Compact and discrete quantum groups

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"Introduction to quantum groups", 2/6

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Plan

(1) Compact quantum groups

(2) Discrete quantum groups

(3) Basic examples, operations

(4) Quantum isometry groups

$\Longrightarrow$ next lecture: representations

Operator algebras

C^*-algebras: with norm and involution, $||aa^*|| = ||a||^2$.

(1) Gelfand theorem: commutative case $A = C(X)$.

(2) Gelfand-Naimark-Segal theorem: $A \subset B(H)$.

(3) Finite dimensions: $A = M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$.

$\implies$ $A = C(X)$, with X "noncommutative compact space"

$\implies$ NC spheres, NC tori. What about quantum groups?

Classical groups

Let G be a compact Lie group. Then $G \subset U_N$. Multiplication:

$$(UV)_{ij} = \sum_k U_{ik} V_{kj}$$

By Stone-Weierstrass we have $C(G) = < u_{ij} >$, where:

$$u_{ij}(U) = U_{ij}$$

The multiplication $G \times G \to G$ transposes as:

$$u_{ij} \to \sum_k u_{ik} \otimes u_{kj}$$

Thus G is well described by $C(G)$, together with $u = (u_{ij})$.

Axioms

Let A be a C^*-algebra, with $u \in M_N(A)$ biunitary (u, u^t unitaries), whose entries generate A, such that:

– $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ defines a morphism $\Delta : A \to A \otimes A$
– $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : A \to \mathbb{C}$
– $S(u_{ij}) = u_{ji}^*$ defines a morphism $S : A \to A^{opp}$

We write then $A = C(G) = C^*(\Gamma)$, and call:

– G a compact quantum group
– Γ a discrete quantum group

[axioms due to Woronowicz, 1987, slightly modified here]

Compact groups 1/2

Theorem. For a closed subgroup $G \subset U_N$, the algebra $A = C(G)$, with the matrix formed by the standard coordinates

$$u_{ij}(g) = g_{ij}$$

is a Woronowicz algebra, with structural maps given by

$$\Delta = m^T \quad , \quad \varepsilon = u^T \quad , \quad S = i^T$$

where m, u, i are the multiplication, unit and inverse of G.

Any commutative Woronowicz algebra appears in this way.

Compact groups 2/2

Proof. We compute m^T, u^T, i^T. We have:

$$m^T(u_{ij})(U \otimes V) = (UV)_{ij} = \sum_k U_{ik}V_{kj} = \sum_k (u_{ik} \otimes u_{kj})(U \otimes V)$$

Regarding now u^T, here we have:

$$u^T(u_{ij}) = 1_{ij} = \delta_{ij}$$

As for the map i^T, this is given by:

$$i^T(u_{ij})(U) = (U^{-1})_{ij} = \bar{U}_{ji} = u_{ji}^*(U)$$

Thus the axioms are satisfied, with $\Delta = m^T, \varepsilon = u^T, S = i^T$.

Finally, the last assertion follows by applying Gelfand.

Group duals 1/2

Theorem. For a discrete group $\Gamma = < g_1, \ldots, g_N >$, the algebra $A = C^*(\Gamma)$, with the diagonal matrix formed by the generators

$$u = diag(g_1, \ldots, g_N)$$

is a Woronowicz algebra, with structural maps given by

$$\Delta(g) = g \otimes g \quad , \quad \varepsilon(g) = 1 \quad , \quad S(g) = g^{-1}$$

for any group element $g \in \Gamma$. This algebra is cocommutative, in the sense that $\Sigma\Delta = \Delta$, where $\Sigma(a \otimes b) = b \otimes a$ is the flip.

Remark. We'll see later that any cocommutative Woronowicz algebra appears in this way (needs representation theory).

Group duals 2/2

Proof. Consider the following unitary representation:

$$\Gamma \to C^*(\Gamma) \otimes C^*(\Gamma) \quad , \quad g \to g \otimes g$$

This produces a map $\Delta : C^*(\Gamma) \to C^*(\Gamma) \otimes C^*(\Gamma)$, given by:

$$\Delta(g) = g \otimes g$$

Similarly, ε comes from the trivial representation:

$$\Gamma \to \{1\} \quad , \quad g \to 1$$

As for S, this comes from the following representation:

$$\Gamma \to C^*(\Gamma)^{opp} \quad , \quad g \to g^{-1}$$

Remark. Note that the use of the opposite algebra is needed.

