Orientability, toral subgroups and matrix models

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"Introduction to quantum groups", 6/6

07/20

Plan

(1) Easiness - review, more

(2) Orientability - questions

(3) Tori - diagonal, spinned

(4) Geometry - axiomatization

(5) Models - general theory

(6) Matrices - Weyl, Fourier

Easiness 1/4

A closed subgroup $G \subset U_N^+$ is called easy when

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \middle| \pi \in D(k,l)\right)$$

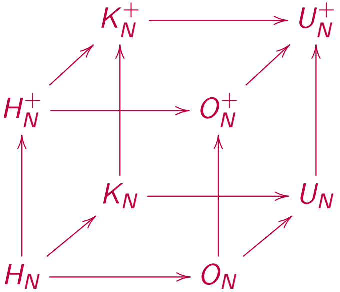
for certain sets of partitions $D(k,l) \subset P(k,l)$, where

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with $\{e_i\}$ = basis of $\mathbb{C}^N$, and δ_π = Kronecker type symbols.

Easiness 2/4

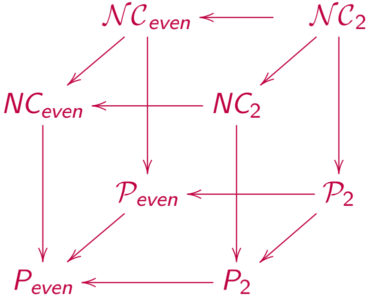
The main examples of easy quantum groups are as follows,



where $H_N = \mathbb{Z}_2 \wr S_N$, $K_N = \mathbb{T} \wr S_N$, $H_N^+ = \mathbb{Z}_2 \wr_* S_N^+$, $K_N = \mathbb{T} \wr_* S_N^+$.

Easiness 3/4

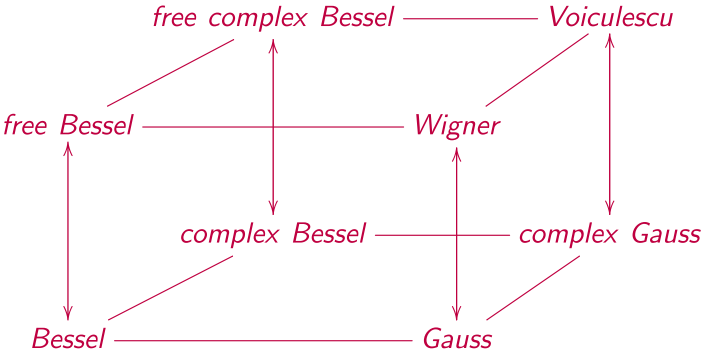
The corresponding categories of partitions are as follows,



with the calligraphic letters standing for "matching".

Easiness 4/4

The asymptotic laws of truncated characters are as follows,



with the vertical arrows standing for the Bercovici-Pata bijection.

Questions

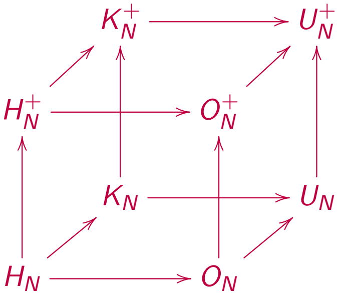
Classify. Compute laws. Find a "contravariant duality" as follows,

$$\begin{array}{ccccccc}
U_N & \longrightarrow & U_N^{(r)} & \longrightarrow & U_N^C & \longrightarrow & U_N^+\\
\vdots & & \vdots & & \vdots & & \vdots\\
H_N^+ & \longleftarrow & H_N^{[r]} & \longleftarrow & H_N^\Gamma & \longleftarrow & H_N
\end{array}$$

between the unitary and real reflection easy quantum groups.

Orientability 1/4

The standard cube is an intersection and generation diagram,



i.e. for any face $P \subset Q, R \subset S$ we have $P = Q \cap R, < Q, R >= S$.

