decomposes as a sum of irreducibles.

(2) The irreducibles appear inside $u^{\otimes k}$, with $k =$ colored integer.

(3) We have $\mathcal{A} = \oplus_{r \in Irr(A)} B(H_r)$, $*$-coalgebra isomorphism, $\perp$.

(4) The characters of irreps form an orthonormal basis of $\mathcal{A}_{central}$.

Quantum groups 5/6

Definition. The Tannakian category of a Woronowicz algebra (A, u) is the following collection $C = (C(k, l))$ of vector spaces:

$$C(k, l) = Hom(u^{\otimes k}, u^{\otimes l})$$

Definition. The Woronowicz algebra associated to a Tannakian category $C = (C(k, l))$ is constructed as follows:

$$A = C^* \left((u_{ij})_{i,j=1\ldots N} \middle| T \in Hom(u^{\otimes k}, u^{\otimes l}), \forall T \in C(k, l) \right)$$

Theorem. These operations produce a bijection $A \leftrightarrow C$, between Woronowicz algebras, and Tannakian categories.

Quantum groups 6/6

Definition. A compact quantum group G is called easy when

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

for a certain category of partitions $D \subset P$, where

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with $\delta_\pi \in \{0, 1\}$ depending on whether the indices fit or not.

Examples. The Brauer theorem says that O_N, U_N are easy, coming from $P_2, \mathcal{P}_2$: the pairings, and the matching pairings.

Unitaries 1/2

Theorem. We have quantum groups defined via

$$C(O_N^+) = C^*\left((u_{ij})_{i,j=1\ldots N}\middle| u=\bar{u}, u^t=u^{-1}\right)$$

$$C(U_N^+) = C^*\left((u_{ij})_{i,j=1\ldots N}\middle| u^*=u^{-1}, u^t=\bar{u}^{-1}\right)$$

called free orthogonal, and free unitary quantum groups.

Proof. If u is biunitary/orthogonal, so are the matrices

$$(u^\Delta)_{ij}=\sum_k u_{ik}\otimes u_{kj} \quad , \quad (u^\varepsilon)_{ij}=\delta_{ij} \quad , \quad (u^S)_{ij}=u_{ji}^*$$

and so we can construct Δ,ε,S, by universality.

Unitaries 2/2

Theorem. The basic unitary quantum groups, namely

$$\begin{matrix} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{matrix}$$

are all easy, coming from the following categories of pairings:

$$\begin{matrix} NC_2 & \longleftarrow & \mathcal{NC}_2 \\ \downarrow & & \downarrow \\ P_2 & \longleftarrow & \mathcal{P}_2 \end{matrix}$$

Proof. This comes from Tannaka (classical case: Brauer).

Permutations 1/2

The coordinates of $S_N \subset O_N$, permutation matrices, are:

$$u_{ij} = \chi\left(\sigma \in S_N \Big| \sigma(j) = i\right)$$

A quick study of u suggests the following definition:

Definition. The quantum permutation group S_N^+ is defined via

$$C(S_N^+) = C^*\left((u_{ij}) \Big| u = N \times N \text{ magic}\right)$$

where "magic" = made of projections, sum 1 on rows/columns.

[the verification of the CQG axioms is routine: Wang 98]

Permutations 2/2

Theorem. We have the following results:

(1) The inclusion $S_N \subset S_N^+$ is an isomorphism at $N = 2, 3$, but not at $N \geq 4$, where S_N^+ is not classical, nor finite.

(2) At $N = 4$ we have $S_4^+ = SO_3^{-1}$. At $N \geq 4$ we have $S_N^+ \sim SO_3$, same fusion rules. At $N \geq 5$ the dual $\widehat{S_N^+}$ is not amenable.

(3) The quantum groups S_N, S_N^+ are easy, coming from P, NC. The main characters are Poisson/free Poisson, with $N \to \infty$.

Proof. Here (1) is elementary, using $u = diag(v, w)$ at $N = 4$, (2) comes from algebra and Tannaka, and (3) from Tannaka.