Comments 1/4

Assume that Γ is abelian, and let $G = \widehat{\Gamma}$ be its Pontrjagin dual, formed by the characters $\chi : \Gamma \to \mathbb{T}$. The isomorphism

$$C^*(\Gamma) \simeq C(G)$$

transforms the structural maps of $C^*(\Gamma)$, given by

$$\Delta(g) = g \otimes g \quad , \quad \varepsilon(g) = 1 \quad , \quad S(g) = g^{-1}$$

into the structural maps of $C(G)$, given by:

$$\Delta\varphi(g, h) = \varphi(gh) \quad , \quad \varepsilon(\varphi) = \varphi(1) \quad , \quad S\varphi(g) = \varphi(g^{-1})$$

Thus, $G = \widehat{\Gamma}$ is a compact quantum group isomorphism.

Comments 2/4

Motivated by this, given a Woronowicz algebra

$$A = C(G) = C^*(\Gamma)$$

we say that G, Γ are <u>dual to each other</u>, and write:

$$G = \widehat{\Gamma} \quad , \quad \Gamma = \widehat{G}$$

This duality extends the usual Pontrjagin duality.

Comments 3/4

Motivated by the compact Lie group case, we have:

Definition. Given $A = C(G)$, we denote by $\mathcal{A} \subset A$ the dense $*$-algebra generated by the coordinates u_{ij}, and we write

$$\mathcal{A} = C^\infty(G)$$

and call it "algebra of smooth functions" on G.

Example. For $A = C^*(\Gamma)$ we have $\mathcal{A} = \mathbb{C}[\Gamma]$.

Comments 4/4

Motivated by the group dual case, we have:

Definition. We agree to identify (A, u) and (B, v) when we have a $*$-algebra isomorphism

$$\mathcal{A} \simeq \mathcal{B}$$

mapping standard coordinates to standard coordinates, $u_{ij} \to v_{ij}$.

Example. This identifies for instance $C^*(\Gamma)$ with $C^*_{red}(\Gamma)$.

Summary

(1) We are looking at pairs (A,u), with $u\in M_N(A)$ biunitary, with:

$$\Delta(u_{ij})=\sum_k u_{ik}\otimes u_{kj}\quad,\quad \varepsilon(u_{ij})=\delta_{ij}\quad,\quad S(u_{ij})=u_{ji}^*$$

(2) We have compact and discrete quantum groups, given by:

$$A=C(G)=C^*(\Gamma)$$

(3) These quantum groups are dual to each other, and we write:

$$G=\widehat{\Gamma}\quad,\quad \Gamma=\widehat{G}$$

(4) We set $C^\infty(G)=<u_{ij}>$, and we use the identifications:

$$C^\infty(G)\simeq C^\infty(H)\quad,\quad u_{ij}\to v_{ij}$$

(5) All this is supported by C^*-algebras, and the above results.

Tech 1/2

Theorem. The comultiplication Δ, counit ε and antipode S satisfy the following conditions,

(1) Coassociativity: $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$.

(2) Counitality: $(id \otimes \varepsilon)\Delta = (\varepsilon \otimes id)\Delta = id$.

(3) Coinversality: $m(id \otimes S)\Delta = m(S \otimes id)\Delta = \varepsilon(.)1$.

on the dense $*$-subalgebra $\mathcal{A} \subset A$ generated by the variables u_{ij}.

Proof. Clear on coordinates, and so on the $*$-algebra $\mathcal{A}$.

Tech 2/2

Remark. In the commutative case, $G \subset U_N$, we have

$$\Delta = m^T \quad , \quad \varepsilon = u^T \quad , \quad S = i^T$$

and the 3 conditions satisfied by Δ, ε, S come by transposition from the basic 3 conditions satisfied by m, u, i, namely

$$m(m \times id) = m(id \times m)$$

$$m(id \times u) = m(u \times id) = id$$

$$m(id \otimes i)\delta = m(i \otimes id)\delta = 1$$

whre $\delta(g) = (g, g)$. In general, the philosophy is the same.

1. Products

Given two compact quantum groups G, H, so is their product $G \times H$, constructed as follows:

$$C(G \times H) = C(G) \otimes C(H)$$

Equivalently, at the level of the associated discrete quantum groups Γ, Λ, which are dual to G, H, we have:

$$C^*(\Gamma \times \Lambda) = C^*(\Gamma) \otimes C^*(\Lambda)$$

As an illustration, we have things of type $G \times \widehat{\Lambda}$, with G, Λ both classical, which are not classical, nor group duals.

