Group actions and fixed point subfactors

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Basic subfactors

Theorem 1. Given a finite group G, acting on a II_1 factor P, in a minimal way, $(P^G)' \cap P = \mathbb{C}$, we have a subfactor

$$P \subset P \rtimes G$$

of index $N = |G|$, called "depth 2 subfactor".

Theorem 2. Given a compact group G, acting minimally on a II_1 factor P, for any finite index subgroup $H \subset G$,

$$P^G \subset P^H$$

is a subfactor of index $N = [G : H]$, called "subgroup subfactor".

Further examples

Theorem 3. Given a compact group G, acting minimally on a II_1 factor P, for any projective representation $G \to PU_n$,

$$P^G \subset (M_n(\mathbb{C}) \otimes P)^G$$

is a subfactor of index $N = n^2$, called "Wassermann subfactor".

Theorem 4. Given a discrete group $\Gamma =< g_1, \ldots, g_n >$, acting on a II_1 factor Q, in an outer way, $Q' \cap Q \rtimes \Gamma = \mathbb{C}$,

$$\left\{ diag(g_1(q), \ldots, g_n(q)) \Big| q \in Q \right\} \subset M_n(Q)$$

is a subfactor of index $N = n^2$, called "diagonal subfactor".

Unification

The main examples of subfactors of integer index, namely

$$P \subset P \rtimes G \ , \ P^G \subset P^H \ , \ P^G \subset M_n(P)^G \ , \ Q^{\curvearrowleft \Gamma} \subset M_n(Q)$$

can be written in a uniform way, as "fixed point subfactors",

$$(C(G) \otimes P)^G \subset (B(l^2(G)) \otimes P)^G$$

$$(\mathbb{C} \otimes P)^G \subset (C(G/H) \otimes P)^G$$

$$(\mathbb{C} \otimes P)^G \subset (M_n(\mathbb{C}) \otimes P)^G$$

$$(\mathbb{C} \otimes (Q \rtimes \Gamma))^{\widehat{\Gamma}} \subset (M_n(\mathbb{C}) \otimes (Q \rtimes \Gamma))^{\widehat{\Gamma}}$$

and so they are of the same nature, namely

$$(A_0 \otimes P)^G \subset (A_1 \otimes P)^G$$

with $A_0 \subset A_1$ being FD algebras, and G being a quantum group.

Quantum groups 1/4

Definition. A Woronowicz algebra is a C^*-algebra A, given with a unitary matrix $u \in M_N(A)$ whose entries generate A, such that:

- $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$ defines a morphism $\Delta : A \to A \otimes A$.
- $\varepsilon(u_{ij}) = \delta_{ij}$ defines a morphism $\varepsilon : A \to \mathbb{C}$.
- $S(u_{ij}) = u_{ji}^*$ defines a morphism $S : A \to A^{opp}$.

Notation. Given a Woronowicz algebra A we write

$$A = C(G) = C^*(\Gamma)$$

and call G, Γ compact and discrete quantum groups.

Quantum groups 2/4

Example 1. Given a compact Lie group $G \subset U_N$, we have

$$A = C(G) \quad , \quad u_{ij}(g) = g_{ij}$$

with $\Delta = m^T, \varepsilon = u^T, S = i^T$ being the transposes of m, u, i.

Example 2. Given a discrete group $\Gamma =< g_1, \ldots, g_N >$, we have

$$A = C^*(\Gamma) \quad , \quad u = diag(g_i)$$

with $\Delta(g) = g \otimes g, \varepsilon(g) = 1, S(g) = g^{-1}$ on group elements.

Quantum groups 3/4

Theorem. Any Woronowicz algebra has a Haar integration,

$$\left(\int_G \otimes id\right)\Delta = \left(id \otimes \int_G\right)\Delta = \int_G (.)1$$

constructed by starting with $\varphi \in A^*$ unital positive, and setting

$$\int_G = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^n \varphi^{*k}$$

with the convolution operation being $\phi * \psi = (\phi \otimes \psi)\Delta$.

Quantum groups 4/4

Definition. A corepresentation of a Woronowicz algebra A is a unitary matrix $v \in M_n(\mathcal{A})$ satisfying

$$\Delta(v_{ij}) = \sum_k v_{ik} \otimes v_{kj} \quad , \quad \varepsilon(v_{ij}) = \delta_{ij} \quad , \quad S(v_{ij}) = v_{ji}^*$$

where $\mathcal{A} \subset A$ is the dense $*$-subalgebra of "smooth elements".

Theorem. The following Peter-Weyl type results hold:

(1) Any corepresentation decomposes as a sum of irreducibles.

(2) The irreducibles appear inside $u^{\otimes k}$, with $k =$ colored integer.