Reflections 1/2

Theorem. We have quantum groups defined via

$$C(H_N^+) = C^*\left((u_{ij})_{i,j=1\ldots N}\Big| u_{ij}=u_{ij}^*,\ (u_{ij}^2)=\mathrm{magic}\right)$$
$$C(K_N^+) = C^*\left((u_{ij})_{i,j=1\ldots N}\Big| [u_{ij},u_{ij}^*]=0,\ (u_{ij}u_{ij}^*)=\mathrm{magic}\right)$$

called quantum hyperoctahedral, and quantum reflection groups.

Proof. If u satisfies the above relations, then so do the matrices $u^\Delta, u^\varepsilon, u^S$. Thus we can construct Δ, ε, S by universality.

Remark. We can alternatively set $H_N^+ = \mathbb{Z}_2 \wr_* S_N^+$, $K_N^+ = \mathbb{T} \wr_* S_N^+$, in analogy with $H_N = \mathbb{Z}_2 \wr S_N$, $K_N = \mathbb{T} \wr S_N$.

Reflections 2/2

<u>Theorem</u>. The basic quantum reflection groups, namely

$$\begin{array}{ccc}
H_N^+ & \longrightarrow & K_N^+\\
\uparrow & & \uparrow\\
H_N & \longrightarrow & K_N
\end{array}$$

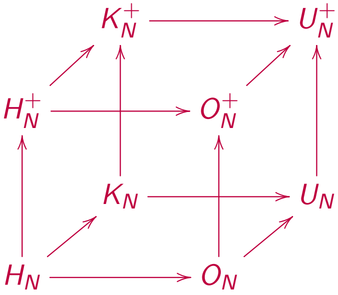
are all easy, coming from the following categories of partitions:

$$\begin{array}{ccc}
NC_{even} & \longleftarrow & \mathcal{NC}_{even}\\
\downarrow & & \downarrow\\
P_{even} & \longleftarrow & \mathcal{P}_{even}
\end{array}$$

<u>Proof</u>. This comes from Tannaka, using the results for U_N, U_N^+.

The cube

Theorem. The basic quantum unitary and reflection groups are all easy, and form a cubic diagram, as follows:



The upper objects appear as liberations of the lower objects. Also, any face $P \subset Q, R \subset S$ has the property $P = Q \cap R$.

Conclusion

We have basic quadruplets (S, T, U, K) as follows,

$$\begin{array}{ccc} S^{N-1}_{\mathbb{R},+} & - & T_N^+ \\ | & & | \\ O_N^+ & - & H_N^+ \end{array} \qquad \begin{array}{ccc} S^{N-1}_{\mathbb{C},+} & - & \mathbb{T}_N^+ \\ | & & | \\ U_N^+ & - & K_N^+ \end{array}$$

called free real and free complex, as well as

$$\begin{array}{ccc} S^{N-1}_{\mathbb{R}} & - & T_N \\ | & & | \\ O_N & - & H_N \end{array} \qquad \begin{array}{ccc} S^{N-1}_{\mathbb{C}} & - & \mathbb{T}_N \\ | & & | \\ U_N & - & K_N \end{array}$$

called classical real and complex $\implies$ construct arrows.

Plan

In the 4 main cases, real/complex and classical/free, we have arrows as follows, given by $T = S \cap \mathbb{T}_N^+$ and $K = U \cap K_N^+$:

$$\begin{array}{ccc} S & \longrightarrow & T \\ \downarrow & & \downarrow \\ U & \longrightarrow & K \end{array}$$

We will complete with the dotted arrows, $S \to U$ and $T \to K$, obtained via QISO constructions. Other arrows for later.

Affine isometries

Theorem. Given an algebraic manifold $X \subset S^{N-1}_{\mathbb{C}}$, the formula

$$G(X) = \left\{ U \in U_N \Big| U(X) = X \right\}$$

defines a compact group of unitary matrices (or isometries), called affine isometry group of X. As basic examples here:

(1) For $S^{N-1}_{\mathbb{R}}, S^{N-1}_{\mathbb{C}}$ we obtain the groups O_N, U_N.

(2) For $T_N, \mathbb{T}_N$ we obtain the groups H_N, K_N.

Proof. All this is clear from definitions.