2. Dual free products

Given two compact quantum groups G,H, so is their dual free product $G\,\hat{*}\,H$, constructed as follows:

$$C(G\,\hat{*}\,H)=C(G)*C(H)$$

Equivalently, at the level of the associated discrete quantum groups Γ,Λ, which are dual to G,H, we have a usual free product:

$$C^*(\Gamma*\Lambda)=C^*(\Gamma)*C^*(\Lambda)$$

This construction always produces non-classical quantum groups, unless of course $G=\{1\}$ or $H=\{1\}$.

3. Free complexification

Given a compact quantum group G, we can construct its free complexification $\widetilde{G}$ as follows, where $z = id \in C(\mathbb{T})$:

$$C(\widetilde{G}) \subset C(\mathbb{T}) * C(G) \quad , \quad \tilde{u} = zu$$

Equivalently, at the level of the associated discrete duals $\Gamma, \widetilde{\Gamma}$, we have the following formula, where $z = 1 \in \mathbb{Z}$:

$$C^*(\widetilde{\Gamma}) \subset C^*(\mathbb{Z}) * C^*(\Gamma) \quad , \quad \tilde{u} = zu$$

We'll see later that the "free analogues" of O_N, U_N are related by free complexification. Simpler than for O_N, U_N themselves (!)

4. Subgroups

Let G be compact quantum group, and let $I \subset C(G)$ be a closed $*$-ideal satisfying the following "Hopf ideal" condition:

$$\Delta(I) \subset C(G) \otimes I + I \otimes C(G)$$

We have then a closed subgroup $H \subset G$, as follows:

$$C(H) = C(G)/I$$

Dually, we obtain a quotient of discrete quantum groups:

$$\widehat{\Gamma} \to \widehat{\Lambda}$$

In all this the Hopf ideal condition is needed for Δ to factorize.

5. Quotients

Let us call "corepresentation" of a Woronowicz algebra $A = C(G)$ any unitary matrix $w \in M_n(\mathcal{A})$ satisfying:

$$\Delta(w_{ij}) = \sum_k w_{ik} \otimes w_{kj} \quad , \quad \varepsilon(w_{ij}) = \delta_{ij} \quad , \quad S(w_{ij}) = w_{ji}^*$$

In this situation, we have a quotient group $G \to H$, given by:

$$C(H) = < w_{ij} >$$

At the dual level we obtain a discrete quantum subgroup:

$$\widehat{\Lambda} \subset \widehat{\Gamma}$$

We will be back later to corepresentations, with a full theory.

6. Projective version

Given a quantum group G, with fundamental corepresentation $u = (u_{ij})$, the $N^2 \times N^2$ matrix given in double indices by

$$w_{ia,jb} = u_{ij} u_{ab}^*$$

is a corepresentation, and the following happen:

(1) The corresponding quotient $G \to PG$ is a quantum group.

(2) In the classical case, $G \subset U_N$, we have $PG = G/(G \cap \mathbb{T}^N)$.

(3) For the group duals, $\Gamma =< g_i >$, we have $\widehat{P\Gamma} =< g_i g_j^{-1} >$.

Summary

The compact quantum groups are subject to making:

1. Products $G \times H$
2. Dual free products $G \,\hat{*}\, H$
3. Free complexification $G \rightsquigarrow \widetilde{G}$
4. Subgroups $H \subset G$
5. Quotients $G \to H$
6. Projective versions $G \to PG$

However, as "basic input" we only have groups, and group duals.

Liberations 1/4

Theorem. We have quantum groups defined via

$$\begin{aligned} C(O_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big| u=\bar{u}, u^t=u^{-1}\right) \\ C(U_N^+) &= C^*\left((u_{ij})_{i,j=1,\ldots,N}\Big| u^*=u^{-1}, u^t=\bar{u}^{-1}\right) \end{aligned}$$

called free orthogonal, and free unitary quantum groups.

Proof. If u is biunitary/orthogonal, so are the matrices

$$(u^\Delta)_{ij}=\sum_k u_{ik}\otimes u_{kj} \quad , \quad (u^\varepsilon)_{ij}=\delta_{ij} \quad , \quad (u^S)_{ij}=u_{ji}^*$$

and so we can construct Δ,ε,S, by universality.

Liberations 2/4

The quantum groups O_N^+, U_N^+ have the following properties:

(1) The closed subgroups $G \subset U_N^+$ are exactly the $N \times N$ compact quantum groups.

(2) As for the closed subgroups $G \subset O_N^+$, these are exactly those satisfying $u = \bar{u}$.

(3) We have embeddings $O_N \subset O_N^+$ and $U_N \subset U_N^+$, obtained by dividing $C(O_N^+), C(U_N^+)$ by their commutator ideals.

