Basic noncommutative geometries

Teo Banica

"Introduction to noncommutative geometry", 4/6

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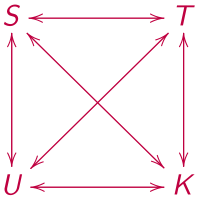
Idea

There is no free $\mathbb{R}^N$, or free $\mathbb{C}^N$. We have quadruplets (S, T, U, K) consisting of a sphere, torus, unitary group and reflection group:



Such quadruplets can be axiomatized. There are 4 main examples of geometries in this sense, namely those of $\mathbb{R}^N$, $\mathbb{C}^N$, $\mathbb{R}^N_+$, $\mathbb{C}^N_+$.

Axioms

A quadruplet (S,T,U,K), between classical real and free complex,

$$\mathbb{R}^N \ < \ (S,T,U,K) \ < \ \mathbb{C}^N_+$$

produces a noncommutative geometry when

$$\begin{matrix} S & = & S_U & & \\ S\cap\mathbb{T}_N^+ & = & T & = & K\cap\mathbb{T}_N^+ \\ G^+(S) & = & < O_N, T > & = & U \\ G^+(T)\cap K_N^+ & = & U\cap K_N^+ & = & K \end{matrix}$$

up to the standard equivalence relation for algebraic manifolds.

Plan

We will complete the basic 4-diagram into a 9-diagram:

$$\begin{array}{ccccc}
\mathbb{R}_+^N & \longrightarrow & \mathbb{TR}_+^N & \longrightarrow & \mathbb{C}_+^N \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}_*^N & \longrightarrow & \mathbb{TR}_*^N & \longrightarrow & \mathbb{C}_*^N \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}^N & \longrightarrow & \mathbb{TR}^N & \longrightarrow & \mathbb{C}^N
\end{array}$$

Then we will discuss classification results, and extensions.

Half-liberation 1/4

Question. Is there a "standard" geometry $\mathbb{R}^N \subset \mathbb{R}^N_* \subset \mathbb{R}^N_+$?

Theorem. The algebraic manifold $S^{(k)} \subset S^{N-1}_{\mathbb{R},+}$ obtained via the relations $a_1 \ldots a_k = a_k \ldots a_1$ is as follows:
(1) At $k = 1$ we have $S^{(k)} = S^{N-1}_{\mathbb{R},+}$.
(2) At $k = 2, 4, 6, \ldots$ we have $S^{(k)} = S^{N-1}_{\mathbb{R}}$.
(3) At $k = 3, 5, 7, \ldots$ we have $S^{(k)} = S^{(3)}$.

Definition. We define the half-classical sphere via the formula

$$C(S^{N-1}_{\mathbb{R},*}) = C(S^{N-1}_{\mathbb{R},+}) \Big/ \Big\langle abc = cba \Big\rangle$$

and call the relations $abc = cba$ half-commutation relations.

Half-liberation 2/4

Definition. We define the real half-classical quadruplet

$$(S^{N-1}_{\mathbb{R},*}, T_N^*, O_N^*, H_N^*)$$

by imposing $abc = cba$ to the coordinates. We define as well

$$(S^{N-1}_{\mathbb{C},*}, \mathbb{T}_N^*, U_N^*, K_N^*)$$

by imposing $abc = cba$ to the coordinates, and their adjoints.

$\Longrightarrow$ To do: find tools for studying these objects, check our NCG axioms for them, establish some further uniqueness results.

Half-liberation 3/4

Theorem. The sphere $S^{N-1}_{\mathbb{R},*}$ has the following properties:

(1) $PS^{N-1}_{\mathbb{R},*}$ is classical, equal to $P^{N-1}_{\mathbb{C}}$.

(2) $S^{N-1}_{\mathbb{R},*} \subset S^{N-1}_{\mathbb{R},+}$ appears as the affine lift of $P^{N-1}_{\mathbb{C}}$.

(3) We have a matrix model $C(S^{N-1}_{\mathbb{R},*}) \subset M_2(C(S^{N-1}_{\mathbb{C}}))$.

(4) Similar results hold for the subspaces $X \subset S^{N-1}_{\mathbb{R},*}$.

Proof. (1) Here $\subset$ is clear, because $abc = aba$ implies $[ab, cd] = 0$, and $\supset$ follows by using the model in (3), namely:

$$x_i = \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}$$

(2) and the faithfulness claim in (3) are related, and follow from some algebra. As for (4), this is something more technical.