Orientability 2/4

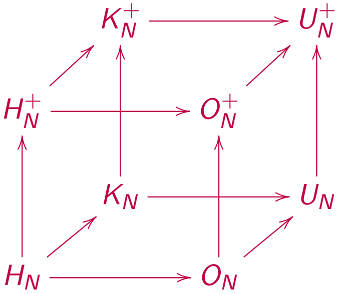
The needed technology here is as follows:

– Intersection $G \cap H$
– Tannaka $C_{G\cap H} =< C_G, C_H >$
– Easy case $D_{G\cap H} =< D_G, D_H >$
$\implies$ and this is OK for our cube problems

– Generation $< G, H >$
– Tannaka $C_{<G,H>} = C_G \cap C_H$
– Easy case $D_{\{G,H\}} = D_G \cap D_H$
– Conjecture $< G, H > \subset \{G, H\}$ iso
$\implies$ ad-hoc methods for 5 faces, 1 face left

Orientability 3/4

Ground Zero: the twistable, easy, uniform, oriented CQG are



where we know what easy means, and:

– uniform means $G = (G_N)$ with $G_{N-1} = G_N \cap U_{N-1}^+$

– twistable means here $H_N \subset G_N$, for any $N \in \mathbb{N}$

– oriented means "not disoriented" with respect to Ox,Oy,Oz

Orientability 4/4

Regarding the oriented CQG, under extra assumptions, mild:

1. classical: O_N, SO_N, U_N^d, H_N^{sd} + bistochastic versions
2. free: the known easy ones, and that's not trivial
3. group duals: abelian + varieties of real reflection groups

Here 1 looks doable, 2 looks hard, 3 is probably the simplest.

Questions

Classification of the "main" closed subgroups $G \subset U_N^+$:

$\Longrightarrow$ use partition methods, and intersection/generation surgery, in 3D or more, in order to "classify" G.

$\Longrightarrow$ the good 3 dimensions are those above, discrete/continuous, real/complex, classical/free.

$\Longrightarrow$ there are 3 more dimensions, “bad”, coming from taking the bistochastic version, special version, diagonal torus.

Conjecture: 6-parameter series + exceptional examples.

Tori 1/4

The diagonal torus $T \subset G$ is the group dual given by

$$C(T) = C(G) \Big/ \left\langle u_{ij} = 0 \middle| \forall i \neq j \right\rangle$$

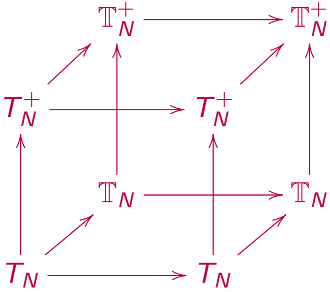
the generators of $\widehat{T}$ being $g_i = u_{ii}$. Equivalently, we can set

$$T = G \cap \mathbb{T}_N^+$$

where $\mathbb{T}_N^+ \subset U_N^+$ is the abstract dual of the free group F_N.

Tori 2/4

The diagonal tori of the main quantum groups are as follows,



where $T_N = \mathbb{Z}_2^N$, $\mathbb{T}_N = \mathbb{T}^N$ and $T_N^+ = \widehat{\mathbb{Z}_2^{*N}}$, $\mathbb{T}_N = \widehat{F_N}$.

Tori 3/4

Given $G \subset U_N^+$, consider its diagonal torus $T = G \cap \mathbb{T}_N^+$, and consider as well its reflection subgroup $K = G \cap K_N^+$:

$$T \subset K \subset G$$

Let also $G_{class} = G \cap U_N$. We say that G appears as:

– a soft liberation, when $G = < G_{class}, K >$

– a hard liberation, when $G = < G_{class}, T >$

$\implies$ OK (hard liberation) for O_N^+, U_N^+, and for O_N^*, U_N^* too.
$\implies$ cannot work for S_N^+, or B_N^+, C_N^+, and H_N^+, K_N^+ fail too.

Tori 4/4

Spinned tori, obtained by using the corepresentation $v = QuQ^*$:

$$\{T_Q \subset G | Q \in U_N\}$$

(1) Generation: $G =< (T_Q)_{Q\in U_N} >$.

(2) Weak generation: $G =< G_{class}, (T_Q)_{Q\in U_N} >$.

(3) Fourier liberation: $G =< G_{class}, (T_F)_{F=Fourier} >$.

(4) Hard liberation: $G =< G_{class}, T_1 >$.

No counterexamples to (1,2). It is known that (3) holds, beyond (4), for S_N^+, and for B_N^+, C_N^+ as well. No easy counterexamples.