Liberations 3/4

Theorem. The following inclusions are proper, at any $N \geq 2$:

$$\begin{array}{ccc} U_N & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & O_N^+ \end{array}$$

Proof. Follows by looking at group dual subgroups. Indeed, we have

$$\widehat{L}_N \subset O_N^+ \quad , \quad \widehat{F}_N \subset U_N^+$$

where $L_N = \mathbb{Z}_2^{*N}$, and where $F_N = \mathbb{Z}^{*N}$ is the free group.

Remark. We have a connection here with the "free tori".

Liberations 4/4

<u>Theorem</u>. We have intermediate liberations as follows,

$$\begin{array}{ccccc}
U_N & \longrightarrow & U_N^* & \longrightarrow & U_N^+\\
\uparrow & & \uparrow & & \uparrow\\
O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+
\end{array}$$

with $*$ meaning that u_{ij}, u_{ij}^* must satisfy the relations $abc = cba$.

<u>Proof</u>. If the entries of u "half-commute", so do the entries of

$$(u^\Delta)_{ij} = \sum_k u_{ik} \otimes u_{kj} \quad , \quad (u^\varepsilon)_{ij} = \delta_{ij} \quad , \quad (u^S)_{ij} = u_{ji}^*$$

so we can construct indeed Δ, ε, S. More can be said here (..)

Affine isometries

Question. Are our quantum groups compatible with the spheres?

Definition. Given an algebraic manifold $X \subset S^{N-1}_{\mathbb{C}}$, the formula

$$G(X) = \left\{ U \in U_N \middle| U(X) = X \right\}$$

defines a compact group of unitary matrices (a.k.a. isometries), called affine isometry group of X.

$\implies$ For the classical spheres $S^{N-1}_{\mathbb{R}}, S^{N-1}_{\mathbb{C}}$ we obtain in this way the classical groups O_N, U_N.

Quantum isometries

Given an algebraic manifold $X \subset S^{N-1}_{\mathbb{C},+}$, the category of the closed subgroups $G \subset U_N^+$ acting affinely on X, in the sense that

$$\Phi(x_i) = \sum_a u_{ia} \otimes x_a$$

defines a morphism of C^*-algebras, as follows,

$$\Phi : C(X) \to C(G) \otimes C(X)$$

has a universal object, denoted $G^+(X)$, and called "affine quantum isometry group" of X. This is indeed routine algebra.

Rotations and spheres 1/2

<u>Theorem.</u> The quantum isometry groups of the basic spheres,

$$\begin{array}{ccccc}
S^{N-1}_{\mathbb{C}} & \longrightarrow & S^{N-1}_{\mathbb{C},*} & \longrightarrow & S^{N-1}_{\mathbb{C},+}\\
\uparrow & & \uparrow & & \uparrow\\
S^{N-1}_{\mathbb{R}} & \longrightarrow & S^{N-1}_{\mathbb{R},*} & \longrightarrow & S^{N-1}_{\mathbb{R},+}
\end{array}$$

are the basic orthogonal and unitary quantum groups, namely:

$$\begin{array}{ccccc}
U_N & \longrightarrow & U_N^* & \longrightarrow & U_N^+\\
\uparrow & & \uparrow & & \uparrow\\
O_N & \longrightarrow & O_N^* & \longrightarrow & O_N^+
\end{array}$$

Rotations and spheres, 2/2

Proof. The variables $X_i = \sum_a u_{ia} \otimes x_a$ satisfy

$$\sum_i X_i X_i^* = \sum_{iab} u_{ia} u_{ib}^* \otimes x_a x_b^* = \sum_a 1 \otimes x_a x_a^* = 1 \otimes 1$$

$$\sum_i X_i^* X_i = \sum_{iab} u_{ia}^* u_{ib} \otimes x_a^* x_b = \sum_a 1 \otimes x_a^* x_a = 1 \otimes 1$$

so we have an action $U_N^+ \curvearrowright S^{N-1}_{\mathbb{C},+}$.

If the variables are u_{ij} are real, or half-commute, or commute, so do the variables X_i. Thus, we have actions everywhere.

Some routine work shows that all these actions are universal.

Conclusion

We have a theory of compact/discrete quantum groups, featuring:

(1) Simple axioms for the algebras $A = C(G) = C^*(\Gamma)$.

(2) The duality formulae $G = \widehat{\Gamma}$ and $\Gamma = \widehat{G}$ well understood.

(3) Manipulations with Δ, ε, S as our main tool, at least so far.

(4) Many examples (various liberations, standard operations).

(5) Compatibility of all this with the noncommutative tori/spheres.

$\Longrightarrow$ next lecture: representation theory