Homogeneous spaces and easy manifolds

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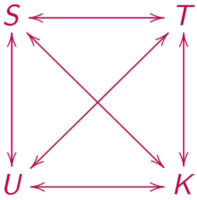
Geometry

There is no free $\mathbb{R}^N$, or free $\mathbb{C}^N$. We have quadruplets (S, T, U, K) consisting of a sphere, torus, unitary group and reflection group:



Such quadruplets (S, T, U, K) can be axiomatized. Technically we need easiness, liberation, twists and the Weingarten formula.

Examples

We have seen that there are 9 main examples of geometries:

$$\begin{array}{ccccc}
\mathbb{R}_+^N & \longrightarrow & \mathbb{T}\mathbb{R}_+^N & \longrightarrow & \mathbb{C}_+^N \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}_*^N & \longrightarrow & \mathbb{T}\mathbb{R}_*^N & \longrightarrow & \mathbb{C}_*^N \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}^N & \longrightarrow & \mathbb{T}\mathbb{R}^N & \longrightarrow & \mathbb{C}^N
\end{array}$$

We must "develop" these geometries $\Longrightarrow$ more manifolds.

Plan

(1) Half-classical manifolds.

(2) Quotients, partial isometries.

(3) Affine homogeneous spaces.

(4) Matrix model techniques.

Half-classical geometry 1/4

Definition. The half-classical sphere $S^{N-1}_{\mathbb{R},*}$ is defined via:

$$C(S^{N-1}_{\mathbb{R},*})=C(S^{N-1}_{\mathbb{R},+})\Big/\Big\langle abc=cba\Big\rangle$$

The relations $abc=cba$ are called half-commutation relations.

Remark. Under suitable assumptions, this is the only sphere:

$$S^{N-1}_{\mathbb{R}}\subset S^{N-1}_{\mathbb{R},*}\subset S^{N-1}_{\mathbb{R},+}$$

For instance, this is the only intermediate "monomial sphere".

Extensions. We can define T_N^*,O_N^*,H_N^* in a similar way, and under suitable assumptions, namely "easiness", we have uniqueness.

Half-classical geometry 2/4

Theorem. The sphere $S^{N-1}_{\mathbb{R},*}$ has the following properties:

(1) $PS^{N-1}_{\mathbb{R},*}$ is classical, equal to $P^{N-1}_{\mathbb{C}}$.

(2) $S^{N-1}_{\mathbb{R},*} \subset S^{N-1}_{\mathbb{R},+}$ appears as the affine lift of $P^{N-1}_{\mathbb{C}}$.

(3) We have a matrix model $C(S^{N-1}_{\mathbb{R},*}) \subset M_2(C(S^{N-1}_{\mathbb{C}}))$.

(4) Similar results hold for the subspaces $X \subset S^{N-1}_{\mathbb{R},*}$.

Proof. (1) Here $\subset$ is clear, because $abc = aba$ implies $[ab, cd] = 0$, and $\supset$ follows by using the model in (3), namely:

$$x_i = \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}$$

(2) and the faithfulness claim in (3) are related, and follow from some algebra. As for (4), the proof here is similar.

Half-classical geometry 3/4

Definition. The half-classical sphere $S^{N-1}_{\mathbb{C},*}$ is defined via

$$C(S^{N-1}_{\mathbb{C},*})=C(S^{N-1}_{\mathbb{C},+})\Big/\Big\langle abc=cba\Big\rangle$$

with $abc=cba$ being now imposed to the variables x_i,x_i^*.

Theorem. The sphere $S^{N-1}_{\mathbb{C},*}$ has the following properties:

(1) $PS^{N-1}_{\mathbb{C},*}$ is classical, equal to $P^{N-1}_{\mathbb{C}}$.

(2) We have a model $C(S^{N-1}_{\mathbb{C},*})\subset M_2(C(S^{N-1}_{\mathbb{C}}\times S^{N-1}_{\mathbb{C}}))$.

(3) Similar results hold for the subspaces $X\subset S^{N-1}_{\mathbb{C},*}$.