(3) We have $\mathcal{A} = \oplus_{r \in Irr(A)} B(H_r)$, $*$-coalgebra isomorphism, $\perp$.

(4) The characters of irreps form an orthonormal basis of $\mathcal{A}_{central}$.

Fixed point subfactors

Theorem. Given a compact quantum group G, acting on a von Neumann algebra P, and acting as well on an inclusion $A_0 \subset A_1$ of finite dimensional algebras, we have the following inclusion:

$$(A_0 \otimes P)^G \subset (A_1 \otimes P)^G$$

When P is a II_1 factor, the action of G on it is minimal, and $A_0 \subset A_1$ is a Markov inclusion, with G being ergodic on both $Z(A_0), Z(A_1)$, this is a subfactor of index $N = [A_1 : A_0]$.

Examples

The main examples of subfactors of integer index, namely

$$(C(G) \otimes P)^G \subset (B(l^2(G)) \otimes P)^G$$

$$(\mathbb{C} \otimes P)^G \subset (C(G/H) \otimes P)^G$$

$$(\mathbb{C} \otimes P)^G \subset (M_n(\mathbb{C}) \otimes P)^G$$

$$(\mathbb{C} \otimes (Q \rtimes \Gamma))^{\widehat{\Gamma}} \subset (M_n(\mathbb{C}) \otimes (Q \rtimes \Gamma))^{\widehat{\Gamma}}$$

all appear as fixed point subfactors, in our sense.

Theory

Consider a fixed point subfactor $(A_0 \otimes P)^G \subset (A_1 \otimes P)^G$.

Theorem 1. The Jones tower of the subfactor is $(A_i \otimes P)^G$, where $\{A_i\}$ is the Jones tower for $A_0 \subset A_1$.

Theorem 2. The standard invariant, taken in the standard lattice sense of Popa, is the lattice $(A_i' \cap A_j)^G$.

Theorem 3. In the case $A_0 = \mathbb{C}$, which covers most of the interesting examples, the planar algebra is $P_k = A_k^G$.

Proof

Consider indeed an arbitrary fixed point subfactor:

$$(A_0 \otimes P)^G \subset (A_1 \otimes P)^G$$

The Jones tower is then obtained as follows:

$$(A_0 \otimes P)^G \subset (A_1 \otimes P)^G \subset (A_2 \otimes P)^G \subset \ldots$$

When computing the relative commutants P dissapears,

$$\left[(A_i \otimes P)^G\right]' \cap (A_j \otimes P)^G = (A_i' \cap A_j)^G$$

("invariance principle") and this gives all the results.

Quantum permutations 1/2

In order to construct new examples, we need actions

$$G \curvearrowright A$$

of compact quantum groups G on FD algebras A. But

$$A = C(X)$$

with X "finite noncommutative space". Thus, we need actions

$$G \curvearrowright X$$

of compact quantum groups G on finite NC spaces X.

$\Longrightarrow$ Needs "quantum permutations", in a very general sense.

Quantum permutations 2/2

Theorem. Let X be a "finite noncommutative space", coming from a FD algebra A, which must be of the form:

$$A = M_{n_1}(\mathbb{C}) \oplus \ldots \oplus M_{n_k}(\mathbb{C})$$

Then X has a quantum automorphism group $G^+(X)$, as follows:

(1) Start with the quantum group U_N^+, whose coordinates form a free $N \times N$ biunitary matrix, where $N = |X| = \dim A$.

(2) Impose the conditions coming from the existence of a coaction map $A \to C(G) \otimes A$, which leaves invariant the canonical trace.

TL and FC

Theorem. Given $G = G^+(X)$ as above, the subfactor

$$P^G \subset (A \otimes P)^G$$

with $A = C(X)$ corresponds to the algebra TL_N.

Remark. This applies in particular to the quantum groups:

$$S_N^+ = G^+(1, \ldots, N) \quad , \quad PO_N^+ = PU_N^+ = G^+(M_n)$$

Theorem. With $G = G^+(X \to Y)$, the subfactor

$$(B \otimes P)^G \subset (A \otimes P)^G$$

with $A = C(X), B = C(Y)$ corresponds to the algebra FC_N.

Finite graphs

(1) The Fuss-Catalan construction is best implemented by using classical finite spaces, of type $\{1,\ldots,N\}$. We are led to

$$H_N^{s+}=\mathbb{Z}_s\wr_* S_N^+$$

called quantum reflection groups, which liberate the classical reflection groups $H_N^s=\mathbb{Z}_s\wr S_N$.

(2) These quantum groups appear as quantum automorphism groups of finite graphs. In the general graph setting

$$P=<\square>$$

is a planar algebra generated by a 2-box, in the sense of Bisch-Jones and Liu, with the Laplacian of the graph being in the box.