Half-liberation 4/4

Theorem. We have full results regarding $S^{N-1}_{\mathbb{R},*}, T_N^*, O_N^*, H_N^*$, and complex analogues as well, regarding $S^{N-1}_{\mathbb{C},*}, \mathbb{T}_N^*, U_N^*, K_N^*$.

Theorem. We have noncommutative geometries, as follows:

$$\begin{array}{ccc}
\mathbb{R}^N_+ & \longrightarrow & \mathbb{C}^N_+ \\
\uparrow & & \uparrow \\
\mathbb{R}^N_* & \longrightarrow & \mathbb{C}^N_* \\
\uparrow & & \uparrow \\
\mathbb{R}^N & \longrightarrow & \mathbb{C}^N
\end{array}$$

Remark. It is possible to prove that O_N^* is the unique intermediate easy quantum group $O_N \subset G \subset O_N^+$. More on this later.

Hybrid geometries 1/4

An intermediate geometry $\mathbb{R}^N \subset X \subset \mathbb{C}^N$ is given by a quadruplet (S, T, U, K), whose components are subject to:

$$S_{\mathbb{R}}^{N-1} \subset S \subset S_{\mathbb{C}}^{N-1}$$

$$T_N \subset T \subset \mathbb{T}_N$$

$$O_N \subset U \subset U_N$$

$$H_N \subset K \subset K_N$$

There are many solutions here, even under strong axioms, such as easiness. We will discuss here the "standard" solution.

Hybrid geometries 2/4

Theorem. We have an intermediate sphere as follows,

$$S^{N-1}_{\mathbb{R}} \subset \mathbb{T}S^{N-1}_{\mathbb{R}} \subset S^{N-1}_{\mathbb{C}}$$

which appears as the affine lift of $P^{N-1}_{\mathbb{R}}$, inside $S^{N-1}_{\mathbb{C}}$.

Theorem. More generally, we have a quadruplet as follows,

$$(\mathbb{T}S^{N-1}_{\mathbb{R}}, \mathbb{T}T_N, \mathbb{T}O_N, \mathbb{T}H_N)$$

which appears in a similar way, by lifting.

Theorem. This quadruplet satisfies our NCG axioms.

$\implies$ A priori $(\mathbb{Z}_r S^{N-1}_{\mathbb{R}}, \mathbb{Z}_r T_N, \mathbb{Z}_r O_N, \mathbb{Z}_r H_N)$ are solutions too.

Hybrid geometries 3/4

Theorem. We have as well half-classical and free quadruplets,

$$(\mathbb{T}S^{N-1}_{\mathbb{R},*}, \mathbb{T}T_N^*, \mathbb{T}O_N^*, \mathbb{T}H_N^*)$$

$$(\mathbb{T}S^{N-1}_{\mathbb{R},+}, \mathbb{T}T_N^+, \mathbb{T}O_N^+, \mathbb{T}H_N^+)$$

obtained via the relations $ab^* = a^*b$.

Theorem. All the above hybrid quantum groups, namely

$$\mathbb{T}O_N, \mathbb{T}O_N^*, \mathbb{T}O_N^+ \quad , \quad \mathbb{T}H_N, \mathbb{T}H_N^*, \mathbb{T}H_N^+$$

are easy, appearing from the partition implementing $ab^* = a^*b$.

Theorem. The hybrid quadruplets satisfy our NCG axioms.

Hybrid geometries 4/4

Theorem. We have noncommutative geometries as follows:

$$\begin{array}{ccccc}
\mathbb{R}^N_+ & \longrightarrow & \mathbb{TR}^N_+ & \longrightarrow & \mathbb{C}^N_+\\
\uparrow & & \uparrow & & \uparrow\\
\mathbb{R}^N_* & \longrightarrow & \mathbb{TR}^N_* & \longrightarrow & \mathbb{C}^N_*\\
\uparrow & & \uparrow & & \uparrow\\
\mathbb{R}^N & \longrightarrow & \mathbb{TR}^N & \longrightarrow & \mathbb{C}^N
\end{array}$$

Proof. This follows by putting together what we have.

