Operator algebras and noncommutative spaces

Teo Banica

"Introduction to quantum groups", 1/6

06/20

Plan

(1) Hilbert spaces, linear operators

(2) Basic spectral/eigenvalue theory

(3) C^*-algebra theory: Gelfand, GNS, FD

(4) Noncommutative spaces: spheres and tori

$\implies$ next lecture: quantum groups

Hilbert spaces

Definition. Complex vector space H with $< x, y >$, satisfying:

(1) $< x, y >$ is linear in x, antilinear in y.
(2) $\overline{< x, y >} =< y, x >$, for any x, y.
(3) $< x, x >> 0$, for any $x \neq 0$.
(4) H is complete with respect to $||x|| = \sqrt{< x, x >}$.

Note that (4) is based on Cauchy-Schwarz. Basic examples:

(1) $H = \mathbb{C}^N$, with $< x, y >= \sum_i x_i \bar{y}_i$.
(2) $H = l^2(\mathbb{N})$, with $< x, y >= \sum_i x_i \bar{y}_i$.
(3) $H = L^2(X)$, with $< f, g >= \int_X f(x)\overline{g(x)}dx$.

Gram-Schmidt $\implies$ $H \simeq l^2(I)$. When I is countable, H is called separable. Example: $H = L^2[0, 1]$, cf. Weierstrass.

Operators

Let H be a Hilbert space, with basis $\{e_i\}_{i\in I}$. We have

$$\mathcal{L}(H)\subset M_I(\mathbb{C})$$

with $T:H\to H$ linear corresponding to the following matrix:

$$M_{ij}=< Te_j,e_i >$$

In particular, when $\dim(H)=N<\infty$, we obtain:

$$\mathcal{L}(H)\simeq M_N(\mathbb{C})$$

Also, in the infinite separable case, we obtain:

$$\mathcal{L}(H)\subset M_\infty(\mathbb{C})$$

$\implies$ However, $H=L^2[0,1]$ suggests not to use all this (..)

Bounded operators 1/2

Theorem. Given a Hilbert space H, the linear operators $T : H \to H$ which are bounded, in the sense that

$$||T|| = \sup_{||x|| \leq 1} ||Tx||$$

is finite, form a complex algebra with unit $B(H)$, which:

(1) is complete with respect to $||.||$ (Banach algebra).

(2) has an involution $T \to T^*$, $< Tx, y > = < x, T^*y >$.

The norm and involution are related by $||TT^*|| = ||T||^2$.

Bounded operators 2/2

Proof. Everything here is quite elementary:

(0) Complex algebra with unit: clear.

(1) Norm closed: set $Tx = \lim_{n\to\infty} T_n x$, for any $x \in H$.

(2) Involution: because $\varphi(x) =< Tx, y >$ is linear.

(3) Formula $||TT^*|| = ||T||^2$: double inequality.

Remark. In the matrix setting, $(M^*)_{ij} = \bar{M}_{ji}$.

C^*-algebras

<u>Definition</u>. A C^*-algebra is a complex algebra with unit A, with:

(1) A norm $a \to ||a||$, making it a Banach algebra.

(2) An involution $a \to a^*$, such that $||aa^*|| = ||a||^2$, $\forall a \in A$.

Basic examples: the closed $*$-subalgebras $A \subset B(H)$.
$\implies$ We'll see that any C^*-algebra is of this form.

Also basic: $C(X)$, with X being a compact space.
$\implies$ We'll see that any commutative C^*-algebra is of this form.

Finite dimensional: sums of matrix algebras, $\oplus_i M_{N_i}(\mathbb{C})$.
$\implies$ We'll see that any FD C^*-algebra is of this form.

Spectral theory

Definition. The spectrum of an element $a \in A$ is the set

$$\sigma(a) = \left\{\lambda \in \mathbb{C} \middle| a - \lambda \notin A^{-1}\right\}$$

where $A^{-1} \subset A$ is the set of invertible elements.

For the matrices, we obtain the eigenvalue set.
For the continuous functions, we obtain the image.

Theorem. $\sigma(ab) = \sigma(ba)$ outside $\{0\}$.

Proof. Indeed, $c = (1 - ab)^{-1} \implies 1 + cba = (1 - ba)^{-1}$.

Remark: in infinite dimensions, $S^*S = 1, SS^* \neq 1$ (shift).

