# Compact and discrete quantum groups

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"Introduction to quantum groups", 2/6

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#### Plan

- (1) Compact quantum groups
- (2) Discrete quantum groups
- (3) Basic examples, operations
- (4) Quantum isometry groups
- ⇒ next lecture: representations

#### Operator algebras

- $C^*$ -algebras: with norm and involution,  $||aa^*|| = ||a||^2$ .
- (1) Gelfand theorem: commutative case A = C(X).
- (2) Gelfand-Naimark-Segal theorem:  $A \subset B(H)$ .
- (3) Finite dimensions:  $A = M_{N_1}(\mathbb{C}) \oplus \ldots \oplus M_{N_k}(\mathbb{C})$ .
- $\implies$  A = C(X), with X "noncommutative compact space"
- ⇒ NC spheres, NC tori. What about quantum groups?

## Classical groups

Let G be a compact Lie group. Then  $G \subset U_N$ . Multiplication:

$$(UV)_{ij} = \sum_{k} U_{ik} V_{kj}$$

By Stone-Weierstrass we have  $C(G) = \langle u_{ij} \rangle$ , where:

$$u_{ij}(U)=U_{ij}$$

The multiplication  $G \times G \rightarrow G$  transposes as:

$$u_{ij} \to \sum_{k} u_{ik} \otimes u_{kj}$$

Thus G is well described by C(G), together with  $u = (u_{ij})$ .

#### **Axioms**

Let A be a  $C^*$ -algebra, with  $u \in M_N(A)$  biunitary  $(u, u^t \text{ unitaries})$ , whose entries generate A, such that:

- $-\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$  defines a morphism  $\Delta: A \to A \otimes A$
- $-\varepsilon(u_{ij})=\delta_{ij}$  defines a morphism  $\varepsilon:A o\mathbb{C}$
- $-S(u_{ij})=u_{ji}^*$  defines a morphism  $S:A o A^{opp}$

We write then  $A = C(G) = C^*(\Gamma)$ , and call:

- G a compact quantum group
- Γ a discrete quantum group

[axioms due to Woronowicz, 1987, slightly modified here]

# Compact groups 1/2

<u>Theorem</u>. For a closed subgroup  $G \subset U_N$ , the algebra A = C(G), with the matrix formed by the standard coordinates

$$u_{ij}(g)=g_{ij}$$

is a Woronowicz algebra, with structural maps given by

$$\Delta = m^T$$
 ,  $\varepsilon = u^T$  ,  $S = i^T$ 

where m, u, i are the multiplication, unit and inverse of G.

Any commutative Woronowicz algebra appears in this way.

# Compact groups 2/2

<u>Proof.</u> We compute  $m^T, u^T, i^T$ . We have:

$$m^{T}(u_{ij})(U \otimes V) = (UV)_{ij} = \sum_{k} U_{ik}V_{kj} = \sum_{k} (u_{ik} \otimes u_{kj})(U \otimes V)$$

Regarding now  $u^T$ , here we have:

$$u^T(u_{ij}) = 1_{ij} = \delta_{ij}$$

As for the map  $i^T$ , this is given by:

$$i^{T}(u_{ij})(U) = (U^{-1})_{ij} = \bar{U}_{ji} = u_{ji}^{*}(U)$$

Thus the axioms are satisfied, with  $\Delta = m^T$ ,  $\varepsilon = u^T$ ,  $S = i^T$ .

Finally, the last assertion follows by applying Gelfand.

## Group duals 1/2

<u>Theorem</u>. For a discrete group  $\Gamma = \langle g_1, \dots, g_N \rangle$ , the algebra  $A = C^*(\Gamma)$ , with the diagonal matrix formed by the generators

$$u = diag(g_1, \ldots, g_N)$$

is a Woronowicz algebra, with structural maps given by

$$\Delta(g) = g \otimes g$$
 ,  $\varepsilon(g) = 1$  ,  $S(g) = g^{-1}$ 

for any group element  $g \in \Gamma$ . This algebra is cocommutative, in the sense that  $\Sigma \Delta = \Delta$ , where  $\Sigma(a \otimes b) = b \otimes a$  is the flip.