Classification 1/4

Definition. A geometry coming from a quadruplet (S, T, U, K) is called easy when both U, K are easy, and

$$U = \{O_N, K\}$$

with the operation on the right being the easy generation operation.

Remark. It is known that if G, H are easy then we have

$$< G, H > \subset < G, H >' \subset \{G, H\}$$

and both these inclusions are conjectured to be isomorphisms.

Classification 2/4

Theorem. An easy geometry is determined by a pair (D, E) of categories of partitions, which must be as follows,

$$\mathcal{NC}_2 \subset D \subset P_2$$

$$\mathcal{NC}_{even} \subset E \subset P_{even}$$

and which are subject to the following conditions,

$$D = E \cap P_2$$

$$E = < D, \mathcal{NC}_{even} >$$

and to the usual axioms for the associated quadruplet (S, T, U, K), where U, K are the easy quantum groups associated to D, E.

Proof. The conditions come from $U = \{O_N, K\}$, $K = U \cap K_N^+$.

Classification 3/4

Remark. In the context of an easy geometry, we have:

$$C(U) = C(U_N^+) \Big/ \left\langle T_\pi \in Hom(u^{\otimes k}, u^{\otimes l}) \Big| \forall k, l, \forall T \in D(k, l) \right\rangle$$

$$C(K) = C(K_N^+) \Big/ \left\langle T_\pi \in Hom(u^{\otimes k}, u^{\otimes l}) \Big| \forall k, l, \forall T \in D(k, l) \right\rangle$$

We have as well the following formula, for the dual of the torus:

$$\Gamma = F_N \Bigg/ \left\langle g_{i_1} \ldots g_{i_k} = g_{j_1} \ldots g_{j_l} \Bigg| \exists \pi \in D(k, l), \delta_\pi \begin{pmatrix} i \\ j \end{pmatrix} \neq 0 \right\rangle$$

As for the sphere, here the situation is a bit more complicated.

Classification 4/4

Theorem. The easy geometries are as follows:

(1) Real case: the 3 geometries that we have are unique.

(2) Classical case: uniqueness again, under an extra axiom.

(3) Other "pure" cases: uniqueness, under an extra axiom.

(4) In general: uniqueness, under an extra “slicing” axiom.

Proof. In terms of the category of pairings $\mathcal{NC}_2 \subset D \subset P_2$, the conditions $D = E \cap P_2$, $E =< D, \mathcal{NC}_{even} >$ reformulate as:

$$D =< D, \mathcal{NC}_{even} > \cap P_2$$

But this equation can be solved by using the known classification results for easy quantum groups, and related techniques.

Monomial spheres 1/2

Reminder. We have seen that the abstract construction

$$C(S^{(k)})=C(S^{N-1}_{\mathbb{R},+})\Big/\Big\langle a_1\ldots a_k=a_k\ldots a_1\Big\rangle$$

produces in practice only 3 spheres, $S^{N-1}_{\mathbb{R}}\subset S^{N-1}_{\mathbb{R},*}\subset S^{N-1}_{\mathbb{R},+}$.

Definition. A monomial sphere is a sphere $S\subset S^{N-1}_{\mathbb{C},+}$ obtained via

$$x_{i_1}^{e_1}\ldots x_{i_k}^{e_k}=x_{i_{\sigma(1)}}^{f_1}\ldots x_{i_{\sigma(k)}}^{f_k}\quad,\quad\forall(i_1,\ldots,i_k)\in\{1,\ldots,N\}^k$$

with $\sigma\in S_k$, and with $e_r,f_r\in\{1,*\}$ being exponents.

Monomial spheres 2/2

Theorem. In the real case, the only monomial spheres are:

$$S^{N-1}_{\mathbb{R}} \subset S^{N-1}_{\mathbb{R},*} \subset S^{N-1}_{\mathbb{R},+}$$

Proof. The idea is that the real monomial spheres are the subsets $S \subset S^{N-1}_{\mathbb{R},+}$ obtained via relations of the form

$$x_{i_1} \ldots x_{i_k} = x_{i_{\sigma(1)}} \ldots x_{i_{\sigma(k)}}, \ \forall (i_1, \ldots, i_k) \in \{1, \ldots, N\}^k$$

associated to certain elements $\sigma \in G_k$, where $G = (G_k)$ is a filtered subgroup of $S_\infty = (S_k)$. But such groups can be classified.

