# Noncommutative algebraic geometry

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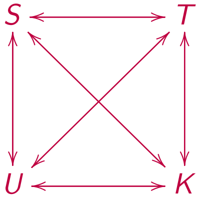
"Introduction to noncommutative geometry", 3/6

07/20

# Plan

We want to axiomatize the quadruplets $(S, T, U, K)$ consisting of a noncommutative sphere, torus, unitary group and reflection group, with a full set of correspondences between them, as follows:




We have 4 main examples in mind, namely classical and free, real and complex. We will finish the work there, by constructing the missing correspondences, then we will do the axiomatization.

$\Longrightarrow$ More examples, classification, development: later.

# Details

We call noncommutative sphere and torus, and quantum unitary and reflection group, the intermediate objects as follows,

$$S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{C},+}$$

$$T_N \subset T \subset \mathbb{T}_N^+$$

$$O_N \subset U \subset U_N^+$$

$$H_N \subset K \subset K_N^+$$

with $S$ being an algebraic manifold, and $T, U, K$ being compact quantum groups. Note that $T$ must be a group dual.

$\implies$ We must axiomatize the correspondences between them.

# Comments

(1) It is a good idea to mix the classical and free cases?

Yes, because there are interesting geometries between classical and free, such as the half-classical one ($abc = cba$).

(2) Is it a good idea to mix the real and complex cases?

Yes, for instance because we have $P^{N-1}_{\mathbb{R},+} = P^{N-1}_{\mathbb{C},+}$. That is, "the free projective geometry is scalarless".

$\Longrightarrow$ More details and comments later, on all this.

# Diagonal tori 1/4

We have quadruplets $(S,T,U,K)$ and arrows as follows,

$$\begin{array}{ccc} S^{N-1}_\mathbb{R} & \longrightarrow & T_N \\ \downarrow & & \downarrow \\ O_N & \longrightarrow & H_N \end{array} \qquad\qquad \begin{array}{ccc} S^{N-1}_\mathbb{C} & \longrightarrow & \mathbb{T}_N \\ \downarrow & & \downarrow \\ U_N & \longrightarrow & K_N \end{array}$$

coming from $\mathbb{R}^N$, $\mathbb{C}^N$, as well as free analogues of them,

$$\begin{array}{ccc} S^{N-1}_{\mathbb{R},+} & \longrightarrow & T_N^+ \\ \downarrow & & \downarrow \\ O_N^+ & \longrightarrow & H_N^+ \end{array} \qquad\qquad \begin{array}{ccc} S^{N-1}_{\mathbb{C},+} & \longrightarrow & \mathbb{T}_N^+ \\ \downarrow & & \downarrow \\ U_N^+ & \longrightarrow & K_N^+ \end{array}$$

which can be thought of as coming from $\mathbb{R}^N_+$, $\mathbb{C}^N_+$.

# Diagonal tori 2/4

Theorem. Given a closed subgroup $G \subset U_N^+$, its diagonal torus is the closed subgroup $T \subset G$ constructed as follows:

$$C(T) = C(G) \Big/ \left\langle u_{ij} = 0 \Big| \forall i \neq j \right\rangle$$

We have then $T = \widehat{\Lambda}$, where $\Lambda = < g_1, \ldots, g_N >$ is the discrete group generated by $g_i = u_{ii}$, which are unitaries inside $C(T)$.

Proof. Since $u$ is unitary, $g_i = u_{ii}$ are unitaries inside $C(T)$. From $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ we obtain, inside the quotient:

$$\Delta(g_i) = g_i \otimes g_i$$

Thus we have $C(T) = C^*(\Lambda)$, and so $T = \widehat{\Lambda}$, as claimed.

# Diagonal tori 3/4

Theorem. The diagonal tori of the unitary quantum groups are

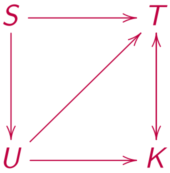
$$\begin{matrix} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{matrix} \qquad \to \qquad \begin{matrix} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{matrix}$$

and for the reflection subgroups, we obtain the same tori.

Proof. The diagonal torus is given by $T = U \cap \mathbb{T}_N^+$, intersection computed inside $U_N^+$, and this gives all the results.

# Diagonal tori 4/4

Theorem. We have correspondences as follows, for the 4 basic examples of quadruplets $(S, T, U, K)$,




which are given by the following formulae:

- $U = G^+(S)$
- $K = U \cap K_N^+ = G^+(T) \cap K_N^+$
- $T = S \cap \mathbb{T}_N^+ = U \cap \mathbb{T}_N^+ = K \cap \mathbb{T}_N^+$

Proof. This is a summary of what we have so far.

# Liberation 1/4

Definition. Given $G \subset U_N^+$, let $T \subset K \subset G$ be its diagonal torus, and its reflection subgroup. The inclusion $G_{class} \subset G$ is called:

(1) A soft liberation, when $G =< G_{class}, K >$.