<u>Remark</u>. We'll see later that any cocommutative Woronowicz algebra appears in this way (needs representation theory).

# Group duals 2/2

<u>Proof</u>. Consider the following unitary representation:

$$\Gamma o C^*(\Gamma) \otimes C^*(\Gamma)$$
 ,  $g o g \otimes g$ 

This produces a map  $\Delta: C^*(\Gamma) \to C^*(\Gamma) \otimes C^*(\Gamma)$ , given by:

$$\Delta(g)=g\otimes g$$

Similarly,  $\varepsilon$  comes from the trivial representation:

$$\Gamma 
ightarrow \{1\}$$
 ,  $g 
ightarrow 1$ 

As for S, this comes from the following representation:

$$\Gamma 
ightarrow \ \ C^*(\Gamma)^{opp} \quad , \quad g 
ightarrow g^{-1}$$

Remark. Note that the use of the opposite algebra is needed.

#### Comments 1/4

Assume that  $\Gamma$  is abelian, and let  $G=\widehat{\Gamma}$  be its Pontrjagin dual, formed by the characters  $\chi:\Gamma\to\mathbb{T}$ . The isomorphism

$$C^*(\Gamma) \simeq C(G)$$

transforms the structural maps of  $C^*(\Gamma)$ , given by

$$\Delta(g) = g \otimes g$$
 ,  $\varepsilon(g) = 1$  ,  $S(g) = g^{-1}$ 

into the structural maps of C(G), given by:

$$\Delta \varphi(g,h) = \varphi(gh)$$
 ,  $\varepsilon(\varphi) = \varphi(1)$  ,  $S\varphi(g) = \varphi(g^{-1})$ 

Thus,  $G = \widehat{\Gamma}$  is a compact quantum group isomorphism.

#### Comments 2/4

Motivated by this, given a Woronowicz algebra

$$A = C(G) = C^*(\Gamma)$$

we say that  $G, \Gamma$  are <u>dual to each other</u>, and write:

$$G = \widehat{\Gamma}$$
 ,  $\Gamma = \widehat{G}$ 

This duality extends the usual Pontrjagin duality.

## Comments 3/4

Motivated by the compact Lie group case, we have:

<u>Definition</u>. Given A = C(G), we denote by  $A \subset A$  the dense \*-algebra generated by the coordinates  $u_{ij}$ , and we write

$$A = C^{\infty}(G)$$

and call it "algebra of smooth functions" on G.

Example. For  $A = C^*(\Gamma)$  we have  $A = \mathbb{C}[\Gamma]$ .

#### Comments 4/4

Motivated by the group dual case, we have:

<u>Definition</u>. We agree to identify (A, u) and (B, v) when we have a \*-algebra isomorphism

$$\mathcal{A} \simeq \mathcal{B}$$

mapping standard coordinates to standard coordinates,  $u_{ij} 
ightarrow v_{ij}.$ 

Example. This identifies for instance  $C^*(\Gamma)$  with  $C^*_{red}(\Gamma)$ .

#### Summary

(1) We are looking at pairs (A, u), with  $u \in M_N(A)$  biunitary, with:

$$\Delta(u_{ij}) = \sum_{i} u_{ik} \otimes u_{kj}$$
 ,  $\varepsilon(u_{ij}) = \delta_{ij}$  ,  $S(u_{ij}) = u_{ji}^*$ 

(2) We have compact and discrete quantum groups, given by:

$$A = C(G) = C^*(\Gamma)$$

(3) These quantum groups are dual to each other, and we write:

$$G = \widehat{\Gamma}$$
 ,  $\Gamma = \widehat{G}$ 

(4) We set  $C^{\infty}(G) = \langle u_{ij} \rangle$ , and we use the identifications:

$$C^{\infty}(G) \simeq C^{\infty}(H)$$
 ,  $u_{ii} \rightarrow v_{ii}$ 

(5) All this is supported by  $C^*$ -algebras, and the above results.