Questions

The family $\{T_Q | Q \in U_N\}$ is the "maximal torus". Conjectures:

(1) Characters: if G is connected, for any nonzero $P \in C(G)_{central}$ there exists $Q \in U_N$ such that $P \neq 0$ inside $C(T_Q)$.

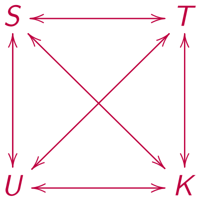
(2) Amenability: G is coamenable if and only if each of the tori T_Q is coamenable, in the usual discrete group sense.

(3) Growth: G has polynomial cogrowth if and only if each T_Q has polynomial cogrowth, in the usual discrete group sense.

$\implies$ OK for groups, group duals, main easy cases.

Geometry 1/4

Step 1. Axiomatize and classify the quadruplets



Step 2. Develop the geometries that you found.

Step 3. Integration theory, Riemannian aspects.

Step 4. Work more, reach to "Nash-Connes Geometry".

Geometry 2/4

A first difficulty is with $T \to U$. The axiom here must be:

$$U =< O_N, T >$$

(1) Classical real case: $O_N =< O_N, T_N >$, clear.

(2) Classical complex case: $U_N =< O_N, \mathbb{T}_N >$, true.

(3) Free real case: $O_N^+ =< O_N, T_N^+ >$. Very technical, by proving first $O_N^+ =< O_{N-1}^+, O_N >$, by recurrence on N.

(4) Free complex case: $U_N^+ =< O_N, \mathbb{T}_N^+ >$. Can be obtained from the free real case formula, by using standard arguments.

Geometry 3/4

A second difficulty is with $T\to K$. We have the following quantum isometry group computations, which are quite surprising:

$$\begin{matrix} T_N^+ & \longrightarrow & \mathbb{T}_N^+ \\ \uparrow & & \uparrow \\ T_N & \longrightarrow & \mathbb{T}_N \end{matrix} \quad \to \quad \begin{matrix} H_N^+ & \longrightarrow & K_N^+ \\ \vdots & & \vdots \\ O_N^{-1} & \longrightarrow & U_N^{-1} \end{matrix}$$

The solution is by saying that $T\to K$ appears as follows:

$$K=G^+(T)\cap K_N^+$$

That is, K must be the "quantum reflection group" of T.

Geometry 4/4

An abstract NCG must come from a quadruplet (S, T, U, K),

$$S^{N-1}_{\mathbb{R}} \subset S \subset S^{N-1}_{\mathbb{C},+} \quad , \quad T_N \subset T \subset \mathbb{T}^+_N$$

$$O_N \subset U \subset U^+_N \quad , \quad H_N \subset K \subset K^+_N$$

such that we can pass from each object to all the other objects,

$$\begin{matrix} S & = & S_{<O_N, T>} & = & S_U & = & S_{<O_N, K>} \\ S \cap \mathbb{T}^+_N & = & T & = & U \cap \mathbb{T}^+_N & = & K \cap \mathbb{T}^+_N \\ G^+(S) & = & < O_N, T > & = & U & = & < O_N, K > \\ G^+(S) \cap K^+_N & = & G^+(T) \cap K^+_N & = & U \cap K^+_N & = & K \end{matrix}$$

with all this being up to the "full=reduced" equivalence relation.

Questions

We have 9 main geometries in our sense, which are all easy:

$$
\begin{array}{ccccc}
\mathbb{R}^N_+ & \longrightarrow & \mathbb{TR}^N_+ & \longrightarrow & \mathbb{C}^N_+ \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}^N_* & \longrightarrow & \mathbb{TR}^N_* & \longrightarrow & \mathbb{C}^N_* \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{R}^N & \longrightarrow & \mathbb{TR}^N & \longrightarrow & \mathbb{C}^N
\end{array}
$$

Under some mild extra axioms (..), these are the only ones.

$\implies$ must first "develop" all these geometries
$\implies$ then go towards "Nash-Connes Geometry"

Models 1/4

We are interested in random matrix models for our algebras:

$$\pi : C(G) \to M_K(C(T))$$

The Hopf image of π is the smallest quotient Hopf C^*-algebra $C(G) \to C(H)$ producing a factorization of type

$$\pi : C(G) \to C(H) \to M_K(C(T))$$

When $H \subset G$ is an isomorphism, we say that π is inner faithful.