Half-classical geometry 4/4

There are many other interesting examples of spheres, tori, unitary and reflection groups, between classical and free complex:

(1) Using the relations $[ab^*, cd^*] = 0$. The sphere here,

$$S^{N-1}_{\mathbb{C}} \subset S^{N-1}_{\mathbb{C},*} \subset S^{N-1}_{\mathbb{C},\times} \subset S^{N-1}_{\mathbb{C},+}$$

is the biggest one whose projective space is classical.

(2) By assuming that $\{ab^*, a^*b\}$ all commute. The sphere here,

$$S^{N-1}_{\mathbb{C}} \subset S^{N-1}_{\mathbb{C},*} \subset S^{N-1}_{\mathbb{C},**} \subset S^{N-1}_{\mathbb{C},\times} \subset S^{N-1}_{\mathbb{C},+}$$

is the biggest one whose both projective spaces are classical.

(3) And many more, cf. $U_N \subset G \subset U_N^+$ easy. Note however that the quadruplets (S, T, U, K) are excluded by our NCG axioms.

Quotient spaces 1/4

Given a quadruplet (S, T, U, K) satisfying our NCG axioms, the sphere S appears by definition as an homogeneous space over U. At the level of the algebras of functions, we have:

$$C(S) \subset C(U) \quad , \quad x_i = u_{i1}$$

The same construction with K at the place of U gives nothing interesting, because the variables $x_i = u_{i1}$ commute. Thus we always obtain the algebra $\mathbb{C}^N$, and the space $\{1, \ldots, N\}$.

$\implies$ Study the spaces $G_{NM} = G_N / G_{N-M}$, over $G = U, K$.

Quotient spaces 2/4

Definition. A family of compact quantum groups $G=(G_N)$, with $G_N\subset U_N^+$ for any $N\in\mathbb{N}$, is called uniform when

$$G_{N-1}=G_N\cap U_{N-1}^+$$

with respect to the standard embedding $U_{N-1}^+\subset U_N^+$, given by:

$$u\to\begin{pmatrix}u&0\\0&1\end{pmatrix}$$

Remark 1. For group duals, $G_N=\widehat{\Gamma}_N$ with $\Gamma_N=<g_1,\ldots,g_N>$, we must have $\Gamma_{N-1}=\Gamma_N/<g_N=1>$, projective limit situation.

Remark 2. Most examples are uniform. However, half-liberations are not, because $abc=cba$ with $c=1$ gives $ab=ba$.

Quotient spaces 3/4

Theorem. For an easy quantum group $G = (G_N)$, coming from a category of partitions D, the following are equivalent:

(1) G is uniform: $G_{N-1} = G_N \cap U_{N-1}^+$ wrt. $U_{N-1}^+ \subset U_N^+$.

(2) $G_{N-1} = G_N \cap U_{N-1}^+$ wrt. all N embeddings $U_{N-1}^+ \subset U_N^+$.

(3) D is stable under removing blocks.

Application. This shows right away that the classical and free U, K are uniform, and that the half-classical ones are not.

Quotient spaces 4/4

<u>Theorem</u>. Given a uniform easy quantum group $G=(G_N)$, and integers $M\leq N$, consider the space $G\to G_{NM}$ given by:

$$C(G_{NM})=\left\langle u_{ij}\Big|i=1,\ldots,N,j=1,\ldots,M\right\rangle\subset C(G)$$

These spaces have then the following properties:

(1) They interpolate between S $(M=1)$ and G $(M=N)$.

(2) They are homogeneous spaces, $G_{NM}=G_N/G_{N-M}$.

(3) The uniform mesure can be computed via Weingarten.

(4) With $G=U,K$ we have BP, in the $M=[tN]\to\infty$ limit.

<u>Proof</u>. Here (1,2,3) go as in the sphere case, and in (4) the variables are sums of non-overlapping coordinates.