Rational functions 1/2

Given $a \in A$, and a rational function $f = P/Q$ having poles outside $\sigma(a)$, we can construct $f(a) = P(a)Q(a)^{-1}$. We write:

$$f(a) = \frac{P(a)}{Q(a)}$$

Theorem. We have the "rational functional calculus" formula

$$\sigma(f(a)) = f(\sigma(a))$$

valid for any $f \in \mathbb{C}(X)$ having poles outside $\sigma(a)$.

Rational functions 2/2

Case $f \in \mathbb{C}[X]$. With $f(X) - \lambda = c(X - r_1)\dots(X - r_n)$:

$$\begin{aligned}
\lambda \notin \sigma(f(a)) \quad &\Longleftrightarrow \quad c(a - r_1)\dots(a - r_n) \in A^{-1} \\
&\Longleftrightarrow \quad a - r_1, \dots, a - r_n \in A^{-1} \\
&\Longleftrightarrow \quad r_1, \dots, r_n \notin \sigma(a) \\
&\Longleftrightarrow \quad \lambda \notin f(\sigma(a))
\end{aligned}$$

Case $f \in \mathbb{C}(X)$. With $f = P/Q$ and $F = P - \lambda Q$:

$$\begin{aligned}
\lambda \in \sigma(f(a)) \quad &\Longleftrightarrow \quad 0 \in \sigma(F(a)) \\
&\Longleftrightarrow \quad 0 \in F(\sigma(a)) \\
&\Longleftrightarrow \quad \exists \mu \in \sigma(a), F(\mu) = 0 \\
&\Longleftrightarrow \quad \lambda \in f(\sigma(a))
\end{aligned}$$

Basic spectra 1/2

Given an element $a \in A$, its spectral radius $\rho(a)$ is the radius of the smallest disk centered at 0 containing $\sigma(a)$.

Theorem. Let A be a C^*-algebra.

(1) The spectrum of a norm 1 element is in the unit disk.

(2) The spectrum of a unitary ($a^* = a^{-1}$) is on the unit circle.

(3) The spectrum of a self-adjoint element ($a = a^*$) is real.

(4) ρ of a normal element ($aa^* = a^*a$) equals its norm.

Basic spectra 2/2

(1) Clear from $(1-a)^{-1} = 1 + a + a^2 + \ldots$, for $||a|| < 1$.

(2) Follows by using $f(z) = z^{-1}$. Indeed, we have:

$$\sigma(a)^{-1} = \sigma(a^{-1}) = \sigma(a^*) = \overline{\sigma(a)}$$

(3) Follows from (2), by using $f(z) = (z + it)/(z - it)$.

(4) By (1) we have $\rho(a) \leq ||a||$. Given $\rho > \rho(a)$, we have:

$$\int_{|z|=\rho} \frac{z^n}{z-a}\, dz = \sum_{k=0}^{\infty} \left(\int_{|z|=\rho} z^{n-k-1} dz \right) a^k = a^{n-1}$$

By applying the norm and taking n-th roots we obtain:

$$\rho \geq \lim_{n \to \infty} ||a^n||^{1/n}$$

When $a = a^*$ we're done. In general, use $||aa^*|| = ||a||^2$.

Gelfand

Theorem. Any commutative C^*-algebra is the form $C(X)$, with its "spectrum" $X = Spec(A)$ consisting of the characters $\chi : A \to \mathbb{C}$.

Proof. Set $X = Spec(A)$, with topology making continuous all the evaluation maps $ev_a : \chi \to \chi(a)$. Then X is a compact space, and $a \to ev_a$ is a morphism of algebras $ev : A \to C(X)$.

(1) ev involutive. Using real + imaginary parts, we must prove that $ev_{a^*} = ev_a^*$ when $a = a^*$. But this follows from $\sigma(a) \subset \mathbb{R}$.

(2) ev isometric. Follows from $||ev_a|| = \rho(a) = ||a||$.

(3) ev surjective. Follows from Stone-Weierstrass.

Continuous calculus

<u>Theorem.</u> Assume that $a \in A$ is normal, and let $f \in C(\sigma(a))$.

(1) We can define $f(a) \in A$, with $f \to f(a)$ being a morphism.

(2) We have the formula $\sigma(f(a)) = f(\sigma(a))$.

<u>Proof</u>. Since a is normal, $B =< a >$ is commutative, and the Gelfand theorem gives $B = C(X)$, with $X = Spec(B)$.

