Von Neumann algebras

Teo Banica

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Basics

Definition. A von Neumann algebra is a $*$-algebra of operators $A \subset B(H)$ which is closed under the weak topology:

$$T_n \in A, T_n x \to Tx \implies T \in A$$

Examples. The usual C^*-algebras, in finite dimensions. Also, the algebras $L^\infty(X) \subset B(L^2(X))$, which are commutative.

Theorem. The commutative von Neumann algebras are those of the form $L^\infty(X)$, with X being a measured space.

Proof. Basic functional analysis and operator theory. The full statement involves as well a multiplicity, in regards with H.

Theory

Theorem. For a $*$-algebra of operators $A \subset B(H)$, the following conditions are equivalent:

(1) A is weakly closed, i.e. is a von Neumann algebra.

(2) A is equal to its algebraic bicommutant, $A = A''$.

This is von Neumann's "bicommutant theorem". As a consequence, the von Neumann algebras appear as commutants, $A = P'$.

Comments. Von Neumann $\implies$ C^*. Conversely, the von Neumann algebras are the C^*-algebras having separable predual. Also,

$$L^\infty(X) = C(\widehat{X})$$

by Gelfand, with $\widehat{X}$ being the Stone-Čech compactification of X.

Finite dimensions

Theorem. Let $A \subset M_N(\mathbb{C})$ be a $*$-algebra.

(1) We have $1 = p_1 + \ldots + p_k$, with $p_i \in A$ minimal projections.

(2) The spaces $A_i = p_i A p_i$ are non-unital $*$-subalgebras of A.

(3) We have a non-unital $*$-algebra sum $A = A_1 \oplus \ldots \oplus A_k$.

(4) Unital $*$-algebra isomorphisms $A_i \simeq M_{N_i}(\mathbb{C})$, $N_i = rank(p_i)$.

(5) Thus, we can decompose $A \simeq M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$.

Proof. (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5).

Reduction theory

Theorem. When writing the center of the algebra as

$$Z(A) = L^\infty(X)$$

wihh X measured space, the algebra decomposes as

$$A = \int_X A_x \, dx$$

with the summands being "factors", $Z(A_x) = \mathbb{C}$.

Example. In finite dimensions the algebra must be

$$A = M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$$

and this is its decomposition as a sum of factors.

Factors

Theorem. The factors, $Z(A) = \mathbb{C}$, fall into 3 classes:

(1) Type I. These are the usual matrix algebras $M_N(\mathbb{C})$ (type I_N), and the algebra $B(H)$, with H separable (type I_∞).

(2) Type II. These are the ∞D factors having a trace $tr : A \to \mathbb{C}$ (type II_1) and their tensor products with $B(H)$ (type II_∞).

(3) Type III. These fall into several classes, III_λ with $\lambda \in [0, 1]$, and appear from II_1 factors, via crossed product type constructions.

Proof. This is heavy, due to Murray and von Neumann, and then Connes, based on ideas of Tomita, Takesaki and others.

$\implies$ The II_1 factors are the "building blocks" of the theory.

II_1 factors

Definition. A II_1 factor is a von Neumann algebra $A \subset B(H)$:

(1) Which is infinite dimensional, $\dim(A) = \infty$.

(2) Has trivial center, $Z(A) = \mathbb{C}$.

(3) And has a faithful positive unital trace, $tr : A \to \mathbb{C}$.

Theorem 1. The trace is unique.

Theorem 2. The trace of projections can take any value in $[0, 1]$.

$\implies$ This is very interesting, "continuous dimension".

The factor R

Theorem 1. The following limiting von Neumann algebra,

$$R = \lim_{k \to \infty} M_{N_k}(\mathbb{C})$$

is a II_1 factor, independent of the limiting procedure.

Theorem 2. R is the unique "hyperfinite" II_1 factor.

Theorem 3. R is the unique "building block" for the whole hyperfinite von Neumann algebra theory.

These results, building on what has been said before, are heavy, due to Murray-von Neumann, Connes, and Connes-Haagerup.

Noncommutative geometry

Definition. The von Neumann algebra of a discrete group Γ is the weak closure of $\mathbb{C}[\Gamma]$ in the left regular representation:

$$L(\Gamma) \subset B(l^2(\Gamma))$$

Comment. When Γ is abelian, we obtain $L^\infty(\widehat{\Gamma})$. This is true in general, with $\widehat{\Gamma}$ being the NC space from the previous lecture:

$$L(\Gamma) = L^\infty(\widehat{\Gamma})$$

Theorem. The algebra $L(\Gamma)$ is a factor (of type II_1) when Γ has ICC. Also, $L(\Gamma) = R$ when Γ has ICC, and is amenable.

More. We can talk as well about $L^\infty(S)$ for the free spheres, but we need here free analogues of O_N, U_N, for integrating. Later.

Random matrices

Definition. A random matrix algebra is an algebra of type:

$$A = M_N(L^\infty(X))$$

The elements of A are called random matrices.

Theorem. The matrices $M \in A$ having i.i.d. normal entries, up to the constraint $M = M^*$, follow with $N \to \infty$ the semicircle law:

$$\gamma_t = \frac{1}{2\pi t}\sqrt{4t^2 - x^2}dx$$

Proof. Moment method. The Wick formula gives with $N \to \infty$ the Catalan numbers, which are the moments of the semicircle law.

Free probability

Definition. Two subalgebras $B, C \subset A$ are called:

(1) Independent, if $tr(b) = tr(c) = 0$ implies $tr(bc) = 0$.

(2) Free, if $tr(b_i) = tr(c_i) = 0$ implies $tr(b_1 c_1 b_2 c_2 \ldots) = 0$.

Theorem. We have the following results:

(1) $C^*(\Gamma), C^*(\Lambda)$ are independent inside $C^*(\Gamma \times \Lambda)$.

(2) $C^*(\Gamma), C^*(\Lambda)$ are free inside $C^*(\Gamma * \Lambda)$.

Theorem. Assuming that $x_1, x_2, x_3, \ldots \in A$ are i/f.i.d., centered, with variance $t > 0$, we have, with $n \to \infty$,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i \sim \mathcal{N}(0, t)/\gamma_t$$

where $\mathcal{N}(0, t)/\gamma_t$ are the normal/Wigner semicircle laws.

Subfactor theory

Definition. Consider an inclusion of II_1 factors $A \subset B$.

(1) Its index is the number $[B : A] = \dim_A B \in [1, \infty]$, defined as a Murray-von Neumann "continuous dimension" quantity.

(2) The "basic construction" is $A \subset B \subset C$, by "reflection", with $C = < B, e >$, where $e : B \to A$ is the orthogonal projection.

Theorem. Let $A_0 \subset A_1$ be a subfactor of finite index $N \in [1, \infty)$, and consider its Jones tower, obtained by basic construction:

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \ldots$$

The Jones projections $e_1, e_2, e_3, \ldots$ generate then a copy of the Temperley-Lieb algebra TL_N, inside the ambient algebra $B(H)$.