Partial isometries 1/4

(1) We have so far homogeneous spaces of the following type:

$$G_{NM}=G_N/G_{N-M}$$

(2) We will add one more parameter, $L\leq M,N$, and look at:

$$G_{NM}^L=(G_N\times G_M)\Big/(G_{N-L}\times G_{M-L}\times G_L)$$

(3) This is a generalization ideed, because at $L=M$ we have:

$$G_{NM}^M=(G_N\times G_M)\Big/(G_{N-M}\times G_0\times G_M)=G_{NM}$$

(4) The spaces G_{NM}^L consist of "quantum partial isometries".

Partial isometries 2/4

Definition. Associated to any integers $L \leq M, N$ are the spaces

$$O_{NM}^L = \left\{ T : E \to F \text{ isometry} \Big| E \subset \mathbb{R}^M, F \subset \mathbb{R}^N, \dim_{\mathbb{R}} E = L \right\}$$

$$U_{NM}^L = \left\{ T : E \to F \text{ isometry} \Big| E \subset \mathbb{C}^M, F \subset \mathbb{C}^N, \dim_{\mathbb{C}} E = L \right\}$$

the notion of isometry being with respect to the usual $<,>$.

Theorem. We have identifications as follows,

$$O_{NM}^L \simeq \left\{ U \in M_{N \times M}(\mathbb{R}) \Big| U^t U = \text{projection of trace } L \right\}$$

$$U_{NM}^L \simeq \left\{ U \in M_{N \times M}(\mathbb{C}) \Big| U^* U = \text{projection of trace } L \right\}$$

by identifying partial isometries with rectangular matrices.

Partial isometries 3/4

Theorem. We have action maps as follows, which are transitive,

$$O_N \times O_M \curvearrowright O_{NM}^L \quad : \quad (A,B)U = AUB^t$$

$$U_N \times U_M \curvearrowright U_{NM}^L \quad : \quad (A,B)U = AUB^*$$

whose stabilizers are $O_{N-L} \times O_{M-L} \times O_L$, $U_{N-L} \times U_{M-L} \times U_L$.

Theorem. We have isomorphisms as follows,

$$\begin{aligned} O_{NM}^L &= (O_N \times O_M)/(O_{N-L} \times O_{M-L} \times O_L) \\ U_{NM}^L &= (U_N \times U_M)/(U_{N-L} \times U_{M-L} \times U_L) \end{aligned}$$

the quotient maps being $(A,B) \to AUB^*$, where $U = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

Partial isometries 4/4

Definition. Associated to any integers $L \leq M, N$ are the algebras

$$C(O_{NM}^{L+}) = C^*\left((u_{ij})_{i=1\ldots N, j=1\ldots M}\Big| u = \bar{u}, u^t u = \text{proj. trace } L\right)$$
$$C(U_{NM}^{L+}) = C^*\left((u_{ij})_{i=1\ldots N, j=1\ldots M}\Big| u^* u, u^t \bar{u} = \text{projs. trace } L\right)$$

with the trace being by definition the sum of the diagonal entries.

Theory. We have results as before, namely homogeneous space structure, Weingarten formula, and Bercovici-Pata in the

$$M = \mu N, L = \lambda N, N \to \infty$$

limit, for sums of non-overlapping coordinates. All this can be done for all our U, K groups, classical/free, and can be twisted as well.

Affine spaces 1/4

Definition. An affine homogeneous space over $G \subset U_N^+$ is a closed subset $X \subset S^{N-1}_{\mathbb{C},+}$, such that there exists $I \subset \{1, \ldots, N\}$ such that

$$\alpha(x_i) = \frac{1}{\sqrt{|I|}} \sum_{j \in I} u_{ij} \quad , \quad \Phi(x_i) = \sum_j u_{ij} \otimes x_j$$

define morphisms of algebras, satisfying the ergodicity condition:

$$\left(\int_G \otimes id \right) \Phi = \int_G \alpha(.)1$$

$\implies$ The formula $(id \otimes \Phi)\Phi = (\Delta \otimes id)\Phi$ is automatic.
$\implies$ We have as well the formula $(id \otimes \alpha)\Phi = \Delta\alpha$.