The map $X \to \sigma(a)$ given by evaluation at a being bijective, we have $X = \sigma(a)$. Thus $B = C(\sigma(a))$, and we are done.

Positivity

Theorem. For an element $a \in A$, the following are equivalent:

(1) a is positive, in the sense that $\sigma(a) \subset [0, \infty)$.

(2) $a = b^2$, for some $b \in A$ satisfying $b = b^*$.

(3) $a = cc^*$, for some $c \in A$.

(1) $\implies$ (2): $\sigma(a) \subset \mathbb{R}$ implies $a = a^*$, so $< a >$ is commutative, and by using the Gelfand theorem, we can set $b = \sqrt{a}$.

(2) $\implies$ (3): this is trivial, because we can set $c = b$.

(3) $\implies$ (1): by contradition. By multiplying c by a suitable element of $< cc^* >$, we are led to the existence of an element $d \neq 0$ satisfying $-dd^* \geq 0$. With $d = x + iy$ we have:

$$dd^* + d^*d = 2(x^2 + y^2) \geq 0$$

Thus $d^*d \geq 0$, contradicting $\sigma(dd^*) = \sigma(d^*d)$ outside $\{0\}$.

NC spaces

Definition. Given an arbitrary C^*-algebra A, we write

$$A = C(X)$$

and call X a "noncommutative compact space".

Equivalently, the category of the noncommutative compact spaces is the category of the C^*-algebras, with the arrows reversed.

The idea is that of studying A, but formulating results in terms of X. For instance whenever we have a morphism $\Phi : A \to B$, we can write $A = C(X), B = C(Y)$, and rather speak of the corresponding morphism $\phi : Y \to X$. And so on, up to technical subtleties.

NC spheres

Definition. We have noncommutative spaces, as follows,

$$C(S^{N-1}_{\mathbb{R},+}) = C^*\left(x_1,\ldots,x_N\Big|x_i = x_i^*, \sum_i x_i^2 = 1\right)$$

$$C(S^{N-1}_{\mathbb{C},+}) = C^*\left(x_1,\ldots,x_N\Big|\sum_i x_i x_i^* = \sum_i x_i^* x_i = 1\right)$$

called free real sphere, and free complex sphere.

Here C^* means "universal C^*-algebra generated by".

These universal algebras are well-defined, because we have

$$\sum_i ||x_i||^2 = \sum_i ||x_i x_i^*|| \leq ||\sum_i x_i x_i^*|| = 1$$

and so the biggest C^*-norms on our algebras exist indeed.

Liberation

<u>Theorem</u>. We have embeddings of NC spaces, as follows,

$$\begin{array}{ccc} S^{N-1}_{\mathbb{C}} & \longrightarrow & S^{N-1}_{\mathbb{C},+} \\ \uparrow & & \uparrow \\ S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{R},+} \end{array}$$

and the free spheres are "liberations" of the classical ones.

<u>Proof</u>. We must establish the following isomorphisms:

$$C(S^{N-1}_{\mathbb{R},+}) = C^*_{comm}\left(x_1,\ldots,x_N\Big|x_i = x_i^*, \sum_i x_i^2 = 1\right)$$

$$C(S^{N-1}_{\mathbb{C},+}) = C^*_{comm}\left(x_1,\ldots,x_N\Big|\sum_i x_i x_i^* = \sum_i x_i^* x_i = 1\right)$$

But these isomorphisms are both clear, by using Gelfand.

Tori

Definition. Given $S \subset S^{N-1}_{\mathbb{C},+}$, the subspace $T \subset S$ given by

$$C(T) = C(S) \Big/ \left\langle x_i x_i^* = x_i^* x_i = \frac{1}{N} \right\rangle$$

is called associated torus. In the real case, we call T cube.

