C*-algebra basics

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"Introduction to operator algebras", 3/6

07/20

C^*-algebras

Definition. A C^*-algebra is a complex algebra A, with:

(1) A norm $a \to ||a||$, making it a Banach algebra.

(2) An involution $a \to a^*$, satisfying $||aa^*|| = ||a||^2$.

Examples. The closed $*$-subalgebras $A \subset B(H)$. We'll see later that any C^*-algebra is of this form (GNS theorem).

Remark. We have seen that $\rho(a) = ||a||$ for $a \in A$ normal. Thus

$$||a|| = \sqrt{||aa^*||} = \sqrt{\rho(aa^*)} \quad , \quad \forall a \in A$$

so the norm is uniquely determined by the $*$-algebra structure.

Gelfand

<u>Theorem</u>. The commutative C^*-algebras are the algebras of the form $C(X)$, with X being a compact space.

<u>Proof</u>. If X is compact, $C(X)$ is indeed a C^*-algebra. Conversely, given A commutative, consider the space of characters

$$X = \{\chi : A \to \mathbb{C}\}$$

with topology making continuous each $ev_a : \chi \to \chi(a)$. Then X is compact, and $a \to ev_a$ is a morphism of algebras $ev : A \to C(X)$.

(1) ev involutive. Using real + imaginary parts, we must prove that $ev_{a^*} = ev_a^*$ when $a = a^*$. But this follows from $\sigma(a) \subset \mathbb{R}$.

(2) ev isometric. Follows from $||ev_a|| = \rho(a) = ||a||$.

(3) ev surjective. Follows from Stone-Weierstrass.

Normal elements

Theorem. Assume that $a \in A$ is normal.

(1) We have $< a >= C(\sigma(a))$.

(2) For $f \in C(\sigma(a))$ we can define $f(a) \in A$.

(3) We have the formula $\sigma(f(a)) = f(\sigma(a))$.

Proof. Since a is normal, the algebra $< a >$ is commutative, and the Gelfand theorem gives $< a >= C(X)$, with:

$$X = \{\chi :< a > \to \mathbb{C}\}$$

The map $X \to \sigma(a)$ given by evaluation at a being bijective, we have $X = \sigma(a)$. Thus we get (1), and (2,3) follow as well.

Remarks. This extends the rational calculus, in the normal case. Also, it applies to any $T \in B(H)$ normal (spectral theorem).

Embeddings

We want to prove that any C^*-algebra appears as $A \subset B(H)$.

Theorem. Assume that A is commutative, $A = C(X)$, and let μ be a positive measure on X. We have then an embedding

$$A \subset B(H)$$

where $H = L^2(X)$, with $f \in A$ corresponding to $T_f : g \to fg$.

Proof. T_f is well-defined, and bounded as well, because:

$$||fg||_2 = \sqrt{\int_X |f(x)|^2 |g(x)|^2 d\mu(x)} \leq ||f||_\infty ||g||_2$$

We obtain in this way $A \subset B(H)$, as claimed.

Positivity

<u>Theorem.</u> For an element $a \in A$, the following are equivalent:

(1) a is positive, in the sense that $\sigma(a) \subset [0, \infty)$.

(2) $a = b^2$, for some $b \in A$ satisfying $b = b^*$.

(3) $a = cc^*$, for some $c \in A$.

<u>(1) $\implies$ (2)</u>: $\sigma(a) \subset \mathbb{R}$ implies $a = a^*$, so $< a >$ is commutative, and by using the Gelfand theorem, we can set $b = \sqrt{a}$.

<u>(2) $\implies$ (3)</u>: this is trivial, because we can set $c = b$.

<u>(3) $\implies$ (1)</u>: by contradition. By multiplying c by a suitable element of $< cc^* >$, we are led to the existence of an element $d \neq 0$ satisfying $-dd^* \geq 0$. With $d = x + iy$ we have:

$$dd^* + d^*d = 2(x^2 + y^2) \geq 0$$

Thus $d^*d \geq 0$, contradicting $\sigma(dd^*) = \sigma(d^*d)$ outside $\{0\}$.

Forms

<u>Definition</u>. Consider a linear map $\varphi : A \to \mathbb{C}$.

(1) φ is called positive when $a \geq 0 \implies \varphi(a) \geq 0$.