#### Tech 1/2

<u>Theorem</u>. The comultiplication  $\Delta$ , counit  $\varepsilon$  and antipode S satisfy the following conditions,

- (1) Coassociativity:  $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ .
- (2) Counitality:  $(id \otimes \varepsilon)\Delta = (\varepsilon \otimes id)\Delta = id$ .
- (3) Coinversality:  $m(id \otimes S)\Delta = m(S \otimes id)\Delta = \varepsilon(.)1$ .

on the dense \*-subalgebra  $\mathcal{A}\subset A$  generated by the variables  $u_{ij}.$ 

<u>Proof.</u> Clear on coordinates, and so on the \*-algebra  $\mathcal{A}$ .

#### Tech 2/2

Remark. In the commutative case,  $G \subset U_N$ , we have

$$\Delta = m^T$$
 ,  $\varepsilon = u^T$  ,  $S = i^T$ 

and the 3 conditions satisfied by  $\Delta, \varepsilon, S$  come by transposition from the basic 3 conditions satisfied by m, u, i, namely

$$m(m \times id) = m(id \times m)$$
  
 $m(id \times u) = m(u \times id) = id$   
 $m(id \otimes i)\delta = m(i \otimes id)\delta = 1$ 

whre  $\delta(g) = (g, g)$ . In general, the philosophy is the same.

#### 1. Products

Given two compact quantum groups G, H, so is their product  $G \times H$ , constructed as follows:

$$C(G \times H) = C(G) \otimes C(H)$$

Equivalently, at the level of the associated discrete quantum groups  $\Gamma$ ,  $\Lambda$ , which are dual to G, H, we have:

$$C^*(\Gamma \times \Lambda) = C^*(\Gamma) \otimes C^*(\Lambda)$$

As an illustration, we have things of type  $G \times \widehat{\Lambda}$ , with  $G, \Lambda$  both classical, which are not classical, nor group duals.

#### 2. Dual free products

Given two compact quantum groups G, H, so is their dual free product  $G \mathbin{\hat{*}} H$ , constructed as follows:

$$C(G \mathbin{\hat{*}} H) = C(G) * C(H)$$

Equivalently, at the level of the associated discrete quantum groups  $\Gamma, \Lambda$ , which are dual to G, H, we have a usual free product:

$$C^*(\Gamma * \Lambda) = C^*(\Gamma) * C^*(\Lambda)$$

This construction always produces non-classical quantum groups, unless of course  $G = \{1\}$  or  $H = \{1\}$ .

## 3. Free complexification

Given a compact quantum group G, we can construct its free complexification  $\widetilde{G}$  as follows, where  $z = id \in C(\mathbb{T})$ :

$$C(\widetilde{G}) \subset C(\mathbb{T}) * C(G)$$
 ,  $\widetilde{u} = zu$ 

Equivalently, at the level of the associated discrete duals  $\Gamma, \widetilde{\Gamma}$ , we have the following formula, where  $z = 1 \in \mathbb{Z}$ :

$$C^*(\widetilde{\Gamma}) \subset C^*(\mathbb{Z}) * C^*(\Gamma)$$
 ,  $\widetilde{u} = zu$ 

We'll see later that the "free analogues" of  $O_N$ ,  $U_N$  are related by free complexification. Simpler than for  $O_N$ ,  $U_N$  themselves (!)

# 4. Subgroups

Let G be compact quantum group, and let  $I \subset C(G)$  be a closed \*-ideal satisfying the following "Hopf ideal" condition:

$$\Delta(I) \subset C(G) \otimes I + I \otimes C(G)$$

We have then a closed subgroup  $H \subset G$ , as follows:

$$C(H) = C(G)/I$$

Dually, we obtain a quotient of discrete quantum groups:

$$\widehat{\Gamma} \to \widehat{\Lambda}$$

In all this the Hopf ideal condition is needed for  $\Delta$  to factorize.