As a basic example, for $S = S^{N-1}_{\mathbb{C}}$ we obtain a torus:

$$S = S^{N-1}_{\mathbb{C}} \implies T = \left\{ x \in \mathbb{C}^N \Big| |x_i| = \frac{1}{\sqrt{N}} \right\}$$

Also, for the real sphere $S = S^{N-1}_{\mathbb{R}}$ we obtain a cube:

$$S = S^{N-1}_{\mathbb{R}} \implies T = \left\{ x \in \mathbb{R}^N \Big| x_i = \pm \frac{1}{\sqrt{N}} \right\}$$

Group algebras

Theorem. Let Γ be a discrete group, and consider the complex group algebra $\mathbb{C}[\Gamma]$, with involution given by:

$$g^* = g^{-1} \quad , \quad \forall g \in \Gamma$$

The maximal C^*-seminorm on $\mathbb{C}[\Gamma]$ is then a C^*-norm, and the corresponding closure of $\mathbb{C}[\Gamma]$ is a C^*-algebra, denoted $C^*(\Gamma)$.

Proof. Let $H = l^2(\Gamma)$, having $\{h\}_{h\in\Gamma}$ as orthonormal basis. Our claim is that we have an embedding, as follows:

$$\pi : \mathbb{C}[\Gamma] \subset B(H) \quad , \quad \pi(g)(h) = gh$$

But this is elementary to check, and gives the result.

Group duals

Theorem. When Γ is abelian, we have an isomorphism

$$C^*(\Gamma) \simeq C(G)$$

where $G = \widehat{\Gamma}$ is its dual, formed by the characters $\chi : \Gamma \to \mathbb{T}$.

Proof. Gelfand gives $A = C(X)$, with $X = Spec(A)$. But the spectrum $X = Spec(A)$, made of characters $\chi : C^*(\Gamma) \to \mathbb{C}$, can be identified with the Pontrjagin dual $G = \widehat{\Gamma}$, as desired.

Definition. Given a discrete group Γ, the space G given by

$$C(G) = C^*(\Gamma)$$

is called abstract dual of Γ, and is denoted $G = \widehat{\Gamma}$.

Back to tori

Theorem. The tori of the basic spheres are all group duals,

$$\begin{array}{ccc}
\mathbb{T}^N & \longrightarrow & \widehat{F_N}\\
\uparrow & & \uparrow\\
\mathbb{Z}_2^N & \longrightarrow & \widehat{\mathbb{Z}_2^{*N}}
\end{array}$$

where F_N is the free group, and $*$ is a free product.

Proof. The diagram formed by the algebras $C(T)$ is:

$$\begin{array}{ccc}
C^*(\mathbb{Z}^N) & \longleftarrow & C^*(\mathbb{Z}^{*N})\\
\downarrow & & \downarrow\\
C^*(\mathbb{Z}_2^N) & \longleftarrow & C^*(\mathbb{Z}_2^{*N})
\end{array}$$

But this gives the result, via some standard identifications.

Summary

(1) C^*-algebras: with norm and involution, $||aa^*|| = ||a||^2$.

(2) Gelfand theorem: commutative case $A = C(X)$.

(3) Noncommutative geometry: write $A = C(X)$ in general.

(4) Examples: NC spheres (real, complex) and tori (group duals).

$\implies$ We'll be back to NCG later, doing quantum groups

Embeddings

We want to prove that any C^*-algebra appears as $A \subset B(H)$.

<u>Theorem</u>. Assume that A is commutative, $A = C(X)$, and let μ be a positive measure on X. We have then an embedding

$$A \subset B(H)$$

where $H = L^2(X)$, with $f \in A$ corresponding to $T_f : g \to fg$.

<u>Proof</u>. T_f is well-defined, and bounded as well, because:

$$||fg||_2 = \sqrt{\int_X |f(x)|^2 |g(x)|^2 d\mu(x)} \leq ||f||_\infty ||g||_2$$

We obtain in this way $A \subset B(H)$, as claimed.

Forms

In general, we can replace the positive measures μ with the corresponding integration functionals.

Definition. Consider a linear map $\varphi : A \to \mathbb{C}$.

(1) φ is called positive when $a \geq 0 \implies \varphi(a) \geq 0$.

(2) φ is called faithful and positive if $a \geq 0, a \neq 0 \implies \varphi(a) > 0$.

In the commutative case, $A = C(X)$, we can write:

$$\varphi(f) = \int_X f(x) d\mu(x)$$

In general, the philosophy is similar.

GNS construction

Theorem. Let $\varphi : A \to \mathbb{C}$ be a positive linear form.

(1) $< a, b >= \varphi(ab^*)$ defines a generalized scalar product on A.

(2) By separating and completing we obtain a Hilbert space H.

(3) $\pi(a) : b \to ab$ defines a representation $\pi : A \to B(H)$.