(2) φ is called faithful and positive if $a \geq 0, a \neq 0 \implies \varphi(a) > 0$.

<u>Theorem</u>. Let $\varphi : A \to \mathbb{C}$ be a positive linear form.

(1) $< a, b >= \varphi(ab^*)$ defines a generalized scalar product on A.

(2) By separating and completing we obtain a Hilbert space H.

(3) $\pi(a) : b \to ab$ defines a representation $\pi : A \to B(H)$.

(4) If φ is faithful in the above sense, then π is faithful.

<u>Proof</u>. Everything here is straightforward, and the last assertion follows from $a \neq 0 \implies \pi(aa^*) \neq 0 \implies \pi(a) \neq 0$.

GNS theorem

Theorem. Let A be a C^*-algebra.

(1) A appears as $A \subset B(H)$, for some Hilbert space H.

(2) When A is separable, H can be chosen to be separable.

(3) When A is FD, the space H can be chosen to be FD.

Proof. We just need a faithful positive linear form $\varphi : A \to \mathbb{C}$, and this can be constructed as in the classical case, as follows:

(1) Any positive linear form $\varphi : A \to \mathbb{C}$ is continuous.

(2) φ is positive iff there is a norm one $h \in A_+$, $||\varphi|| = \varphi(h)$.

(3) $\forall a \in A$ there exists φ positive of norm 1, $\varphi(aa^*) = ||a||^2$.

(4) There exists a faithful positive linear form $\varphi : A \to \mathbb{C}$.

Noncommutative spaces

Definition. Given an arbitrary C^*-algebra A, we write

$$A = C(X)$$

and call X a "noncommutative compact space".

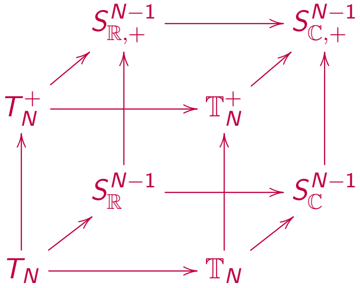
Equivalently, the category of noncommutative compact spaces is the category of C^*-algebras, with the arrows reversed.

Example 1. Given a morphism $\Phi : A \to B$, we write $A = C(X)$, $B = C(Y)$, and speak of the morphism $\phi : Y \to X$.

Example 2. Given a product $A = B \otimes C$, we write $A = C(X)$, $B = C(Y)$, $C = C(Z)$, and speak of $X = Y \times Z$.

Spheres and tori

Definition. We have noncommutative spheres and cubes/tori,



with the free complex sphere being defined by the formula

$$C(S^{N-1}_{\mathbb{C},+}) = C^*\left(x_1,\ldots,x_N\Big|\sum_i x_ix_i^* = \sum_i x_i^*x_i = 1\right)$$

and $S^{N-1}_{\mathbb{R},+}$, $\mathbb{T}_N^+$, T_N^+ being obtained via $x_i = x_i^*$, $x_ix_i^* = x_i^*x_i = 1/N$.

Group duals

Definition. The group algebra $C^*(\Gamma)$ of a discrete group Γ is the enveloping C^*-algebra of $\mathbb{C}[\Gamma]$, with involution $g^* = g^{-1}$.

Theorem. When Γ is abelian, we have an identification

$$C^*(\Gamma) = C(G)$$

where $G = \widehat{\Gamma}$ is the group formed by the characters $\chi : \Gamma \to \mathbb{T}$.

Proof. This follows from Gelfand, because the algebra characters $\chi : C^*(\Gamma) \to \mathbb{C}$ must come from group characters $\chi : \Gamma \to \mathbb{T}$.

Definition. Given a discrete group Γ, the space G given by

$$C(G) = C^*(\Gamma)$$

is called abstract dual of Γ, and is denoted $G = \widehat{\Gamma}$.

Cubes and tori

Theorem. The basic cubes and tori are all group duals,

$$\begin{array}{ccc} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{array} \quad = \quad \begin{array}{ccc} \widehat{\mathbb{Z}_2^{*N}} & \longrightarrow & \widehat{F_N} \\ \uparrow & & \uparrow \\ \mathbb{Z}_2^N & \longrightarrow & \mathbb{T}^N \end{array}$$

where F_N is the free group, and $*$ is a free product.

Proof. The various algebras $C(T)$ are generated by unitaries, with certain relations between them, and this gives the result.