(2) A hard liberation, when $G =< G_{class}, T >$.

Remark. We have the following intersection diagram:

$$\begin{array}{ccccc} T & \longrightarrow & K & \longrightarrow & G \\ \uparrow & & \uparrow & & \uparrow \\ T_{class} & \longrightarrow & K_{class} & \longrightarrow & G_{class} \end{array}$$

Soft liberation means generation for the square on the right.
Hard liberation means generation for the whole rectangle.

# Liberation 2/4

Theorem. The following happen:

(1) $O_N^+, U_N^+$ appear as soft liberations of $O_N, U_N$.

(2) $O_N^+, U_N^+$ appear as well as hard liberations of $O_N, U_N$.

(3) $H_N^+, K_N^+$ appear as soft liberations of $H_N, K_N$.

(4) $H_N^+, K_N^+$ do not appear as hard liberations of $H_N, K_N$.

Proof. (1) follows from (2), and (2) follows by recurrence from

$$O_N^+ =< O_N, O_{N-1}^+ >$$

which itself follows by recurrence (Chirvasitu). (3) is trivial, and (4) follows from the fact that "hard liberation stops at $H_N^{[\infty]}, K_N^{[\infty]}$".

# Liberation 3/4

Theorem. We have the following formulae:

(1) $O_N =< O_N, T_N >$.

(2) $U_N =< O_N, \mathbb{T}_N >$.

(3) $O_N^+ =< O_N, T_N^+ >$.

(4) $U_N^+ =< O_N, \mathbb{T}_N^+ >$.

Proof. (1) is trivial. (2) follows from $\mathbb{T}O_N \subset U_N$ maximal. (3) is the hard liberation property of $O_N$, discussed above, and which is highly non-trivial. As for (4), this follows from (3).

# Liberation 4/4

Theorem. We have correspondences as follows, obtained by adding the "soft" and "hard" generation formulae

$$U =< O_N, K >=< O_N, T >$$

to the various isometry and intersection formulae that we have,

$$\begin{array}{ccc} S & \longrightarrow & T \\ \downarrow & \nearrow\!\!\swarrow & \updownarrow \\ U & \longleftrightarrow & K \end{array}$$

which work for the basic quadruplets $(S, T, U, K)$.

# Spheres 1/4

Regarding $U \to S$, in the classical case the situation is very simple, because $S$ appears by rotating $x = (1, 0, \ldots, 0)$ by the elements of $U$. In fact, the spheres are homogeneous spaces, as follows:

$$S^{N-1} = U_N / U_{N-1}$$

In functional analytic terms, the correspondence $U \to S$ appears, at the level of algebras of functions, as follows:

$$C(S^{N-1}) \subset C(U_N) \quad , \quad x_i \to u_{i1}$$

Thus, we must check if this works in the free cases too.

# Spheres 2/4

Theorem. For the basic spheres, we have a diagram as follows,

$$\begin{array}{ccc} C(S) & \xrightarrow{\ \Phi\ } & C(U)\otimes C(S) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle id\otimes\pi} \\ C(U) & \xrightarrow{\ \Delta\ } & C(U)\otimes C(U) \end{array}$$

where $\Phi$ is the affine coaction map, and where $\pi(x_i) = u_{i1}$.

Proof. This diagram commutes indeed on the standard generators.

## Spheres 3/4

Theorem. We have a quotient map and an inclusion as follows,

$$U \to S_U \subset S$$

with $S_U$ being the first column space of $U$, given by

$$C(S_U) =< u_{i1} >\subset C(U)$$

at the level of the corresponding algebras of functions.

Proof. We have an inclusion and a quotient map as follows:

$$C(S) \to C(S_U) \subset C(U)$$

Thus, we obtain the result, by transposing.

$\implies$ We must prove that $S_U \subset S$ is an isomorphism.

# Spheres 4/4

In order to investigate the faithfulness of $S_U \subset S$, we will use the faithfulness properties of the integration over $S$.

Definition. We endow $C(S)$ with its integration functional

$$\int_S : C(S) \to C(U) \to \mathbb{C}$$

obtained by composing $x_i \to u_{i1}$ with the Haar integral of $U$.

In the real and complex classical cases, we obtain the integration with respect to the uniform measure on $S^{N-1}_{\mathbb{R}}, S^{N-1}_{\mathbb{C}}$.

# Weingarten 1/4

Theorem. The integration over $S$ has the ergodicity property

$$\left(\int_U \otimes id\right)\Phi(x) = \int_S x$$

where $\Phi : C(S) \to C(U) \otimes C(S)$ is the coaction map.