Affine spaces 2/4

Theorem. When α is injective we must have $X = X_{G,I}^{min}$, where:

$$C(X_{G,I}^{min}) = \left\langle \frac{1}{\sqrt{|I|}} \sum_{j\in I} u_{ij} \Big| i = 1,\ldots,N \right\rangle \subset C(G)$$

Also, we must have $X \subset X_{G,I}^{max}$, as subsets of $S_{\mathbb{C},+}^{N-1}$, where

$$C(X_{G,I}^{max}) = C(S_{\mathbb{C},+}^{N-1}) \Big/ \left\langle (Px^{\otimes k})_{i_1\ldots i_k} = \frac{1}{\sqrt{|I|^k}} \sum_{j_1\ldots j_k\in I} P_{i_1\ldots i_k, j_1\ldots j_k} \right\rangle$$

with P being the orthogonal projection onto $Fix(u^{\otimes k})$.

$\Longrightarrow$ In general, we have $X_{G,I}^{min} \subset X \subset X_{G,I}^{max}$, as for group algebras.

Affine spaces 3/4

Theorem. In the classical case, $G \subset U_N$, we have

$$X = G/(G \cap C_N^I)$$

where $C_N^I \subset U_N$ is the group fixing $\xi_I = \frac{1}{\sqrt{|I|}}(\delta_{i\in I})_i$.

Theorem. For group duals, $G = \widehat{\Gamma}$ with $\Gamma = < g_1, \ldots, g_N >$,

$$X = \widehat{\Gamma}_I \quad , \quad \Gamma_I = < g_i | i \in I > \subset \Gamma$$

when identifying as usual full and reduced group algebras.

Theorem. The quantum groups G_N themselves, the spaces G_{NM}, and the spaces G_{NM}^L as well, are affine homogeneous.

Affine spaces 4/4

Several questions, regarding the affine homogeneous spaces:

(1) In the easy case, we have the Bercovici-Pata bijection for sums of non-overlapping coordinates. Unify with Meixner/free Meixner.

(2) In fact, this tehnically requires the passage to product groups, $G_N \times G_N$, which are not exactly easy. New axiomatics needed.

(3) What is an easy manifold? Or a free manifold? Must get rid of G, in the definition of the AHS. Needs good Tannaka duality.

Matrix models 1/4

An idea that already appeared, in connection with half-liberation:

Definition. A matrix model for a noncommutative algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$ is a morphism of C^*-algebras

$$\pi : C(X) \to M_K(C(T))$$

with T being a compact space, and $K \in \mathbb{N}$ being an integer.

We are mostly interested in the case where T has an integration functional, and π is faithful, commuting with the traces.

Matrix models 2/4

Theorem. Let $X \subset S^{N-1}_{\mathbb{C},+}$ be algebraic, satisfying $X_{class} \neq \emptyset$. Then we have an increasing sequence of algebraic submanifolds

$$X_{class} = X^{(1)} \subset X^{(2)} \subset X^{(3)} \subset \ldots\ldots \subset X^{(\infty)} \subset X$$

with $X^{(K)}$ with $K < \infty$ being the part of X which is realizable with $K \times K$ random matrix models, and with $X^{(\infty)} = \cup_K X^{(K)}$.

Proof. Using the algebraic relations defining X, we can construct a universal $K \times K$ model space T_K, and then factorize

$$\pi_{univ} : C(X) \to M_K(C(T_K))$$

as to obtain our submanifold $X^{(K)} \subset X$. Technically, $X_{class} \neq \emptyset$ is needed. Finally, we can set $X^{(\infty)} = \cup_K X^{(K)}$, as above.

Matrix models 3/4

The available results on matrix models are as follows:

(1) $X_{half-class} \subset X^{(2)}$, cf. half-liberation.

(2) $X_{1/K-class} \subset X^{(K)}$, by cyclic extension.

(3) Hadamard models for quantum partial permutations.

(4) Many interesting results for CQG $\implies$ AHS?

Matrix models 4/4

Many interesting questions, on models, and in general:

(1) What is a free manifold? Or an easy manifold?

(2) How to unify Bercovici-Pata with Meixner/free Meixner?

(3) How to extend the CQG modelling theory to the AHS?

(4) How to unify easiness and matrix/polynomial relations?

Plus and of course, go towards Nash-Connes Geometry.