Models 2/4

The inner faithful models $\pi : C(G) \to M_K(C(T))$ are conjectured to exist in general, and remind the quantum group:

(1) The Tannakian category of G is given by the formula

$$C_{kl} = Hom(U^{\otimes k}, U^{\otimes l})$$

where $U_{ij} = \pi(u_{ij})$, with formal intertwiner spaces on the right.

(2) The Haar integration over G is given by the formula

$$\int_G = \lim_{k\to\infty} \frac{1}{k} \sum_{r=1}^{k} \int_G^r$$

where $\int_G^r = (\varphi \circ \pi)^{*r}$, with $\varphi = tr \otimes \int_T$ being the standard trace.

Models 3/4

A model $\pi : C(G) \to M_K(C(T))$ is called stationary when:

$$\int_G = \left(tr \otimes \int_T\right)\pi$$

In this case, the model must be faithful. We have as basic example

$$C(O_N^*) \to M_2(C(U_N)) \quad , \quad u_{ij} \to \begin{pmatrix} 0 & v_{ij} \\ \bar{v}_{ij} & 0 \end{pmatrix}$$

where v is the fundamental corepresentation of $C(U_N)$, as well as

$$C(U_N^*) \to M_2(C(U_N \times U_N)) \quad , \quad u_{ij} \to \begin{pmatrix} 0 & v_{ij} \\ w_{ij} & 0 \end{pmatrix}$$

with v, w corresponding to the two copies of $C(U_N)$.

Models 4/4

As another basic example, we have a stationary matrix model

$$\pi : C(S_4^+) \to M_4(C(SU_2))$$

given on the standard coordinates by the formula

$$\pi(u_{ij}) = [x \to Proj(c_i x c_j)]$$

where $x \in SU_2$, and c_1, c_2, c_3, c_4 are the Pauli matrices.

Questions

(1) Half-liberation.

(2) Weyl matrix models.

(3) Universal flat models.

(4) Sinkhorn and other.

Matrices 1/4

A pair of orthogonal MASA is a pair of maximal abelian subalgebras

$$B, C \subset A$$

which are orthogonal: $tr(bc) = tr(b)tr(c)$, for any $b \in B, c \in C$.

Popa: up to a unitary, the pairs of orthogonal MASA in the simplest von Neumann factor, namely $M_N(\mathbb{C})$, are

$$A = \Delta \quad , \quad B = H\Delta H^*$$

with $\Delta =$ diagonal matrices, and $H \in M_N(\mathbb{C})$ being Hadamard (entries on the unit circle, rows pairwise orthogonal).

Matrices 2/4

(1) Given $H \in M_N(\mathbb{C})$ Hadamard, the associated pair of MASA fit into a "commuting square" in the sense of subfactor theory:

$$\begin{array}{ccc} \Delta & \longrightarrow & M_N(\mathbb{C}) \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & H\Delta H^* \end{array}$$

(2) By "basic construction" we obtain a subfactor $Q \subset R$, whose invariants can be computed using "Ocneanu compactness".

(3) Work of Jones shows that Q must appear as fixed point algebra under some kind of "quantum permutation group" action.

Matrices 3/4

Given an Hadamard matrix $H \in M_N(\mathbb{C})$, the rank 1 projections

$$P_{ij} = Proj\left(\frac{H_i}{H_j}\right)$$

where $H_1, \ldots, H_N \in \mathbb{T}^N$ are the rows of H, form a magic unitary.

$\implies$ We associate to H the quantum permutation group $G \subset S_N^+$ given by the following Hopf image factorization,

$$\begin{array}{ccc} C(S_N^+) & \xrightarrow{\quad\pi\quad} & M_N(\mathbb{C}) \\ & \searrow \qquad \nearrow & \\ & C(G) & \end{array}$$

where $\pi(u_{ij}) = Proj(H_i/H_j)$ are the above rank 1 projections.

Matrices 4/4

The main results regarding $H \to G$ are as follows:

(1) Fourier: $F_G \to G$. Also $H' \otimes H'' \to G' \times G$".

(2) Various abstract results: Haar, Tannaka, duality.

(3) Relation with commuting squares and subfactors.

(4) Diță deformations of F_G, with various parameters.

(5) Extensions to the partial Hadamard matrix setting.

Questions

Physics!