Quantum isometries

Theorem. Given an algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$, the category of closed subgroups $G \subset U_N^+$ acting affinely on X, in the sense that

$$\Phi(x_i) = \sum_a u_{ia} \otimes x_a$$

defines a morphism of C^*-algebras

$$\Phi : C(X) \to C(G) \otimes C(X)$$

has a universal object $G^+(X)$, called affine QISO group of X.

Proof. In order for Φ to exist, the variables $z_i = \sum_a u_{ia} \otimes x_a$ must satisfy the polynomial relations defining X. Thus, we can construct $G^+(X)$ by starting with U_N^+, and imposing these relations.

Quantum rotations

Theorem. We have the following QISO computations,

$$\begin{matrix} S^{N-1}_{\mathbb{R},+} & \longrightarrow & S^{N-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{C}} \end{matrix} \quad \to \quad \begin{matrix} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{matrix}$$

modulo identifying, as usual, the various C^*-algebraic completions.

Proof. This is clear in the free cases. In the classical cases the trick of Bhowmick-Goswami ("apply S, relabel, process") applies.

Schur-Weyl twists

Definition. The Schur-Weyl twist of an easy quantum group

$$H_N \subset G \subset U_N^+$$

with corresponding category of partitions $\mathcal{NC}_2 \subset D \subset P_{even}$ is the quantum group $H_N \subset \bar{G} \subset U_N^+$ given by the formula

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(\bar{T}_\pi \Big| \pi \in D(k,l)\right)$$

where the twisted implementation of the partitions is given by

$$\bar{T}_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \bar{\delta}_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

where $\bar{\delta}_\pi \in \{-1, 0, 1\}$ is $\bar{\delta}_\pi = \varepsilon(\tau)$ if $\tau \geq \pi$, and $\bar{\delta}_\pi = 0$ otherwise, with $\tau = \ker\binom{i}{j}$, and $\varepsilon : P_{even} \to \{\pm 1\}$ being the signature.

Isometries of tori

Theorem. We have the following QISO computations,

$$\begin{matrix} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{matrix} \qquad \to \qquad \begin{matrix} H_N^+ & \longrightarrow & K_N^+ \\ \vdots & & \vdots \\ \bar{O}_N & \longrightarrow & \bar{U}_N \end{matrix}$$

where $\bar{O}_N, \bar{U}_N$ are the Schur-Weyl twists of O_N, U_N.

Proof. This is similar to the computation for the spheres, with the BG trick ("apply S, relabel, process") applying as well.

Reflections of tori

Theorem. We have correspondences as follows,

$$\begin{matrix} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{matrix} \qquad \to \qquad \begin{matrix} H_N^+ & \longrightarrow & K_N^+ \\ \uparrow & & \uparrow \\ H_N & \longrightarrow & K_N \end{matrix}$$

obtained via the operation $T \to G^+(T) \cap K_N^+$.

Proof. Follows from the previous result, by intersecting with K_N^+, because "commutation + anticommutation $\implies$ vanishing".

Conclusion

We have quadruplets (S, T, U, K) as follows,

$$\begin{array}{ccc} S^{N-1}_{\mathbb{R}} & \longrightarrow & T_N \\ \downarrow & & \downarrow \\ O_N & \longrightarrow & H_N \end{array} \qquad\qquad \begin{array}{ccc} S^{N-1}_{\mathbb{C}} & \longrightarrow & \mathbb{T}_N \\ \downarrow & & \downarrow \\ U_N & \longrightarrow & K_N \end{array}$$

coming from $\mathbb{R}^N$, $\mathbb{C}^N$, as well as free analogues of them,

$$\begin{array}{ccc} S^{N-1}_{\mathbb{R},+} & \longrightarrow & T_N^+ \\ \downarrow & & \downarrow \\ O_N^+ & \longrightarrow & H_N^+ \end{array} \qquad\qquad \begin{array}{ccc} S^{N-1}_{\mathbb{C},+} & \longrightarrow & \mathbb{T}_N^+ \\ \downarrow & & \downarrow \\ U_N^+ & \longrightarrow & K_N^+ \end{array}$$

which can be thought of as coming from $\mathbb{R}^N_+$, $\mathbb{C}^N_+$.

$\Longrightarrow$ Construct the missing arrows, axiomatize.