#### 5. Quotients

Let us call "corepresentation" of a Woronowicz algebra A = C(G) any unitary matrix  $w \in M_n(A)$  satisfying:

$$\Delta(w_{ij}) = \sum_{k} w_{ik} \otimes w_{kj}$$
 ,  $\varepsilon(w_{ij}) = \delta_{ij}$  ,  $S(w_{ij}) = w_{ji}^*$ 

In this situation, we have a quotient group  $G \rightarrow H$ , given by:

$$C(H) = \langle w_{ij} \rangle$$

At the dual level we obtain a discrete quantum subgroup:

$$\widehat{\Lambda} \subset \widehat{\Gamma}$$

We will be back later to corepresentations, with a full theory.

# 6. Projective version

Given a quantum group G, with fundamental corepresentation  $u = (u_{ij})$ , the  $N^2 \times N^2$  matrix given in double indices by

$$w_{ia,jb} = u_{ij}u_{ab}^*$$

is a corepresentation, and the following happen:

- (1) The corresponding quotient  $G \rightarrow PG$  is a quantum group.
- (2) In the classical case,  $G \subset U_N$ , we have  $PG = G/(G \cap \mathbb{T}^N)$ .
- (3) For the group duals,  $\Gamma = \langle g_i \rangle$ , we have  $\widehat{P\Gamma} = \langle g_i g_i^{-1} \rangle$ .

#### Summary

The compact quantum groups are subject to making:

- 1. Products  $G \times H$
- 2. Dual free products  $G \hat{*} H$
- 3. Free complexification  $G \leadsto \widetilde{G}$
- 4. Subgroups  $H \subset G$
- 5. Quotients  $G \rightarrow H$
- 6. Projective versions  $G \rightarrow PG$

However, as "basic input" we only have groups, and group duals.

#### Liberations 1/4

Theorem. We have quantum groups defined via

$$C(O_N^+) = C^* \left( (u_{ij})_{i,j=1,\dots,N} \middle| u = \bar{u}, u^t = u^{-1} \right)$$

$$C(U_N^+) = C^* \left( (u_{ij})_{i,j=1,\dots,N} \middle| u^* = u^{-1}, u^t = \bar{u}^{-1} \right)$$

called free orthogonal, and free unitary quantum groups.

<u>Proof.</u> If u is biunitary/orthogonal, so are the matrices

$$(u^{\Delta})_{ij} = \sum_{k} u_{ik} \otimes u_{kj}$$
 ,  $(u^{\varepsilon})_{ij} = \delta_{ij}$  ,  $(u^{S})_{ij} = u_{ji}^{*}$ 

and so we can construct  $\Delta, \varepsilon, S$ , by universality.

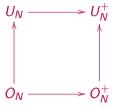
## Liberations 2/4

The quantum groups  $O_N^+$ ,  $U_N^+$  have the following properties:

- (1) The closed subgroups  $G \subset U_N^+$  are exactly the  $N \times N$  compact quantum groups.
- (2) As for the closed subgroups  $G \subset O_N^+$ , these are exactly those satisfying  $u = \bar{u}$ .
- (3) We have embeddings  $O_N \subset O_N^+$  and  $U_N \subset U_N^+$ , obtained by dividing  $C(O_N^+)$ ,  $C(U_N^+)$  by their commutator ideals.

## Liberations 3/4

<u>Theorem.</u> The following inclusions are proper, at any  $N \ge 2$ :



Proof. Follows by looking at group dual subgroups. Indeed, we have

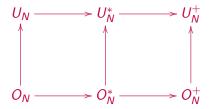
$$\widehat{L}_N \subset O_N^+$$
 ,  $\widehat{F}_N \subset U_N^+$ 

where  $L_N = \mathbb{Z}_2^{*N}$ , and where  $F_N = \mathbb{Z}^{*N}$  is the free group.