$\implies$ The complex analogue of this is not known yet.

Projective spaces 1/2

Theorem. The projective spaces of our 9 geometries collapse to

$$\begin{array}{ccccc}
P^{N-1}_+ & \longrightarrow & P^{N-1}_+ & \longrightarrow & P^{N-1}_+ \\
\uparrow & & \uparrow & & \uparrow \\
P^{N-1}_{\mathbb{C}} & \longrightarrow & P^{N-1}_{\mathbb{C}} & \longrightarrow & P^{N-1}_{\mathbb{C}} \\
\uparrow & & \uparrow & & \uparrow \\
P^{N-1}_{\mathbb{R}} & \longrightarrow & P^{N-1}_{\mathbb{R}} & \longrightarrow & P^{N-1}_{\mathbb{C}}
\end{array}$$

where P^{N-1}_+ is the free projective space, $P^{N-1}_{\mathbb{R},+} = P^{N-1}_{\mathbb{C},+}$.

$\Longrightarrow$ Interesting trichotomy here, "real, complex, free".

Projective spaces 2/2

Definition. A monomial space is a subset $P \subset P^{N-1}_+$ obtained via

$$p_{i_1 i_2} \ldots p_{i_{k-1} i_k} = p_{i_{\sigma(1)} i_{\sigma(2)}} \ldots p_{i_{\sigma(k-1)} i_{\sigma(k)}}, \ \forall i \in \{1, \ldots, N\}^k$$

with σ ranging over a subset of $\bigcup_{k \in 2\mathbb{N}} S_k$, stable under $\sigma \to |\sigma|$.

Theorem. We have only 3 monomial projective spaces, namely:

$$P^{N-1}_{\mathbb{R}} \subset P^{N-1}_{\mathbb{C}} \subset P^{N-1}_+$$

$\Longrightarrow$ How to axiomatize the quadruplets (P, PT, PU, PK)?

Twisting

By Schur-Weyl twisting we obtain potential geometries as follows,

$$\begin{array}{ccccc}
\mathbb{R}^N_+ & \longrightarrow & \mathbb{T}\mathbb{R}^N_+ & \longrightarrow & \mathbb{C}^N_+ \\
\uparrow & & \uparrow & & \uparrow \\
\bar{\mathbb{R}}^N_* & \longrightarrow & \mathbb{T}\bar{\mathbb{R}}^N_* & \longrightarrow & \bar{\mathbb{C}}^N_* \\
\uparrow & & \uparrow & & \uparrow \\
\bar{\mathbb{R}}^N & \longrightarrow & \mathbb{T}\bar{\mathbb{R}}^N & \longrightarrow & \bar{\mathbb{C}}^N
\end{array}$$

but the axioms must be fine-tuned, e.g. due to QISO problems.

Intersections

An interesting problem is that of intersecting the twisted and untwisted geometries. There are $9 \times 9 = 81$ cases here.

In the real case we only have $3 \times 3 = 9$ cases. The spheres are non-smooth, "polygonal", and the QISO groups are

$$\begin{array}{ccccc}
O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+ \\
\uparrow & & \uparrow & & \uparrow \\
H_N & \longrightarrow & H_N^{[\infty]} & \longrightarrow & \bar{O}_N^* \\
\uparrow & & \uparrow & & \uparrow \\
H_N^+ & \longrightarrow & H_N & \longrightarrow & \bar{O}_N
\end{array}$$

where $H_N^* \subset H_N^{[\infty]} \subset H_N^+$ is the standard higher liberation of H_N.

Other extensions

Besides twisting, and taking intersections, we have:

(1) Super-easiness.

(2) Partition quantum groups.

(3) Other easiness-related theories.

(4) Other types of noncomutative spheres.

Conclusion

We have 9 main examples of geometries, as follows:

$$\begin{array}{ccccc}
\mathbb{R}^N_+ & \longrightarrow & \mathbb{T}\mathbb{R}^N_+ & \longrightarrow & \mathbb{C}^N_+ \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}^N_* & \longrightarrow & \mathbb{T}\mathbb{R}^N_* & \longrightarrow & \mathbb{C}^N_* \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}^N & \longrightarrow & \mathbb{T}\mathbb{R}^N & \longrightarrow & \mathbb{C}^N
\end{array}$$

The problem now is that of "developing" these geometries.