Subfactors of small index and big index

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"Introduction to subfactor theory", 6/6

07/20

Subfactors

Theorem. Consider a subfactor $A_0 \subset A_1$, of finite index $N \in [1, \infty)$. Build by "basic construction" the associated Jones tower:

$$A_0 \subset_{e_1} A_1 \subset_{e_2} A_2 \subset_{e_3} A_3 \subset \ldots$$

The Jones projections generate then a copy of TL_N:

$$TL_N = < e_1, e_2, e_3, \ldots >$$

The planar algebra structure of TL_N extends into a planar algebra structure of the graded algebra $P = (P_k)$, where

$$P_k = A_0' \cap A_k$$

are the higher relative commutants, which are FD algebras.

Theory

Theorem 1. The subfactors $A_0 \subset A_1$ having "finite depth" are classified by their planar algebras $P = (P_k)$.

Theorem 2. More generally, the "amenable" subfactors $A_0 \subset A_1$ are classified by their planar algebras $P = (P_k)$.

Theorem 3. In general, any planar algebra produces a subfactor (complementing "any subfactor produces a planar algebra").

Question. What are the planar algebras of the subfactors of the Murray-von Neumann hyperfinite factor R?

Invariants

The good. The spectral measure μ, having as moments:

$$M_k = \dim(P_k)$$

The bad. The Poincaré series, Stieltjes transform of μ:

$$f(z) = \sum_{k=0}^{\infty} \dim(P_k) z^k$$

The ugly. The principal graph, Bratelli diagram of

$$P_0 \subset P_1 \subset P_2 \subset \dots$$

with the reflections coming from basic constructions removed.

Small index 1/4

Theorem. The subfactors of index $N \leq 4$, which must satisfy

$$N \in \left\{ 4\cos^2\left(\frac{\pi}{n}\right) \Big| n \in \mathbb{N} \right\}$$

are subject to an ADE classification result.

For spin model subfactors, and at $N = 4$, this is related to:

Theorem. The quantum groups $G \subset S_4^+$ appear via

$$S_4^+ = SO_3^{-1}$$

as twists of the usual ADE subgroups of SO_3.

Small index 2/4

Invariants. In general, $N \leq 4$, these can be computed as follows:

(1) The principal graphs are ADE.

(2) The Poincaré series coefficients count loops on these graphs.

(3) The spectral measures can be recovered by Stieltjes.

At $N = 4$ we can simply compute laws of characters.

Advanced. The Jones manipulation on the Poincaré series,

$$\Theta(q) = q + \frac{1-q}{1+q} f\left(\frac{q}{(1+q)^2}\right)$$

blows up the spectral measure on $\mathbb{T}$. Very simple formulae.

Small index 3/4

Theorem. The subfactors of index $N \leq 5$ and a bit higher can be fully classified, by using advanced planar algebra techniques.

For spin model subfactors, and at $N = 5$, this is related to:

Theorem. The quantum groups $G \subset S_5^+$ can be fully classified, by using the above subfactor classification result.

Small index 4/4

Question 1. What is the correct blowup of the spectral measure, in index 5, and more generally, in the "understood" index range?

Question 2. As a consequence, the inclusion $S_N \subset S_N^+$ follows to be maximal at $N = 4, 5$. What about $N = 6$, and in general?

Question 3. What is the natural extra assumption to be added, as for the subfactors of index 6 to become classifiable?

Big index 1/6

Motivation. The various mathematical "objects", once classified by classification theorems, fall into two classes:

(1) Serial.

(2) Exceptional.

This happens for instance for the simple Lie algebras, or for the complex reflection groups. There are many other examples.

Big index 2/6

There are many "uniform" constructions of subfactors. In the quantum group context, the uniformity comes via:

Definition. A closed subgroup $G \subset U_N^+$ is called easy when

$$Hom(u^{\otimes k}, u^{\otimes l}) = span\left(T_\pi \Big| \pi \in D(k,l)\right)$$

for a certain category of partitions $D \subset P$, where

$$T_\pi(e_{i_1} \otimes \ldots \otimes e_{i_k}) = \sum_{j_1 \ldots j_l} \delta_\pi \begin{pmatrix} i_1 & \ldots & i_k \\ j_1 & \ldots & j_l \end{pmatrix} e_{j_1} \otimes \ldots \otimes e_{j_l}$$

with $\delta_\pi \in \{0,1\}$ depending on whether the indices fit or not.

Big index 3/6

<u>Theorem</u>. The basic unitary quantum groups, namely

$$\begin{array}{ccc} O_N^+ & \longrightarrow & U_N^+ \\ \uparrow & & \uparrow \\ O_N & \longrightarrow & U_N \end{array}$$

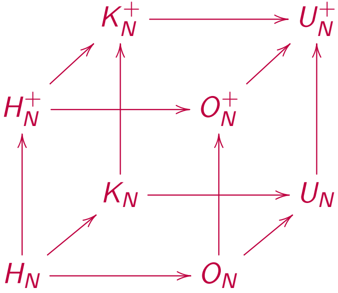
are all easy, coming from the following categories of pairings:

$$\begin{array}{ccc} NC_2 & \longleftarrow & \mathcal{NC}_2 \\ \downarrow & & \downarrow \\ P_2 & \longleftarrow & \mathcal{P}_2 \end{array}$$

The spectral measures are normal $\mathbb{R}/\mathbb{C}$ and $\cap/\bigcirc$ with $N \to \infty$.

Big index 4/6

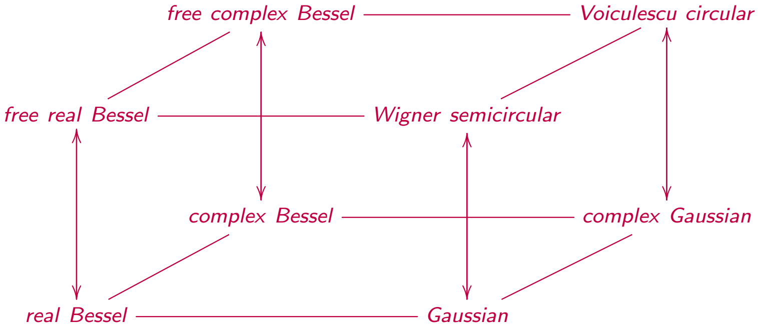
Theorem. The main unitary and reflection quantum groups



are all easy, coming from various basic categories of partitions.

Big index 5/6

<u>Theorem</u>. The asymptotic laws of truncated characters are



with the vertical arrows standing for the Bercovici-Pata bijection.

Big index 6/6

Various questions, which are under current investigation:

(1) Classification of the easy quantum groups.

(2) Various extensions of the easiness theory.

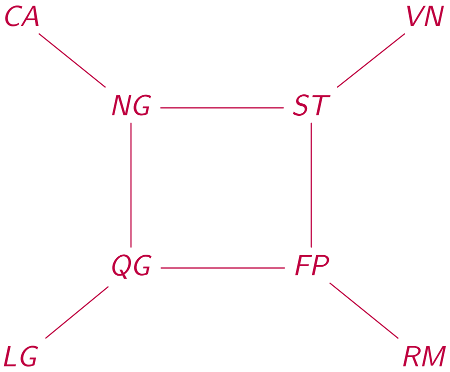
(3) In particular, the super-quizziness problem.

(4) Extensions covering the noncommutative tori.

All this is a mixture of QG, NG, FP, of interest for ST.

Conclusion

When looking for "serial subfactors", we are led to the scheme



for operator algebras in general, with the hot stuff in the middle.

Question

What about R?