Remark. We have a connection here with the "free tori".

## Liberations 4/4

<u>Theorem</u>. We have intermediate liberations as follows,



with \* meaning that  $u_{ij}$ ,  $u_{ii}^*$  must satisfy the relations abc = cba.

<u>Proof.</u> If the entries of u "half-commute", so do the entries of

$$(u^{\Delta})_{ij} = \sum_{k} u_{ik} \otimes u_{kj}$$
 ,  $(u^{\varepsilon})_{ij} = \delta_{ij}$  ,  $(u^{S})_{ij} = u_{ji}^{*}$ 

so we can construct indeed  $\Delta, \varepsilon, S$ . More can be said here (..)

#### Affine isometries

Question. Are our quantum groups compatible with the spheres?

<u>Definition.</u> Given an algebraic manifold  $X\subset S^{N-1}_{\mathbb C}$ , the formula

$$G(X) = \left\{ U \in U_N \middle| U(X) = X \right\}$$

defines a compact group of unitary matrices (a.k.a. isometries), called affine isometry group of X.

 $\implies$  For the classical spheres  $S_{\mathbb{R}}^{N-1}, S_{\mathbb{C}}^{N-1}$  we obtain in this way the classical groups  $O_N, U_N$ .

#### Quantum isometries

Given an algebraic manifold  $X \subset S^{N-1}_{\mathbb{C},+}$ , the category of the closed subgroups  $G \subset U_N^+$  acting affinely on X, in the sense that

$$\Phi(x_i) = \sum_a u_{ia} \otimes x_a$$

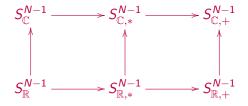
defines a morphism of  $C^*$ -algebras, as follows,

$$\Phi: C(X) \to C(G) \otimes C(X)$$

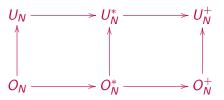
has a universal object, denoted  $G^+(X)$ , and called "affine quantum isometry group" of X. This is indeed routine algebra.

## Rotations and spheres 1/2

Theorem. The quantum isometry groups of the basic spheres,



are the basic orthogonal and unitary quantum groups, namely:



## Rotations and spheres, 2/2

<u>Proof</u>. The variables  $X_i = \sum_a u_{ia} \otimes x_a$  satisfy

$$\sum_{i} X_{i} X_{i}^{*} = \sum_{iab} u_{ia} u_{ib}^{*} \otimes x_{a} x_{b}^{*} = \sum_{a} 1 \otimes x_{a} x_{a}^{*} = 1 \otimes 1$$

$$\sum_{i} X_{i}^{*} X_{i} = \sum_{iab} u_{ia}^{*} u_{ib} \otimes x_{a}^{*} x_{b} = \sum_{a} 1 \otimes x_{a}^{*} x_{a} = 1 \otimes 1$$

so we have an action  $U_N^+ \curvearrowright S_{\mathbb{C},+}^{N-1}$ .

If the variables are  $u_{ij}$  are real, or half-commute, or commute, so do the variables  $X_i$ . Thus, we have actions everywhere.

Some routine work shows that all these actions are universal.

#### Conclusion

We have a theory of compact/discrete quantum groups, featuring:

- (1) Simple axioms for the algebras  $A = C(G) = C^*(\Gamma)$ .
- (2) The duality formulae  $G = \widehat{\Gamma}$  and  $\Gamma = \widehat{G}$  well understood.
- (3) Manipulations with  $\Delta, \varepsilon, S$  as our main tool, at least so far.
- (4) Many examples (various liberations, standard operations).
- (5) Compatibility of all this with the noncommutative tori/spheres.
  - ⇒ next lecture: representation theory