(4) If φ is faithful in the above sense, then π is faithful.

Proof. Almost everything here is straightforward, and the last assertion follows from a positivity trick, namely:

$$a \neq 0 \implies \pi(aa^*) \neq 0 \implies \pi(a) \neq 0$$

Existence

In order to establish the GNS theorem, it remains to prove that any C^*-algebra has a faithful and positive linear form $\varphi : A \to \mathbb{C}$.

Theorem. Let A be a C^*-algebra.

(1) Any positive linear form $\varphi : A \to \mathbb{C}$ is continuous.

(2) φ is positive iff there is a norm one $h \in A_+$, $||\varphi|| = \varphi(h)$.

(3) $\forall a \in A$ there exists φ positive of norm 1, $\varphi(aa^*) = ||a||^2$.

(4) If A is separable there is a faithful positive form $\varphi : A \to \mathbb{C}$.

Proof of (1,2)

(1) This follows from $|\varphi(a)| \leq ||\pi(a)||\varphi(1) \leq ||a||\varphi(1)$.

(2) Let $a \in A_+$, $||a|| \leq 1$. We have then:

$$|\varphi(h) - \varphi(a)| \leq ||\varphi|| \cdot ||h - a|| \leq \varphi(h)1 = \varphi(h)$$

Thus $Re(\varphi(a)) \geq 0$. We must prove $a = a^* \implies \varphi(a) \in \mathbb{R}$.
We can assume $h = 1$. With $a = a^*$, for $t \in \mathbb{R}$ we have:

$$|\varphi(1 + ita)|^2 \leq \varphi(1)^2(1 + t^2||a||^2)$$

On the other hand with $\varphi(a) = x + iy$ we have:

$$|\varphi(1 + ita)| \geq (\varphi(1) - ty)^2$$

We therefore obtain that for any $t \in \mathbb{R}$ we have:

$$\varphi(1)^2(1 + t^2||a||^2) \geq (\varphi(1) - ty)^2$$

Thus we have $y = 0$, and this finishes the proof.

Proof of (3,4)

(3) This follows from (2), and from Hahn-Banach.

(4) Let (a_n) be a dense sequence inside A. For any n we construct a positive form satisfying $\varphi_n(a_n a_n^*) = ||a_n||^2$, and then we set:

$$\varphi = \sum_{n=1}^{\infty} \frac{\varphi_n}{2^n}$$

Let $a \in A$ be a nonzero element. Pick a_n close to a and consider the GNS pair (H, π) associated to (A, φ_n). We have:

$$\begin{aligned}
\varphi_n(aa^*) &= ||\pi(a)1|| \\
&\geq ||\pi(a_n)1|| - ||a - a_n|| \\
&= ||a_n|| - ||a - a_n|| \\
&> 0
\end{aligned}$$

Thus $\varphi_n(aa^*) > 0$, and so $\varphi(aa^*) > 0$, and we are done.

GNS theorem

Theorem. Let A be a C^*-algebra.

(1) A appears as $A \subset B(H)$, for some Hilbert space H.

(2) When A is separable, H can be chosen to be separable.

(3) When A is FD, the space H can be chosen to be FD.

Proof. Follows indeed by performing the GNS construction. $\square$

Finite dimensions

Theorem. Let $A \subset M_N(\mathbb{C})$ be a C^*-algebra.

(1) We have $1 = p_1 + \ldots + p_k$, with $p_i \in A$ minimal projections.

(2) The spaces $A_i = p_i A p_i$ are non-unital $*$-subalgebras of A.

(3) We have a non-unital $*$-algebra sum $A = A_1 \oplus \ldots \oplus A_k$.

(4) Unital $*$-algebra isomorphisms $A_i \simeq M_{N_i}(\mathbb{C})$, $N_i = rank(p_i)$.

(5) Thus, we can decompose $A \simeq M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$.

(6) This holds in fact for any finite dimensional C^*-algebra.

Proof. (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5) $\implies$ (6).

Conclusions

C^*-algebras: algebras with norm and involution, $||aa^*|| = ||a||^2$.

(1) Gelfand theorem: commutative case $A = C(X)$.

(2) Gelfand-Naimark-Segal theorem: $A \subset B(H)$.

(3) Finite dimensions: $A = M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$.

$\implies$ More basic theory: von Neumann algebras.

$\implies$ $A = C(X)$. Spheres and tori. What about groups?