Proof. This is something non-trivial, coming from the knowledge of the integration over $U$, via the Weingarten formula:

$$\int_U u_{i_1 j_1}^{k_1} \ldots u_{i_p j_p}^{k_p} = \sum_{\pi,\sigma\in D(k)} \delta_\pi(i)\delta_\sigma(j) W_{kN}(\pi,\sigma)$$

Here $\delta \in \{0,1\}$ are Kronecker type symbols, and $W_{kN} = G_{kN}^{-1}$ is the inverse of the Gram matrix $G_{kN}(\pi,\sigma) = N^{|\pi\vee\sigma|}$.

# Weingarten 2/4

Theorem. There is a unique trace $tr : C(S) \to \mathbb{C}$ satisfying

$$(id \otimes tr)\Phi(x) = tr(x)1$$

and this is the canonical integration, constructed above.

Proof. Let $tr$ be as in the statement. We have:

$$tr\left(\int_U \otimes\, id\right)\Phi(x) = \int_U (id \otimes tr)\Phi(x) = tr(x)$$

On the other hand, by ergodicity we have as well:

$$tr\left(\int_U \otimes\, id\right)\Phi(x) = tr\left(\int_S x\right) = \int_S x$$

Thus $tr$ equals the standard integration, as claimed.

# Weingarten 3/4

Theorem. The construction $U \to S_U$ makes correspond:

$$\begin{array}{ccc} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{array} \quad \to \quad \begin{array}{ccc} S^{N-1}_{\mathbb{R},+} & \longrightarrow & S^{N-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{C}} \end{array}$$

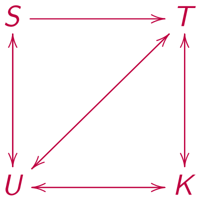
Proof. We use the ergodicity formula established above:

$$\left(\int_U \otimes id\right)\Phi = \int_S$$

Since $\int_U$ is faithful on $C(U)$ and $(\varepsilon \otimes id)\Phi = id$, the coaction map $\Phi$ is faithful as well. Thus $\int_S xx^* = 0$ with $x \in C(S)$ implies $x = 0$, and so $\int_S$ is faithful on $C(S)$. Thus we have $S = S_U$.

# Weingarten 4/4

<u>Theorem</u>. We have arrows as follows, obtained by adding the first column space construction $U \to S$ to what we already have,




which work for the basic quadruplets $(S, T, U, K)$.

$\implies$ And we'll stop here, $T \to S$ and $S \leftrightarrow K$ being non-trivial.

# Axiomatization 1/2

We call noncommutative sphere and torus, and quantum unitary and reflection group, the intermediate objects as follows,

$$S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{C},+}$$

$$T_N \subset T \subset \mathbb{T}_N^+$$

$$O_N \subset U \subset U_N^+$$

$$H_N \subset K \subset K_N^+$$

with $S$ being an algebraic manifold, and $T, U, K$ being compact quantum groups. Note that $T$ must be a group dual.

# Axiomatization 2/2

A quadruplet $(S,T,U,K)$ produces a noncommutative geometry when one can pass from each object to all the other ones,

$$\begin{matrix} S & = & S_{<O_N,T>} & = & S_U & = & S_{<O_N,K>}\\ S\cap\mathbb{T}_N^+ & = & T & = & U\cap\mathbb{T}_N^+ & = & K\cap\mathbb{T}_N^+\\ G^+(S) & = & <O_N,T> & = & U & = & <O_N,K>\\ G^+(S)\cap K_N^+ & = & G^+(T)\cap K_N^+ & = & U\cap K_N^+ & = & K \end{matrix}$$

with the usual convention that all this is up to the equivalence relation, namely isomorphism of $*$-algebras of coordinates.

# Compact form

A quadruplet $(S, T, U, K)$, between classical real and free complex,

$$(S^{N-1}_{\mathbb{R}}, T_N, O_N, H_N) \ < \ (S, T, U, K) \ < \ (S^{N-1}_{\mathbb{C},+}, \mathbb{T}_N^+, U_N^+, K_N^+)$$

produces a noncommutative geometry when

$$\begin{matrix} S & = & S_U & & \\ S \cap \mathbb{T}_N^+ & = & T & = & K \cap \mathbb{T}_N^+ \\ G^+(S) & = & < O_N, T > & = & U \\ G^+(T) \cap K_N^+ & = & U \cap K_N^+ & = & K \end{matrix}$$

up to the standard equivalence relation for algebraic manifolds.

# Conclusion

(1) We have NCG axioms, and 4 basic examples, as follows:

$$\begin{array}{ccc} \mathbb{R}^N_+ & \longrightarrow & \mathbb{C}^N_+ \\ \uparrow & & \uparrow \\ \mathbb{R}^N & \longrightarrow & \mathbb{C}^N \end{array}$$

(2) More examples, and classification results, coming soon.

(3) Technology used: basics, twists, liberation, Weingarten.

(4) We must develop these NCG: more manifolds, integration.